# Research Synopsis

June 2018

*Solid mechanics provides the canvas for my research projects, both out of personal interest and because that field is at the root of my educational background.*

*Four underlying themes are presented below: Homogenization, Dimensional reduction, Defect mechanics and Elasto-Plasticity which can also be thought to be part of defect mechanics, the defect being the plastic slip.*

*For the sake of brevity, I will not elaborate here on three themes: Nonlinear parabolic equations and Monotone multi-valued graphs, [13,17,40,59]*[1]*; Material conservation laws [2,9,42]; Passage from discrete to continuous models [46].*

*Reference [56] is both an expository article and a book describing the state of the art – at least up to the end of 2006 – of the rapidly expanding Variational theory of brittle fracture which is the topic of Subsection 3.2.*

*Finally please bear in mind that this synopsis may be a bit out of sync, although I am trying to fill in the gaps every so often ......*

[1] Throughout this synopsis the numbers in brackets refer to the corresponding publication in my publication list; see detailed CV.

# 1 Homogenization

From a mathematical standpoint, homogenization can be seen as the study of P.D.E.'s with highly oscillating coefficients. A model equation is

$$-\text{div}(A^\varepsilon(x)\nabla u^\varepsilon) = f \text{ in } \Omega,$$
$$u^\varepsilon = 0 \text{ on } \partial\Omega,$$

where $\Omega$ is a bounded open subset of $\mathbb{R}^N$, $f \in H^{-1}(\Omega)$, and $A^\varepsilon(x)$ is an $\varepsilon$-indexed sequence of $N \times N$ matrices with measurable coefficients satisfying

$$A^\varepsilon(x)\xi \cdot \xi \geq \alpha|\xi|^2,$$
$$(A^\varepsilon(x))^{-1}\xi \cdot \xi \geq \beta|\xi|^2,$$

and $0 < \alpha, \beta < \infty$.

We propose to investigate the ($W_0^{1,2}(\Omega)$-weak) limits $u$ of converging subsequences of $u^\varepsilon$. A fundamental compactness result due to F. Murat & L. Tartar or, when restricted to symmetric $A^\varepsilon$'s , to S. Spagnolo, asserts the existence of a subsequence $A^{\varepsilon'}$ of $A^\varepsilon$ and of a (possibly $x$-dependent) $A^0$, endowed with the same boundedness and coercivity properties as $A^\varepsilon$, and such that, for any $f \in H^{-1}(\Omega)$:

$$u^{\varepsilon'} \rightharpoonup u, \quad \text{weakly in } W^{1,2}(\Omega),$$
$$A^{\varepsilon'}\nabla u^{\varepsilon'} \rightharpoonup A^0\nabla u, \quad \text{weakly in } L^2(\Omega; \mathbb{R}^N),$$

with

$$-\text{div}(A^0(x)\nabla u) = f \text{ in } \Omega,$$
$$u = 0 \text{ on } \partial\Omega.$$

The matrix $A^{\varepsilon'}$ is said to H-converge to $A^0$. If thinking in terms of a mixture of conducting materials, $A^\varepsilon(x)$ stands for the conductivity tensor at the point $x$, and $\varepsilon$ is a small parameter representing the fineness of the mixture; in turn $A^0$ is the macroscopic conductivity tensor and $u$ the average temperature field. In particular, if considering periodic mixtures, $A^\varepsilon(x) = A(x/\varepsilon)$, with $A(y)$ defined on the unit torus, in which case the whole sequence H-converges to a constant $A^0$.

Also, if $A^\varepsilon \stackrel{H}{\rightharpoonup} A^0$, the solution $u$ can be corrected, so as to obtain a strong approximation for $u^\varepsilon$: there exists a matrix $P^\varepsilon(x) \in L^2(\Omega; \mathbb{R}^{N\times N})$ which only depends upon $A^\varepsilon$ and $\Omega$ and which is such that

$$\nabla u^\varepsilon - P^\varepsilon\nabla u \longrightarrow 0, \text{ strongly in } L^1(\Omega; \mathbb{R}^{N\times N}),$$

and even a bit better than that.

Within such a framework, I have studied several types of problems detailed in the subsections below.

The above-mentioned compactness result can be extended to much more general elliptic settings: first and foremost, linear elasticity, but also monotone operators. L. TARTAR established a similar compactness result for equations of the type

$$
\begin{aligned}
-\mathrm{div}(a^\varepsilon(x, \nabla u^\varepsilon)) &= f \text{ in } \Omega,\\
u^\varepsilon &= 0 \text{ on } \partial\Omega,
\end{aligned}
$$

where $\Omega$ is a bounded open subset of $I\!R^N$, $f \in H^{-1}(\Omega)$, and $(x,e) \in \Omega \times I\!R^N \to a^\varepsilon(x,e) \in I\!R^N$ is an $\varepsilon$-indexed sequence of strongly monotone Carathéodory maps , i.e., such that

$$
\begin{aligned}
(a^\varepsilon(x,e) - a^\varepsilon(x,e')) \cdot (e-e') &\geq \quad \alpha|e-e'|^2,\\
((a^\varepsilon)^{-1}(x,e) - (a^\varepsilon)^{-1}(x,e')) \cdot (e-e') &\geq \beta^{-1}|e-e'|^2,
\end{aligned}
$$

with $0 < \alpha, \beta < \infty$. In [40] and its sequel [59], F. MURAT, L. TARTAR and I investigate the compactness under a homogenization process of elliptic systems where the pair $(e, a^\varepsilon(x,e))$ is replaced by a $x$-parameterized maximal monotone graph in $e$; compactness is established under hypotheses that I will not detail now. This is to my knowledge the most general framework for which existence and stability under homogenization holds true. As an additional note intended to readers that are familiar with monotonicity methods, the mathematical tools that we use are much more straightforward than those usually at work when dealing with maximal monotone graphs; in particular, the delicate measurability issues introduced by the $x$-dependence are completely eschewed; this is new even as far as existence at fixed $\varepsilon$ is concerned.

## 1.1 Dynamics

Evolution problems with spatially oscillating coefficients are investigated in [1,3,5,6,8,12,15,19,21]. The model problem is the wave equation

$$
\begin{aligned}
\frac{\partial^2 u^\varepsilon}{\partial t^2} - \mathrm{div}(A^\varepsilon(x)\nabla u^\varepsilon) &= 0, && \text{in } \Omega,\\
u^\varepsilon(x,0) = u_0^\varepsilon &\rightharpoonup u_0, && \text{weakly in } H_0^1(\Omega),\\
\frac{\partial u^\varepsilon}{\partial t} = v_\varepsilon &\rightharpoonup v_0, && \text{weakly in } L^2(\Omega)\\
u^\varepsilon &= 0, && \text{on } \partial\Omega,
\end{aligned}
$$

with $A^\varepsilon \overset{H}{\rightharpoonup} A^0$. The solution $u^\varepsilon$ is easily seen to converge (weakly in the appropriate topology) to $u$, the solution of the wave equation with $A^0$ as acoustic matrix and $u_0, v_0$ as initial conditions. Correcting the solution $u$, so as to obtain, as in the elliptic case, a better approximation of $u^\varepsilon$ is not so straightforward. In fact, it cannot be accomplished unless $u_0^\varepsilon$ satisfies a compatibility condition. If such is not the case, some of the energy has been lost during the limit process [19]. It is not possible as of yet to recover the lost energy in all generality. This is achieved in [21] in the very simple case where $A^\varepsilon(x) = A(x)$, at the expense of a non trivial study of the propagation of the microlocal defect measure (a tool introduced by P. Gérard, and under the name of H-measure by L. Tartar) associated to the space-time gradient of $u^\varepsilon$.

In a different direction, references such as [3,5,6,8,15] examine more complex evolution systems – coupled evolution equations originating in thermomechanics – the form of which is altered by the homogenization process. Various pathologies are evidenced: non-locality in time of the homogenized equations , change of initial conditions, ..... As an example, let us mention that the mixture of two Kelvin-Voigt viscoelastic materials gives rise to a material with fading memory and a rather complex thermal dissipation [8]. Satisfactory stability criteria for an evolution system undergoing homogenization are lacking at the present time.

## 1.2 Bounds and exact relations

The precise geometric structure of a mixture is often unknown even in the case of binary mixtures. Bounds on the possible macroscopic behaviors may thus prove useful. In the language of H-convergence the question may be phrased as follows (for the model case of linear conduction). Consider

$$\begin{aligned} A^\varepsilon(x) = \chi^\varepsilon(x)A + (1-\chi^\varepsilon(x))B, &\quad \chi^\varepsilon(x) \in \{0,1\}\text{a.e.},\\ A^\varepsilon \overset{H}{\rightharpoonup} A^0&,\\ \chi^\varepsilon \rightharpoonup \theta, &\quad \text{weak-* in } L^\infty(\Omega). \end{aligned}$$

Can one characterize the set of all such matrices $A^0$? That set is called the $G$-closure. Or still, can one characterize the set of all such matrices $A^0$ if $\theta \in L^\infty(\Omega;[0,1])$ is given? That set is the $G_\theta$-closure. The answer is known since the work of Z. Hashin & S. Shtrikman (formal argument in the case of an isotropic macroscopic behavior), F. Murat & L. Tartar (general case), K. Lurie & A. Cherkaev (two-dimensional case) in the specific setting of two-phase linear *and isotropic* conduction, i.e., $A = \alpha I, B = \beta I$.

