LOSS OF ELLIPTICITY THROUGH HOMOGENIZATION IN LINEAR ELASTICITY

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Abstract. It was shown in [6] that, in the setting of linearized elasticity, a Γ-convergence result holds for highly oscillating sequences of elastic energies whose functional coercivity constant in $\mathbb{R}^N$ is zero while the corresponding coercivity constant on the torus remains positive. We illustrate the range of applicability of that result by finding sufficient conditions for such a situation to occur. We thereby justify the degenerate laminate construction of [7]. We also demonstrate that the predicted loss of strict strong ellipticity resulting from the construction in [7] is unique within a "laminate-like" class of microstructures. It will only occur for the specific micro-geometry investigated there. Our results thus confer both a rigorous, and a canonical character to those in [7].

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1. Introduction

In the canonical scalar second order elliptic setting, that is when attempting to solve

$$\left\{\begin{aligned} -\operatorname{div}\big(A(x)\nabla u\big) &= f \quad \text{in } \Omega\\ u &= 0 \quad \text{on } \partial\Omega,\end{aligned}\right. \tag{1.1}$$

with $A \in L^\infty(\Omega;\mathbb{R}^{N\times N})$, $\mathbb{R}^{N\times N}$ being the set of $N\times N$ matrices, the coercivity of the matrix A,

$$\operatorname*{ess-inf}_{x\in\Omega}\Big(\min\big\{A(x)\xi\cdot\xi : \xi\in\mathbb{R}^N,\ |\xi|=1\big\}\Big) > 0, \tag{1.2}$$

is a necessary and sufficient condition for a successful application of Lax-Milgram's lemma, hence for an existence and uniqueness statement for the solution to (1.1), and this for all right hand-sides $f \in H^{-1}(\Omega;\mathbb{R}^N)$.

This is so because the bilinear form

$$\int_\Omega A(x)\nabla v\cdot\nabla v\,dx$$

is coercive over $H^1_0(\Omega)$ if, and only if (1.2) holds true.

In the case of a system, the situation may become a bit less pristine. As far as linearized elasticity is concerned, the analogue of the matrix-valued function A is a L^∞-mapping

$$\mathbb{L} : \Omega \to \mathscr{L}_s(\mathbb{R}^{N\times N}_s),$$

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where $\mathbb{R}_s^{N\times N}$ is the subspace of symmetric elements of $\mathbb{R}^{N\times N}$, and $\mathscr{L}_s(\mathbb{R}_s^{N\times N})$ denotes the space of symmetric endomorphisms on $\mathbb{R}_s^{N\times N}$. Then, application of Lax-Milgram's lemma necessitates that

$$\int_\Omega \mathbb{L}(x)\nabla v\cdot\nabla v\,dx$$

be coercive, in other words that

$$\inf\left\{\int_\Omega \mathbb{L}(x)\nabla v\cdot\nabla v\,dx : v\in C_c^\infty(\Omega;\mathbb{R}^N),\ \int_\Omega |\nabla v|^2\,dx=1\right\}>0. \tag{1.3}$$

That condition is implied by very strong ellipticity, that is by the pointwise condition

$$\operatorname*{ess-inf}_{x\in\Omega}\left(\min\left\{\mathbb{L}(x)M\cdot M : M\in\mathbb{R}_s^{N\times N}, |M|=1\right\}\right)>0,\ (^1) \tag{1.4}$$

but it only implies (strict) strong ellipticity, that is the pointwise condition

$$\operatorname*{ess-inf}_{x\in\Omega}\left(\min\left\{\mathbb{L}(x)(a\otimes b)\cdot(a\otimes b) : a,b\in\mathbb{R}^N, |a|=|b|=1\right\}\right)>0. \tag{1.5}$$

(See Remark 2.1 below.)

This apparently innocuous discrepancy strongly impacts linearly elastic behavior and endows it with features that prove to be drastically at odds with its scalar sibling: lack of maximum principle, differences between wave speeds and