Reference [7] is devoted to the case of two-phase linear isotropic elasticity under the additional assumption that the mixture remains isotropic. Z. Hashin & S. Shtrikman had suggested bounds (named after them), which we prove to be optimal. Note however that, to this day, the set of isotropic elements of the $G_\theta$-closure has yet to be fully characterized.

References [11,14,58] produce the two-dimensional $G$-closure associated to mixtures of linear and anisotropic conductors. The $G_\theta$-closure of even binary mixtures of anisotropic conductors is unknown at this time. The only complete answer, provided by V. Nesi, is in two dimensions and when one of the materials is isotropic.

Yet another field of investigation consists in obtaining homogenization-invariant algebraic relations between various coefficients of a given problem or between different problems. The first situation is described in [16] where a remark of R. Hill is used in the study of multiphase mixtures of linearly elastic and isotropic materials with the same shear modulus. The limit material is also isotropic with that same shear modulus. The second situation is studied in the framework of two-dimensional incompressible linear elasticity in [20] and in its nonlinear analogue in [37]. The following result is an amusing (and already known) byproduct of the study: the material resulting from the periodic homogenization of two conductors respectively occupying the black and white squares of a checkerboard and with respective Fourier laws $\alpha^p/p|\nabla u|^p$ and $\alpha^{-p'}/p'|\nabla u|^{p'}(1/p+1/p'=1, 1<p<\infty)$, is *linear*, isotropic, with conductivity 1.

## 1.3 Topological optimization

From a mechanical standpoint, the design of optimal structures consists, in the simplest case, in finding the best structure made of a linearly elastic and isotropic material for a given load and for a given weight or volume, within a preset design volume. In other words, let $\Omega$ denote the design domain, $f$ the imposed surface forces on $\partial\Omega$ (satisfying some obvious equilibrium conditions), $\Theta$ the acceptable volume fraction that the material can use and $A_{ijkh}=\lambda\delta_{ij}\delta_{kh}+\mu(\delta_{ik}\delta_{jh}+\delta_{ih}\delta_{jk})$ the Hooke's law for that material. The unknown is a characteristic function $\chi$ (of an open subset of $\Omega$ whose boundary contains the subset of $\partial\Omega$ where $f\neq 0$) such that the solution $u$ of

$$\begin{aligned} -\mathrm{div}\sigma &= 0, && \text{in } \Omega,\\ \sigma &= \chi(x)Ae(u), && \text{in } \Omega,\\ \sigma\cdot n &= f, && \text{on } \partial\{x\in\Omega:\chi(x)=1\}\cap\partial\Omega,\\ \sigma\cdot n &= 0, && \text{on } \partial\{x\in\Omega:\chi(x)=1\}\setminus\partial\Omega, \end{aligned}$$

minimizes $\int_{\partial\Omega} f\cdot u\,dx$ under the constraint $\int_\Omega \chi\,dx=\Theta$. I recall that $e(u):=1/2(\nabla u+\nabla^T u)$.

The problem is formally equivalent to the following problem of the calculus of variations:

$$\sup_\chi \inf_u \Big\{1/2\int_\Omega \chi Ae(u)\cdot e(u)\,dx+\lambda\int_\Omega \chi\,dx-\int_{\partial\Omega} f\cdot u\,d\mathcal{H}^{N-1}\Big\},$$

where $\lambda$ is a Lagrange multiplier associated to the volume constraint. Convex duality permits to rewrite the latter variational problem as

$$\inf_\chi \inf_s \Big\{1/2\int_\Omega \chi A^{-1}s\cdot s\,dx+\lambda\int_\Omega \chi\,dx : -\mathrm{div}(\chi s)=0 \text{ in } \Omega,\ \chi s\cdot n=f \text{ on } \partial\Omega\Big\}.$$

Filling the voids in $\Omega$ with a very weak elastic material, with say $\eta A$ ($\eta<<1$) as elasticity, provides in turn a rigorous approximation of that problem. Finally, the task boils down to the analysis of

$$\inf_\chi \inf_s \Big\{\frac{1}{2}\int_\Omega (\chi+(1-\chi)\eta)A^{-1}s\cdot s\,dx+\lambda\int_\Omega \chi\,dx : \\ -\mathrm{div}s=0 \text{ in } \Omega, s\cdot n=f \text{ on } \partial\Omega\Big\}.$$

There are generically no exact minimizers and the behavior of minimizing sequences $\{\chi_n\}$ is investigated. A homogenization problem naturally arises and the solutions live in a design domain which is larger than the one we started with, namely that of all mixtures of materials $A$ and $\eta A$. Bounding techniques – see the previous subsection – permit to derive a very thorough understanding of the (relaxed) optimal designs; cf. [22,28,29,30], as well as various works of G. Allaire, R.V. Kohn and G. Strang. In [22,28,29], we develop numerical algorithms that are by now, I believe, known for their efficiency.

Links with quasiconvexity (à la J. M. Ball, J. Morrey) are numerous; see [30] in particular. In that work, the existence of classical solutions to the following non-quasiconvex minimization problem is discussed:

$$\inf_u \{ \int_\Omega f(\nabla u)\, dx : u = \xi \cdot x \text{ on } \partial\Omega \},$$

with $f : I\!R^{nN} \to I\!R^+$ defined as

$$f(\eta) = \begin{cases} \lambda + \alpha|\eta|^2, & \eta \neq 0, \\ 0, & \eta = 0, \end{cases} .$$

The existence result, which is achieved through bounding techniques, provides a generalization of that of B. Dacorogna & P. Marcellini to dimensions strictly greater to 2.

The design of optimal structures subject to multiple loads can also be tackled, thanks to a theoretical result obtained in [27], which permits to get a good grasp on optimal microstructures – 6 imbedded levels of layers of material with void in 3d – in such a setting.

## 1.4 Loss of ellipticity

At first glance, homogenization in linear elasticity is just an offspring of its conductivity counterpart. Simply replace $\nabla u^\varepsilon$ by the linear strain $e(u^\varepsilon) := 1/2(\nabla u^\varepsilon + \nabla^T u^\varepsilon)$, use Korn's inequality to derive $H^1$-bounds – which depends on the smoothness assumptions on the domain $\Omega$ – and voilà... This is the tenet of most works dealing with homogenization in linear elasticity (see e.g. the previous subsection).

Closer inspection somewhat obfuscates the picture. Indeed, in conductivity, uniform pointwise ellipticity of the form

$$A^\varepsilon(x)\xi \cdot \xi \geq \alpha|\xi|^2,\ \xi \in I\!R^{N\times N},$$

is equivalent to $H^1_0$-coercivity of the associated quadratic form $\int_\Omega A^\varepsilon(x)\nabla u^\varepsilon \cdot \nabla u^\varepsilon dx$. Not so in elasticity for which very strong ellipticity, that is

$$A^\varepsilon(x)\xi \cdot \xi \geq \alpha|\xi|^2,\ \xi \text{ symmetric } N \times N \text{ matrix},$$

implies, but is *not implied by* $H^1_0$-coercivity of the associated quadratic form. The latter only implies uniform strict strong ellipticity, that is

$$A^\varepsilon(x) a \otimes b \cdot a \otimes b \geq \alpha|a|^2|b|^2,\ a, b \in I\!R^N.$$

In the periodic case (that is when $A^\varepsilon(x) := A(x/\varepsilon)$, with $A(y)$ defined on the unit torus), and under the sole assumption of uniform strict strong ellipticity, it is possible – under additional conditions that will not be elaborated upon – to establish a $\Gamma(H_0^1)$-convergence result of the functional

$$\int_\Omega A^\varepsilon(x)\nabla u^\varepsilon \cdot \nabla u^\varepsilon dx$$

to a functional of the same type (involving a homogeneous stiffness tensor $A^0$). This was established by G. Geymonat, S. Müller & N. Triantafyllidis. The question becomes whether $A^0$ can lose strict strong ellipticity (it cannot lose strong ellipticity because the $\Gamma$-limit must be rank-one convex). No examples were given in the above mentioned work.

In an apparently disconnected context, S. Gutiérrez has produced an example of a layering construction involving one very strongly and one strongly elliptic material for which the resulting homogenized stiffness loses strict (and even non-strict) strong ellipticity, *provided that* the homogenization process has been validated. In [70], M. Briane and I reconcile those two results. We demonstrate that S. Gutiérrez' construction does fall within the scope of the results of G. Geymonat, S. Müller & N. Triantafyllidis and that it is generically the only layering type microstructure for which strict strong ellipticity can be lost.

If however one cares to contemplate periodic mixtures of a very strongly elliptic material and of a strongly elliptic material in an inclusion-like situation, then A. Gloria and I demonstrate in [77] that no such pathology can occur. We believe that identical results hold in a stochastic setting but have failed to secure a complete proof thus far.

The exploration of wave propagation, pursued in collaborative work with M. Briane in [83], demonstrates highly unusual features of the "Gutiérrez medium": for example, a bounded domain with Dirichlet boundary conditions can accommodate plane waves; or still, no dilatational waves can propagate in the layering direction. This strongly evokes some kind of labile aether as it was understood in the pre-Lorentzian XIXth century when attempting to explain the propagation of light.

# 2 Dimensional reduction

3d-2d dimensional reduction, the only type I have studied, consists in deriving a limit model for a P.D.E. or a system of P.D.E.'s formulated on a domain $\Omega^\varepsilon := \omega(\subset \mathbb{R}^2) \times (-\varepsilon, \varepsilon)$ when $\varepsilon$ tends to 0. Considering once again (homogeneous) linear conduction as a model problem, we are thus led to the analysis of

$$\begin{aligned} -\mathrm{Div}(A\nabla U^\varepsilon) = F^\varepsilon, &\quad \text{in } \Omega^\varepsilon, \\ U^\varepsilon = 0, &\quad \text{on } \partial\omega \times (-\varepsilon, \varepsilon), \\ A\nabla U^\varepsilon \cdot n = 0, &\quad \text{on } \omega \times \{-\varepsilon, \varepsilon\}. \end{aligned}$$

I wish to emphasize the importance of the Neumann conditions at the top and bottom of the domain; by contrast, the Dirichlet conditions on the lateral boundary are unessential in most cases.

This type of model is relevant in many applications. The goal is to achieve a better understanding of those 2d-models that naturally arise as limits of 3d-models when the thickness of the sample becomes asymptotically 0. In other words, we propose to derive appropriate plate or membrane models. Such models, which have been extensively studied by the mechanics community till the 60's, have regained notoriety thanks in particular to the progress in thin film technology. That is because the impact of the mechanical properties on the performance of the microchips increases with the decrease of the size of those chips. The magnitude of the contracts obtained by a few of my colleagues in the U.S. on that topic is a clear indicator of the relevance of such studies.

A $1/\varepsilon$-dilation of the vertical variable $X_3$ is implemented. Indexing by $\alpha$ the horizontal components and setting $u^\varepsilon(x) := U^\varepsilon(x_\alpha, \varepsilon x_3)$, $f^\varepsilon(x) := F^\varepsilon(x_\alpha, \varepsilon x_3)$, we are led to the following rewriting of the original problem:

$$\begin{aligned} -(\mathrm{div}_\alpha + 1/\varepsilon \frac{\partial}{\partial x_3}) \left( A \begin{pmatrix} \nabla_\alpha u^\varepsilon \\ \frac{1}{\varepsilon} \frac{\partial u^\varepsilon}{\partial x_3} \end{pmatrix} \right) &= f^\varepsilon, \quad \text{in } \Omega := \omega \times (-1,1), \\ u^\varepsilon &= 0, \quad \text{on } \partial\omega \times (-1,1), \\ A\nabla u^\varepsilon \cdot n &= 0, \quad \text{on } \omega \times \{-1,1\}. \end{aligned}$$

The limit in $\varepsilon$ is easily computed under boundedness assumptions like

$$f^\varepsilon \rightharpoonup f, \text{ weakly in } L^2(\Omega).$$

In such a case,

$$u^\varepsilon \rightharpoonup u(x_\alpha), \text{ weakly in } H^1_0(\Omega),$$

and $u$ satisfies

$$\begin{aligned} -\mathrm{div}_\alpha(B\nabla_\alpha u) &= \overline{f}, \quad \text{in } \omega, \\ u &= 0, \quad \text{on } \partial\omega, \end{aligned}$$

where $\overline{f} := 1/2 \int_{-1}^{1} f \, dx_3$ and

$$B_{\alpha\beta} := A_{\alpha\beta} - \frac{A_{\alpha 3}A_{3\beta}}{A_{33}}.$$

In the setting of linear elasticity, P. Ciarlet et P. Destuynder have carried out the same type of analysis and shown that a proper choice of the $\varepsilon$-magnitude of the applied loads (vertical loads = $\varepsilon\times$ horizontal loads) yields a membrane/flexural model usually referred to as that of Kirchoff-Love. The membrane effect corresponds to $x_\alpha$-dependent displacement fields, while the flexural effect corresponds to displacement fields of the form $(-x_3 \frac{\partial w}{\partial x_\alpha}, w)$ where

$w(x_\alpha)$ solves a fourth order P.D.E. If adopting in nonlinear elasticity the standpoint of potential energy minimization, one is correspondingly led to the investigation of a problem of the type

$$\min_U \Big\{ \int_{\Omega^\varepsilon} W(\nabla U)\, dX - \int_{\Omega^\varepsilon} F^\varepsilon \cdot U\, dX \Big\},$$

for elastic energies $W : I\!\!R^{3\times 3} \to I\!\!R$; those energies will be assumed to behave like $|M|^p$ at infinity. Note that the case where $W(M) \nearrow \infty$ whenever $\det M \searrow 0$ is poorly handled at present. Further, those energies are taken to be quasiconvex if existence of a minimizer at fixed $\varepsilon$ is desired. The $1/\varepsilon$-dilation then leads to the study of

$$\min_u \Big\{ \int_\Omega W\Big(\nabla_\alpha u | 1/\varepsilon \frac{\partial u}{\partial x_3}\Big)\, dx - \int_\Omega f^\varepsilon \cdot u\, dx \Big\}.$$

Under the $p$-growth assumption on $W$ and if $f^\varepsilon$ remains bounded in $L^p(\Omega; I\!\!R^3)$, the problem reduces to the study of the $\Gamma(L^p)$-convergence (in the sense of E. De Giorgi) of

$$E_\varepsilon(u) := \int_\Omega W\Big(\nabla_\alpha u | 1/\varepsilon \frac{\partial u}{\partial x_3}\Big)\, dx.$$

The homogeneous case – an energy that does not depend upon $x$ – has been solved by H. Le Dret & A. Raoult. They have shown that $E_\varepsilon \overset{\Gamma(L^p)}{\rightharpoonup} E$ defined as

$$E(u) = 2 \begin{cases} \int_\omega Q_{2,3}\overline{W}\Big(\nabla_\alpha u\Big)\, dx_\alpha, & u \in W^{1,p}(\omega; I\!\!R^3), \\ \infty, & \text{otherwise,} \end{cases}$$

where

$$\overline{W}(\overline{F}) := \inf_{z \in I\!\!R^3} W(\overline{F}|z), \overline{F} \in I\!\!R^{3\times 2},$$

and $Q_{2,3}\overline{W}$ is the quasiconvexification of $W$, the definition of which I now recall. By definition,

$$Q_{2,3}\overline{W}(M) := \inf_\varphi \left\{ \int_{(0,1)^2} \overline{W}(M + D\varphi)\, dx : \varphi \in \mathcal{C}_0^\infty\left((0,1)^2; I\!\!R^3\right) \right\},$$

for any $M \in I\!\!R^{3\times 2}$. I wish to draw the reader's attention to the membrane-like character of the obtained model. The object of [36], is to study more complex situations in which the energy density $W$ is of the form $W(\varepsilon)(X_\alpha, X_3; F)$ and in which the domain itself may be profiled in the sense that

$$\Omega^\varepsilon := \{(X_\alpha, X_3) : |X_3| \le \varepsilon f_\varepsilon(X_\alpha)\}.$$

Setting

$$W^\varepsilon(x; F) := W(\varepsilon)(x_\alpha, \varepsilon x_3; F),$$

leads to the study of

$$E_\varepsilon(u) := \int_{\omega_\varepsilon} W^\varepsilon\Big(x; \nabla_\alpha u | 1/\varepsilon \frac{\partial u}{\partial x_3}\Big)\, dx,$$

with

$$\omega_\varepsilon := \{(x_\alpha, x_3) : x_\alpha \in \omega, |x_3| < f_\varepsilon(x_\alpha)\}.$$

Assuming non-degeneracy of the profile and $\varepsilon$-uniform $p$-growth of $W^\varepsilon$ , a subsequence of $E^\varepsilon$ is shown to $\Gamma(L^p)$-converge to

$$E(u) = 2 \begin{cases} \int_\omega W\Big(x_\alpha; \nabla_\alpha u\Big)\, dx_\alpha, & u \in W^{1,p}(\omega; I\!R^3), \\ \infty, & \text{otherwise,} \end{cases}$$

where $W$ has yet to be determined. Once again, the limit model is membrane-like. The energy density $W$ is explicitly computed in two cases: that of an energy $W^\varepsilon(x; F) = W(x_3; F)$ without profile (akin to an non-homogeneous version of the type of energy considered in H. Le Dret & A. Raoult) and that of an energy $W^\varepsilon(x; F) = W(x_\alpha/\varepsilon, x_3; F)$ with $f_\varepsilon(x_\alpha) = f(x_\alpha/\varepsilon)$ (the case of periodic homogenization). In both cases, $W$ does not depend upon the specific choice of the subsequence; thus, the whole sequence $E^\varepsilon$ $\Gamma$-converges. A third case will be mentioned in the next section.

In [48], the fully heterogeneous case, that is that of $W^\varepsilon(x; F) = W(x_\alpha, x_3; F)$, is treated, as well as the fully heterogeneous case for the Cosserat-type model first studied by G. Bouchitté, I. Fonseca & L. Mascarenhas, in which one keeps track, through the $\Gamma$-convergence process, not only of $u^\varepsilon$, but also of the moment $1/\varepsilon(u^\varepsilon(x_3 = 1) - u^\varepsilon(x_3 = -1))$, a must whenever order-one bending moments are applied to the thin 3d-domain.

In [55], the impact of a van der Waals type interfacial energy on the asymptotic behavior of a thin plate is investigated. Both mid-plane and out of plane deformations – via the Cosserat vector – are monitored and the limit behavior is shown to heavily depend on the ratio between the strength of the interfacial energy and the thickness of the sample. This is reminiscent of a similar study performed in [24] in the setting of homogenization. In both settings, there is a critical ratio for which the limit behavior combines dimensional reduction (resp. homogenization) effects with singular perturbations (interfacial) effects. However, that behavior is more mysterious here because G. Dal Maso, I. Fonseca & G. Leoni have shown that it is non-local as a function of the out of plane deformation.

In the light of the previous results, it would seem that 3d-2d models obtained through techniques of $\Gamma$-convergence are unable to generate flexural effects, which a linear analysis à la P. Ciarlet & P. Destuynder does: this is because gradient symmetrization, a specificity of linearized elasticity, produces a limit kinematics of the Kirchhoff-Love type and not of the membrane type. Further, that kinematics is also the result of the relative magnitude of the in-plane versus transverse applied load, the latter being of order $\varepsilon$ times the order of the former (note that, in a nonlinear setting, one should prescribe the absolute order of magnitude of each component of the loads, and not only their relative magnitude). When in a nonlinear setting, the previous considerations suggest to try and implement the change of dependent variable that was performed in linearized elasticity so as to preserve the symmetrized gradient structure, namely,

$$u_3(x_\alpha, x_3) = \varepsilon U_3(x_\alpha, \varepsilon x_3).$$

This is the object of [38]. It is shown there that the $\Gamma(L^p)$-$\liminf$, $F(v)$, of the resulting rescaled model,

$$F_\varepsilon(v) := \int_\Omega W \begin{pmatrix} D_\beta v_\gamma & \frac{1}{\varepsilon} D_3 v_\gamma \\ \frac{1}{\varepsilon} D_\beta v_3 & \frac{1}{\varepsilon^2} D_3 v_3 \end{pmatrix} dx,$$

cannot remain impervious to the lateral boundary conditions imposed on the sequences considered in the computation of the $\Gamma(L^p)$-$\liminf$. I recall that $F$ is defined as follows:

$$F(v) := \inf_{\{v^\varepsilon\}} \Big\{ \liminf_\varepsilon F_\varepsilon(v^\varepsilon) : v^\varepsilon \to v \ \text{ strongly in } L^p(\Omega; I\!R^3) \Big\},$$

with – or without – the boundary condition $v^\varepsilon = v$ on $\partial\omega \times (-1, 1)$. This result is roughly equivalent to the non-local character of that $\Gamma(L^p)$-$\liminf$, that is to the non-existence of an integral representation with a limit energy density. But the very notion of limit mechanical model becomes dubious because a constitutive model which is boundary condition dependent is an oxymoron.

Note that G. Friesecke, R.D. James & S. Muller have successfully recovered a flexural effect upon a proper rescaling of a frame indifferent energy functional under loading conditions that kill any membrane-like effect; their argument is based on a beautiful extension of F. John's rigidity result for transformation gradients that are pointwise close to rotations and it has in turn given rise to many results in mechanics as well as in geometry.

At the conclusion of this section, I wish to point out that pathologies of the type encountered in homogenization also appear in a 3d-2d context when studying evolution problems [10]. In [41], P. Gérard and I investigate the propagation of oscillations in a 3d-2d setting for the wave equation. The spirit of that study is identical to that of [21], but the necessary tools are both H-measures and semi-classical measures, because, in contrast to the homogenization setting, the 3d-2d setting exhibits a characteristic length, the thickness of the sample $\varepsilon$. That study permits to derive a new observability result for the damped wave equation on thin 3d domains (that is an $\varepsilon$-uniform control of the initial potential energy in terms of the energy dissipated over some time interval). In the usual setting of control theory for the wave equation, observability is easily seen to be equivalent to stabilization (the exponential decay of the potential energy); see the work of G. Lebeau and of E. Zuazua. By contrast, is not clear that such an equivalence holds true in our setting.

## 3 Defect mechanics

Defect mechanics studies the onset and propagation of a defect in a healthy material. A variety of defects may occur: the material, assumed linearly elastic in nearly all that follows, may become weaker through a local decrease in its elastic properties; we speak of damage. A crack, or cracks, may appear and grow; this is fracture. Dislocations may be generated in a crystalline structure and plastify part of the sample ..... Each of those mechanisms is the topic of such an abundant literature that any attempt at even a brief overview would be hopeless.

Schematically, the essence of the classical modeling of a defect mechanism in a quasi-static framework is always the same. (Note that very little is known of the interplay between kinetic energy and defect growth.) A parameter $\beta$, assumed to characterize the defect, is introduced. It can be scalar or tensor-valued, or even a global variable like the crack set. The potential energy $P(u, \beta, t)$ is taken as a function of time $t$, of the displacement field $u$, and of $\beta$. It it first minimized, for a fixed $t$ (quasi-staticity hypothesis), but also for a fixed $\beta$, so as to obtain the displacement field $u(\beta, t)$ at equilibrium, for that time and that value of the defect parameter. The value of $\beta(t)$ at $t$ is determined through an evolution law of the type

$$-\frac{\partial P}{\partial \beta}(t) \in \partial_{\dot{\beta}} \mathcal{D}(\beta(t), \dot{\beta}(t)).$$

The dissipation potential $\mathcal{D}$ is convex in the variable $\dot{\beta}$. This ensures that the mechanical dissipation remains non-negative and thus that the second law of thermodynamics holds. We further specialize $\mathcal{D}$ to be 1-homogeneous, so as to guarantee rate independence, a necessary feature of the evolution if the analysis is to proceed as exposed below.

The approach that J.J. Marigo and I have initiated in the past eighteen years, first for damage, then for fracture is variational, and, as such, also confined to quasi-static problems. We want to depart as little as feasible from the classical theory. The first step is to remark that the initial problem – quasi-static equilibrium + positivity of the dissipation – is equivalent to the following two statements:

(1) $u(t), \beta(t)$ satisfies a first order optimality condition for the local minimality of $P(v, \gamma) + K(\beta(t); \gamma)$, among all admissible pairs $v$'s and$\gamma$'s that respect the prior defect states, with $K(\beta; \gamma)$ defined as $\int_{\text{the domain}} \mathcal{D}(\beta, \gamma - \beta) dx$ if $\beta$ is a local variable, and as $\mathcal{D}(\beta; \gamma - \beta)$ if $\beta$ is a global variable;

(2) $\dfrac{dE}{dt}(t) = \displaystyle\int_{\text{the domain}} \{$ time derivative of the loads$\} \times u(t) dx$, where

$$E(t) := P(u(t), \beta(t)) + \int_0^t \left[ \left( \int_{\text{the domain}} \right) \mathcal{D}(\beta(s), \dot{\beta}(s))(dx) \right] ds,$$

which is precisely the mechanical form of the second principle in the terminology of M. Gurtin.

As such the problem is generically intractable because of the paucity of available tools for the handling of (1). This has led us to require that item (1) be replaced by a global minimality condition. This is certainly a drastic and often unrealistic selection mechanism, but it will permit to construct evolution paths that satisfy (1), (2).

In the sequel, I will assume, for simplicity sake, that the material fills a domain $\Omega$ and that it undergoes an imposed time-dependent displacement $U(x,t)$ on the part $\partial\Omega_d$ of its boundary.

## 3.1 Damage

In [18,23], a brutal damage model is investigated. It is characterized by two energy densities $W^s(e) := 1/2 A_s e \cdot e \geq W^e(e) := 1/2 A_e e \cdot e$. The characteristic function of the damaged zone is denoted by $\chi$; thus the potential energy associated to a kinematically admissible field $u$ at time $t$ is

$$P(u,\chi,t) = \int_\Omega (\chi(x)W^e + (1-\chi(x))W^s)(e(u)(x))\, dx.$$

In the spirit of A. Griffith, the following dissipated energy is introduced:

$$K(\chi) = k\int_\Omega \chi(x)\, dx.$$

We propose to implement the scheme outlined at the onset of this section under the constraint that $\chi(t)$ should monotonically increase with $t$, a translation of the irreversible character of the decrease of the mechanical properties of the material. We then operate a temporal discretization of the problem. At the first time step $t_1$, the problem becomes

$$\inf_u \{\int_\Omega \Psi_1(e(u)(x))\, dx \ , u \in W^{1,p}(\Omega; I\!R^N);\ u = U_1 \text{ on } \partial\Omega_d\},$$

where $U_1$ denotes the displacement field imposed on $\partial\Omega_d$ at $t_1$ and

$$\Psi_1(\xi) := \inf_{\chi\in\{0,1\}} \{(\chi W^e + (1-\chi)W^s)(\xi) + k\chi\}.$$

The integrand $\Psi_1$ is not quasiconvex and consequently the infimum is not necessarily attained. The quasiconvexification of $\Psi_1$ introduces, in a manner similar to that of Subsection 1.3, a problem of bounds in homogenization, precisely that of the minimal energy for mixtures of two elastic materials with respective Hooke's laws $A_e$ et $A_s$ in given volume fraction. This problem can be solved, which permits in turn to carry through the analysis of the first time step.

Such is not the case for the following time steps because of the complexity introduced by the irreversibility constraint: indeed, once a microstructure – that whose overall stiffness produces the minimal energy at a given point – is formed, then it cannot be undone, so that the future evolution is constrained by the presence of that microstructure. A. Garroni and I [54] have derived a time-continuous evolution which accomodates both the irreversibility constraint and the relaxation process by only allowing at each time mixtures that add weak material to the already formed microstructures. At each time global minimality is obtained for the relaxed problem among admissible tests, while "energy conservation" (item (2) in the proposed scheme) is satisfied.

As in Subsection 1.3, the relaxed formulation is nicely amenable to numerics.

I wish to emphasize that our model naturally leads to considering the volume fraction of damaged material (that associated to $A_e$) as "damage variable" and permits to compute the dependence of the Hooke's law – the homogenized tensor that is optimal from the standpoint of the bounds – upon that variable, whereas the other damage models must a priori postulate that relation. In [33] as well as in the last part of [36], the same procedure is used, together with a 3d-2d analysis of the type presented in Section 2, to study the damaging behavior of a thin film.

Note that C.J. Larsen & A. Garroni have more recently introduced a formulation that remembers the minimizing sequences at each time, and not only the effective elasticities. This allows one to impose a more natural irreversibility constraint, and also to consider more complex settings, like that of more than two materials for exemple. In so doing, they have also improved the minimality property of the obtained Hooke's law at each time of the evolution process.

## 3.2 Fracture

In [31,32,34,43,50,52,56,57,60,61], the problem of brittle fracture is considered. There is no restriction on the shape of the possible cracks, in the domain as well as on its boundary (so as to allow for possible debonding). The parameter $\beta$ is then very complex because it is exactly all closed sets $\Gamma$ of $\overline{\Omega}$. A kinematically admissible field $u$ at time $t$, and for a given $\Gamma$, is a field that takes the value $U$ on $\partial\Omega_d \setminus \Gamma$ – $\Gamma$ can "chew" part of $\partial\Omega_d$ – and the associated potential energy is

$$P(u, \Gamma, t) = \int_{\Omega\setminus\Gamma} W(e(u)(x))\, dx,$$

where $W$ denotes the elastic energy density. In agreement with A. Griffith's theory, the dissipated energy is given by

$$K(\Gamma) = k\mathcal{H}^{N-1}(\Gamma \cap (\Omega \cup \partial\Omega_d)),$$

where $\mathcal{H}^{N-1}$ denotes the $N-1$-dimensional Hausdoorf measure. I wish to draw the reader's attention to the specific form of the surface energy; the set $\Gamma \cap (\Omega \cup \partial\Omega_d)$ for which surface energy is paid must indeed take into account the surface energy dissipated in crack creation and/or extension both in the interior and on the part of the boundary where the displacement field is imposed, but not on the part of the boundary which is traction free.

Once again the goal is to minimize $P + K$ at each time $t$ under the constraint that $\Gamma(t)$ should monotonically increase with $t$ (a translation of the irreversible character of the cracking process), while also ensuring that "energy conservation" is satisfied. A time discretization is performed. At the first time step $t_1$, the problem becomes

$$\inf\Big\{\int_{\Omega\setminus\Gamma} W(e(u))\,dx + k\mathcal{H}^{N-1}(\Gamma\setminus(\partial\Omega\setminus\partial\Omega_d)) : \Gamma(\text{closed })\subset\overline{\Omega};\ u\in W^{1,2}(\Omega\setminus\Gamma);\ u=U_1 \text{ on } \partial\Omega_d\setminus\Gamma\Big\}.$$

In the case of an antiplane problem and of a quadratic and isotropic elastic energy with Lamé coefficients $\lambda, \mu$, the minimization reduces to

$$\inf\Big\{\mu/2\int_{\Omega\setminus\Gamma}|\nabla u|^2\,dx + k\mathcal{H}^{N-1}(\Gamma\setminus(\partial\Omega\setminus\partial\Omega_d))\Big\}.$$

This is very close to D. Mumford & J. Shah's formulation for image segmentation.

The arguments introduced by E. De Giorgi in the framework of image segmentation can be adapted to the present setting and the equivalence of the formulation with a weak formulation in $SBV(\Omega)$ – the subspace of functions of bounded variation such that their gradient has no Cantor part, a space introduced by L. Ambrosio & E. De Giorgi – can be established. The problem was studied by M. Carriero & A. Leaci in the case where $\partial\Omega_d = \partial\Omega$. The case where only part of the boundary is constrained – the realistic case from a mechanical standpoint – is investigated in B. Bourdin's thesis. The problem is as follows:

$$\inf\Big\{\int_{\Omega}|\nabla u|^2\,dx + k\mathcal{H}^{N-1}(S_u\setminus(\partial\Omega\setminus\partial\Omega_d)) : u\in SBV(\hat\Omega);\ u=U_1 \text{ in } \hat\Omega\setminus\overline{\Omega}\Big\}.$$

Here, $\hat\Omega$ is a domain which compactly contains $\Omega$ and the boundary condition $U_1$ – or rather its extension to $\mathbb{R}^N$ – is assumed to be Lipschitz.

The existence of a minimum is straightforward upon application of lower semi-continuity results of L. Ambrosio. The critical result, obtained by B. Bourdin upon adaptation of the results of M. Carriero & A. Leaci, is a regularity result on the jump set associated to the solution of the weak problem : Any minimum $u$ for the weak formulation is such that $\mathcal{H}^{N-1}((\overline{S_u\setminus(\partial\Omega\setminus\partial\Omega_d)})\setminus(S_u\setminus(\partial\Omega\setminus\partial\Omega_d)) = 0$. Equipped with that result, the equivalence between the original problem and the weak formulation proceeds as in the case of image segmentation. A solution $u_1, \Gamma_1 := \overline{S_{u_1}\setminus(\partial\Omega\setminus\partial\Omega_d)}$ of the original minimization problem is found.

The analysis is iterated for the following time steps. The irreversibility constraint must be accounted for. In its discretized form, it reduces to

$$\Gamma_i\supset\Gamma_{i-1}.$$

The case of two or three-dimensional elasticity is a nightmare. An energy density, quadratic in the symmetrized gradients, is considered. The problem reduces to a minimization of

$$\int_{\Omega \setminus \Gamma} 1/2 Ae(u)(x) \cdot e(u)(x)\, dx + k\mathcal{H}^{N-1}(\Gamma \setminus (\partial\Omega \setminus \partial\Omega_d)),$$

among all fields $(u, \Gamma)$ such that $\Gamma(\text{closed }) \subset \overline{\Omega}$ and $u \in W^{1,2}(\Omega \setminus \Gamma)$; $u = U_1$ on $\partial\Omega_d \setminus \Gamma$.

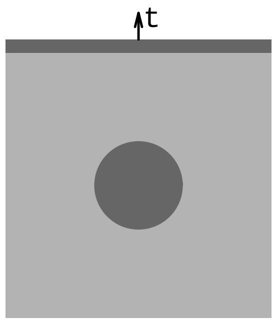
Once again, we would like to introduce a weak formulation. The space $SBV(\Omega)$ is however inadequate because Korn's inequality – the usual tool in linearized elasticity – does not hold true in $BV$. The space $SBD(\Omega)$ of elements of $L^1(\Omega; \mathbb{R}^N)$, the symmetrized gradient of which is a measure with no Cantor part, seems more appropriate. It is unfortunately a more pathological space than its analogue $SBV(\Omega; \mathbb{R}^N)$ and it is only very recently that progress has been made (see the end of this paragraph).

Putting aside the mathematical hurdles, I would like to point out that our formulation is the first to allow for initiation of a crack in an unspoiled sample and to predict both the initiation time and the crack path. Furthermore, it permits to resolve the dichotomy between progressive and brutal propagation, a source of numerous problems in the classical theory of brittle fracture. Of course, there is a price: the formulation will always fail in the presence of applied tensile forces in $\Omega$, or on $\partial\Omega$. One could try to overcome the problem by appealing to surface energy of the cohesive type, and by investigating local minima in lieu of global ones. This is the source of many mathematical difficulties. Preliminary results in a one-dimensional setting are presented in [44]. Note that A. Chambolle, A. Giacomini & M. Ponsiglione have shown that, at least in 2d and for connected cracks, local minimality – instead of global minimality – prohibits initiation in the case of a Griffith type energy, so that a combination of local minimality and of a different type of surface energy is indeed required.

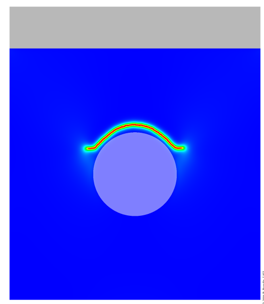
The numerical treatment of such minimization problems is not trivial because the test fields can be – and will be if the loading level is high enough – discontinuous. Several approximation methods can be developed as shown in [35], together with actual computations. The most successful, based on a scheme first proposed by L. Ambrosio & V.M. Tortorelli in the setting of image segmentation, consists in "smearing" the crack through the introduction of a supplemental variable $v$ which will be, say 1, everywhere except very near the crack. The size of the localization zone – the set where $v \neq 1$ – is then decreased and the resulting functional, a combination of the elastic energy with an energy à la L. Modica & S. Mortola that approximates perimeter, is shown to $\Gamma$-converge (in the right topology) to the sum of elastic and surface energies.

Using this kind of approach, B. Bourdin has now developed a 3-d code that allows to tackle crack propagation in realistic settings: see his most recent work on diffusive cooling on his *website*.

As a more academic example of such computations, the six figures below address the case of a brittle elastic plate, perfectly bonded to a (fixed and) rigid disk in its center, initially crack-free, and subject to a monotonically increasing displacement "load" on its northern side. The results demonstrate a wide array of behaviors: brutal onset of the crack, smooth extension of that crack, sudden asymmetry of the cracking process, ....

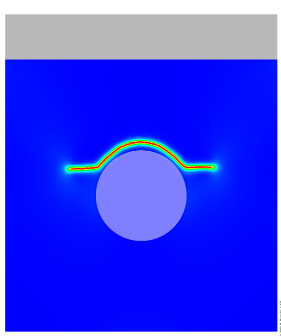



**Phase 1 : Elastic response.** If $t < 0.28$, the matrix remains purely elastic.



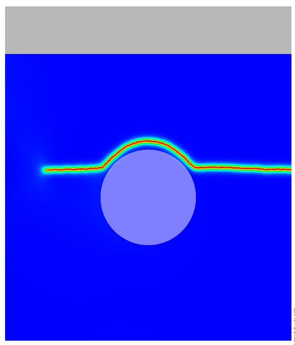
**Phase 2 : Brutal onset.** At $t \sim 0.28$, a crack of finite length brutally appears near the top of the inclusion. The crack is symmetric with respect to the 2 axis; it is not straight.

Such an evolution, which qualitatively agrees with experimental observations by D. Hull, could not be reproduced through any kind of algorithm based on the classical methods of brittle fracture.

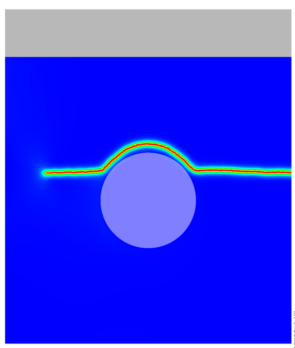


**Phase 3 : Progressive and symmetric evolution of the crack.** When $t$ varies between 0.28 and 0.38, the crack progressively grows in the matrix. The evolution is smooth: the surface energy increases smoothly, while the bulk energy is nearly constant. The propagation is symmetric but not straight.

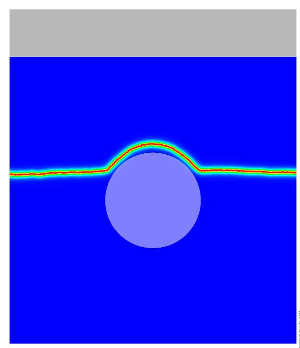
The formulation adopted by B. Bourdin brings forth new challenges. Indeed, the approximating two-field functional $(u, v)$ is separately convex in $u$ and $v$, but not globally convex. The alternate minimization algorithm, used in the actual computations, cannot guarantee convergence to a a global minimizer at each time step, but only convergence to a critical point.



**Phase 4 : Rupture of the right ligament.** At $t \sim 0.38$, the right hand-side of the matrix is brutally cut. The brutal character of the phenomenon is evidenced by a jump discontinuity of both surface and bulk energies. The crack is no longer symmetric.



**Phase 5 : Arrest of the left branch.** When $t$ varies between 0.38 and 0.40, the left part of the crack does not grow, or grows too slowly for growth to be detected.



**Phase 6 : Rupture of the left ligament.** At $t \sim 0.40$, the crack brutally severs the remaining filament of uncracked material. The corresponding energy jumps can be evidenced.

The resulting evolutions might prove more faithful to the "real picture", even in the absence of bona fide existence theorems similar to those evoked further below. But this will be so only if the critical points of the approximating functional are close to those of the fracture functional. Such a statement is certainly not implied by the Γ-convergence of the approximating functional. In the case of the functional introduced by L. MODICA & S. MORTOLA in the minimal surface problem, recent work of Y. TONEGAWA provides an essentially positive answer to the convergence of critical points, provided that the associated energies are uniformly bounded. In joint work with N. LE and S. SERFATY [57], we conclude to the validity of the assertion in the one-dimensional setting for the functional of D. MUMFORD & J. SHAH; its multi-dimensional counterpart remains completely open.

Many questions have yet to be completely answered, although the existence of the time-continuous solution for the model with global minimality is now fully established in the gradient setting. In the antiplane case, G. DAL MASO & R. TOADER have given an existence result in the antiplane 2d-case under a restrictive connectedness hypothesis on the possible crack sites. A. CHAMBOLLE has gone one step further in analyzing the same problem in plane elasticity. The method does not use the weak formulation; it is based on continuity results for the solution of the Neumann problems on varying domains.

In [45], the existence of the time-continuous solution in the general 3d-case for gradients and quadratic energies is established; the method does use the weak $SBV$-formulation and does not restrict in any manner the geometry of the possible cracks. The success of the argument relies on a geometric measure theoretic result on the transfer of jumps, which should have applications beyond the specific problem at hand. The highly non-trivial extension of the results of [45] to finite elasticity (the quasiconvex case) is the object of [47,49]; the lack of uniqueness inherent to the analysis of quasiconvex problems is a major source of trouble and necessitates new tools, on top of the previously mentioned jump transfer result. However, we do not handle the constraint that $W(F) \nearrow \infty$ as $\det F \searrow 0$, but impose $p$-growth on the energy density $W$; the determinant constraint has been treated in recent work by G. DAL MASO & G. LAZZARONI. To my knowledge, those are the first existence results for a time-indexed family of minimization problems with a monotonicity constraint on the minimization variable.

In [39], a minimization approach to the possible debonding of a thin film from its substrate is undertaken in the spirit of this Subsection, and with the help of a 3d-2d analysis of the type presented in Section 2. The resulting model is shown to drastically depend on the type of surface energy – cohesive versus brittle – at the interface between film and substrate. The question of the existence of a critical film thickness under which debonding does not occur – an experimental observation – has been answered in the positive by K. BHATTACHARYA & C. LARSEN.

This kind of analysis is not confined to defect mechanics; it can be used for other kinds of quasi-static dissipative evolutions. This is the object of [53], a work in collaboration with with A. MIELKE, in which we study a phase-field model in a similar light. In particular, we obtain the existence of an evolution for a phase-field model in an authentic non-linear elastic setting, that is one where there is no upper bound on the energy density; this is done by using a multiplicative decomposition of the finite strain to factor out the time dependent displacement boundary conditions, in lieu of the more usual additive decomposition. That decomposition is also at the root of the argument used in the work of G. Dal Maso & G. Lazzaroni alluded to earlier.

Coming back to fracture, a recent thrust in that area has been on crack kinking (in a Griffith setting). The prediction of a sudden change in the direction of crack propagation has been a vexing issue for many years, even in a 2d isotropic setting. The mechanics literature has been conflicted and various competing criteria have been proposed, most notably that which would have the crack kink in the direction of maximal energy release, versus that which will choose the direction where the crack lips are in pure traction (the principle of local symmetry); recent literature seems to exhibit a strong bias for the latter. In [60,61], A. Chambolle, J.-J. Marigo and I investigate the 2d connected setting. We show in essence that, if accepting local stability of the crack at each time, then kinking will only occur when both criteria are met simultaneously. But this is impossible, thanks to results of M. Amestoy & J.-B. Leblond, so that 2d kinking in the form usually envisioned in the literature is actually forbidden. We conjecture, without proof at present, that kinking may be the result of branching with the immediate arrest of one branch, perhaps because of non-interpenetration.

Finally, the setting of linear elasticity, the original setting of linear fracture mechanics, has at last been "vanquished", thanks to a slew of new results which have all been obtained in the couple of years. As was mentioned before, the lack of intimate knowledge of $SBD$ was a formidable hurdle. It was for example not known whether the complement $S_u$ of the set of Lebesgue points of a function $u \in SBD(\Omega)$ is strictly larger than the jump set $J_u$ of $u$ – the set of points where $u$ admits unequal left and right approximate limits (for a well chosen direction) – in the sense that $\mathcal{H}^{N-1}(S_u \setminus J_u) > 0$; see the work of L. Ambrosio, G. Belletini, A. Coscia & G. Dal Maso. This, and other similar issues, has been adjudicated in various works; see a joint work with A. Chambolle and S. Conti [74] for a Poincaré-Korn type inequality in $SBD$, as well as various subsequent works by S. Conti, I. Flaviano & M. Foccardi and also independent 2d work by M. Friedrich. Of particular relevance is the preprint by M. Friedrich & F. Solombrino which adresses the 2d quasi-static fracture evolution in the setting of linearized elasticity and which uses in particular Friedrich's recent result that $SBD^2(\Omega) \cap L^\infty(\Omega) \subset SBV(\Omega; \mathbb{R}^2)$ where $SBD^2(\Omega)$ stands for those functions in $SBD(\Omega)$ such that the absolutely continuous part of their symmetrized gradient is square integrable.

As a brand new application of those equally new results, A. Chambolle, S. Conti and I have shown in [80] that the already mentioned Ambrosio-Tortorelli variational approximation of the crack evolution problem remains valid in 2d elasticity if a non-interpenetration constraint is imposed on the crack lips. This was quite a challenging problem which took us over 10 years to complete!

### 3.3 Cavitation

Cavitation is my most recent endeavor. It commonly refers to the nucleation and growth of cavities in materials that are usually nearly incompressible elastomers, which means that they can undergo finite nearly isochoric deformations. Traditionally, that is since the famous experiments of A.N. Gent & P.B. Lindley, cavitation is being thought of as a purely elastic phenomenon. In his seminal work on cavitation, J.M. Ball posited that hyper-elasticity can, in and of itself, create cavities through solutions of the type $x/|x|$ that are good Sobolev functions, provided that the growth at infinity of the elastic energy be subcritical, that is less than the spatial dimension. More recent experiments do not concur and point to the necessity of adding some kind of surface energy in the mix, thereby viewing cavitation as a fracture type phenomenon for which the cavities are cracks in their elastically deformed states.

However, there is added complexity for at least two reasons. On the one hand, cavity initiation tends to happen in regions of high hydrostatic stress but low stored energy (which is possible because of near incompressibility). If that is so, the potential energy introduced at the onset of Section 3 might not be the right object for investigating cavitation. On the other hand, healing is part and parcel of cavitation; in other words, cavities do heal. So the fracture process is not irreversible.

In [82] A. Giacomini, O. Lopez-Pamies and I investigate a dual fracture/healing evolution problem in the spirit of the classical fracture problem expounded in the previous subsection. Irreversibility is dropped and, if $\Gamma$ is the crack at time $t$, then both crack extension and crack repair are possible, creating a new crack $\Gamma'$ which will carry a change in surface energy of the form

$$k\mathcal{H}^{N-1}(\Gamma' \setminus \Gamma) - k'\mathcal{H}^{N-1}(\Gamma \setminus \Gamma').$$

We prove the existence of a time-continuous globally minimizing evolution, albeit only in 2d and with the topological restrictions already evoked in the previous subsection, that is the "connectedness" of the cracks.

In [81], A. Kumar, O. Lopez-Pamies and I take a more engineering-oriented approach which aims at capturing the complexity of cavitation through a phase field model which is amenable to computations and thus can be compared with the recent and striking experiments of K. Ravi-Chandar. The hope, unsubstantiated for now, is that such a phase field model converges, in some appropriate sense, to a sharp interface model of the type that we analized in [82] as the size of the process zone tends to 0.

The issue of initiation is a very delicate one and, as of yet, we lack a complete and satisfactory answer in spite of promising numerical investigations performed by O. Lopez-Pamies. I will hopefully be in a position to report on this in the near future.

# 4 Plasticity

In the past few years, I have also focussed on evolution problems in elasto-plasticity. As briefly mentioned in the previous section, the quasi-static rate independent evolutions presented there are not confined to the creation and/or growth of a defect. Macroscopic elasto-plasticity – by contrast to dislocation induced plasticity – is a case in point.

We consider an elasto-plastic material with Hooke's law $A$ that fills a domain $\Omega$ and undergoes an imposed time-dependent displacement $U(x,t)$ on an open part $\partial_d\Omega$ of the boundary of $\Omega$, the remaining part being traction-free for simplicity. If $u$ is a kinematically admissible field at time $t$, then $e(u)$ decomposes into a sum – the setting being that of small strain elasto-plasticity – of an elastic strain $e$ and a plastic strain $\beta$ (the internal variable here), i.e. $e(u) = e + \beta$; further the boundary condition might not be satisfied by $u$, and relaxes into $\beta = (U(t) - u) \odot \nu\mathcal{H}^{N-1}$ on $\partial_d\Omega$, thereby allowing boundary slips. The potential energy associated to $u$ at time $t$ is

$$P(u,\beta,t) = 1/2 \int_\Omega Ae \cdot e\, dx = 1/2 \int_\Omega A(e(u) - \beta) \cdot (e(u) - \beta)\, dx,$$

while the dissipated energy is

$$\mathcal{D}(\beta) = \text{“} \int_{\Omega \cup \partial_d \Omega} H(\beta)\text{”},$$

where $H(\cdot)$ is the convex conjugate to the indicatrix function of the set of admissible stresses $K$, a convex set which is generally unbounded in the direction of hydrostatic stresses (multiples of the identity matrix),

The correct functional setting for $u$ is that of $BV(I\!R^N; I\!R^N)$, so that the plastic strain is generically a Radon measure at all times. Then the expression for $H(\beta)$ has to be understood in the sense of convex functions of a measure, i.e.,

$$\int_{\Omega \cup \partial_d \Omega} H(\beta) := \int_{\Omega \cup \partial_d \Omega} H\left(\frac{d\beta}{d|\beta|}\right) d|\beta|,$$

where $|\beta|$ is the variation measure of $\beta$ and $d\beta/d|\beta|$ the Radon-Nykodim derivative of $\beta$ with respect to $|\beta|$.

The above setting was shown by G. Dal Maso, A. De Simone & M. G. Mora to give rise to a well-posed evolution which is precisely that of classical elasto-plasticity, namely,

$$\begin{gathered} e(u(t)) = e(t) + \beta(t) \text{ in } \Omega,\ \beta(t) = (U(t) - u(t)) \odot \nu\mathcal{H}^{N-1} \text{ on } \partial_d\Omega \\ \operatorname{div} \sigma(t) = 0 \text{ in } \Omega,\ \sigma(t)\nu = 0 \text{ on } \partial\Omega \setminus \overline{\partial_d\Omega} \\ \sigma(t) := Ae(t) \in K \\ \dot\beta(t) = \begin{cases} 0, & \sigma(t) \in \operatorname{int}(K) \\ \parallel \text{ normal to } \partial K, & \sigma(t) \in \partial K \end{cases} \quad (\text{ the flow rule }) \end{gathered}$$

Note that the original proof, due to P. SUQUET, of existence of an evolution in that framework was a P.D.E. type approach.

Since that seminal work, a few more sophisticated plastic models have been analyzed, most notably those that contain plastic hardening (the easy case), or plastic softening (the difficult case). Most of this work is due to the same authors, together with M. MORINI.

A joint work with J. F. BABADJIAN and M. G. MORA [62] has been focussing on issue of non-associativity, that is on models where plastic flow occurs in a direction other than that of the normal to the set of admissible stresses. This is the case in both rock and soil mechanics. The (geo-)mechanics literature is excruciatingly technical and conventional wisdom has it that such problems lack a variational structure. Yet, thanks to a rather lonely work of P. LABORDE, we have managed to impart a variational structure on such a problem and to derive a rate independent evolution close in spirit to that obtained in very recent work of G. DAL MASO, A. DE SIMONE & F. SOLOMBRINO on Cam-Clay Plasticity.

A similar idea has enabled U. STEFANELLI and I to provide in [68] what we believe to be the first existence theorem for elasto-plastic evolutions involving what is called "non linear kinematic hardening" (a notion that I will not explicit in this synopsis). The corresponding non-associative model, which goes by the name of Armstrong-Frederick, is undoubtedly, together with its variants, the most widely used plasticity model in the engineering literature because it is thought to provide an adequate rendering of many complex features of real plastic evolutions like the detailed features of the Bauschinger effect. Yet our work is the first to produce a mathematical analysis of such an evolution, although, in all fairness, a partially successful attempt was made earlier by K. CHELMIŃSKI without the benefit of the variational structure that we uncovered.

As detailed in [78], it thus seems that non-associativity can always be incorporated into a variational framework, although, in all fairness, the resulting models have to be tweaked before a successful proof of mathematical well-posedness of the evolution can be produced. In particular, the existence theorems obtained in e.g. [62], [68], or in the works of G. DAL MASO, A. DE SIMONE & F. SOLOMBRINO all pertain to a time-rescaled evolution where the rescaling at time $t$ is in essence the total plastic dissipation $\int_0^t \|\dot\beta(s)\|_{L^1}\ ds$ up to that time. In [79] M.G. MORA and I revisit the general framework of non-associativity and demonstrate that one can actually obtain existence of an evolution in the original time variable.

In a different direction, A. GIACOMINI and I undertook in [63] a revisiting of heterogeneous small strain elasto-plasticity. The topic had recently been broached, within the framework of rate-independent evolutions, by F. SOLOMBRINO. We have improved on those results by allowing for a more realistic class of multi-phase composites which arises naturally as a limit of elasto-plastic models with vanishingly small hardening.

In a nutshell, the issue is as follows. As seen above, the dissipated energy acts on the plastic strain $\beta$, a bounded Radon measure which may see sets of co-dimension 1. Because of that, it is necessary to define $H$ on the interfaces between the various phases. When the associated sets of admissible stresses are well-ordered for the inclusion (as in the work of Solombrino), then it is a simple matter; just take the infimum of the two dissipations at the interface. When such is not the case, then the right interfacial dissipation is actually an inf-convolution, and this creates in turn serious problems when proving that the resulting dissipation functional is lower-semicontinuous, an essential ingredient in establishing existence of an evolution.

In doing so, we have also exhibited an interfacial flow rule which seems to be a missing ingredient in the classical modeling of elasto-plasticity used by the mechanics and engineering community. That flow rule states that, at a point $x$ of the interface between phases 1 and 2, the velocities $\dot{u}_i(t)$ in phase $i$ at that point satisfy

$$\frac{\dot{u}_1(t) - \dot{u}_2(t)}{|\dot{u}_1(t) - \dot{u}_2(t)|} \in \partial I_{\{(K_1\nu)_\tau \cap (K_2\nu)_\tau\}}((\sigma_D\nu)_\tau),$$

where $I_A$ stands for the indicatrix function of a set $A$, $K_i$ is the admissible set of stresses for phase $i$, $\nu$ is the normal to the interface at $x$, the subscript $\tau$ means the tangential component along the interface at $x$, and $\sigma_D$ is the deviatoric part of the Cauchy stress at $x$. It would remain to come up with an example where the missing flow rule arbitrates between competing solutions.

On a different note, our study has also led us to a revisiting and generalization of the duality between (rate of) plastic strain and deviatoric stress without which the flow rules cannot be recovered from the mathematical analysis of elasto-plastic evolutions. Even in the homogeneous case, our results uncover strict restrictions on the stress field for plastic slips (jumps) to appear. For example, A. Giacomini, J.-J. Marigo and I show in [72] that, in a Von Mises setting (that where the set of admissible stresses $K$ is of the form $\mathbb{R}i \oplus \{|\sigma_D| \le \sigma_c\}$ where $\sigma_D$ stands for the deviatoric (trace-free) part of the stress $\sigma$), no plastic slips can appear if the middle eigenvalue of $\sigma_D$ is not identically 0 along the slip. Note that the stress is well defined in that case because of a regularity result due to A. Bensoussan & J. Frehse and to A, Demyanov.

In a related direction, no results of uniqueness of the plastic strain had been obtained, except in the one dimensional case. In [73], [75], the same authors tackle the uniqueness issue. We relate it to a spatially hyperbolic problem and show that the restrictions alluded to in the previous paragraph can be instrumental in attaining such uniqueness. Conversely, we also demonstrate that uniqueness can be drastically violated through the onset, at arbitrary times of arbitrarily nonsmooth plastic strains. The geometry of the characteristics associated with hyperbolicity and their interaction with the boundary data prove to be the determining factors. For the time being, our results are, unfortunately, on a case by case basis.

In [67], we consider the issue of homogenization in plasticity, viewed as a rate independent evolution. For lack of better tools, we have to restrict our focus to the periodic case and to resort to two-scale convergence. We then derive a two-scale rate independent evolution as homogenization limit. The resulting model is rather complex because it uses an infinite number of internal variables, namely the two-scale plastic strains at each point $y$ of the microstructure. The derivation of the associated flow rules is quite involved. Indeed, stress-strain duality takes place in the $y$ variable, and this at almost every macroscopic point $x$ in the domain. But the "almost everywhere" is not the same because stresses are only defined Lebesgue-a.e., while plastic strains $\beta$ are measures that can concentrate in both the macro- and the microscopic variables, only the latter being taken into account by the usual duality evoked earlier. This has to be reconciled through adequate disintegration procedures. Nevertheless, the resulting model is found to fit squarely within the confines of the thermodynamical theory of generalized standard materials.

The investigations described in the preceding paragraphs are exclusively concerned with small strain elasto-plasticity. Finite strain plasticity is an unsettled field which arose out of fierce battles among many mechanicians of the 80's and 90's. Many issues are yet undecided, starting with the most basic one, that of the multiplicative decomposition of the strain $F$ (the gradient of the transformation) into a product of the elastic strain $E$ and of the plastic strain $P$. Some like P.M. Nagdhi were skeptical from the getgo. Other contentious issues involved the existence, or not, of an intermediate configuration, that, or not, of plastic frame indifference, and so on. In any case, the many proposed models shared a common feature, the absence of any mathematical justification.

To our knowledge, the first systematic attempt at providing a sound mathematical framework for finite plasticity may be found in the work of A. Mielke, together with various co-authors. To do so, he adopts the $F = EP$ representation. He further introduces both a plastic gradient and a strain hardening regularization without which even the functional framework is unclear. Indeed, as was the case for small strain elasto-plasticity, the plastic strain $P$ resulting from his formulation is expected to be a measure, while the elastic strain $E$ has the typical integrability property required for the elastic energy to be finite, typically an $L^p$-type regularity. But then, the product is non-sensical, hence the need for some amount of regularization. Finally, he also must tweak the dissipation functional associated with the onset of plasticity because the natural dissipation requires the introduction of the logarithm of $P$, a difficult notion when $P$ is merely a matrix with positive singular values. Doing so, he cannot recover a *bona fide* plasticity model because his dissipation might prove too low.

This motivated E. Davoli and I to propose a different route in [69]. While keeping a multiplicative decomposition, we privilege the inverse decomposition $F = PE$. We then use the polar decomposition of $P$ and material frame indifference to rewrite $F$ as

$$F = RDE$$

where $R$ is a rotation and $D$ is diagonal. Taking *both* $R$ and $D$ as internal variables, we show that thermodynamics imposes that the only dissipation arises from $D$ and involves $\log D$. The problem faced by A. Mielke is then alleviated in our setting because $P$ has become a positive diagonal matrix, namely $D$. We then proceed to prove the existence of a finite strain elasto-plastic evolution under the same kind of regularization as that mentioned earlier.

We also show that, barring such a regularization, any multiplicative model leads to immediate softening under loading, at least in the rigid-plastic setting. This is in our opinion a demonstration of the ill-posedness of most non-regularized models of finite plasticity, and this as soon a multiplicative decomposition is adopted!

I cannot resist mentioning that this last paper has had a rocky path to publication. The unusual vigor of the many reports that were generated throughout its evaluation bears testimony to the contentiousness of any model that pertains to finite plasticity.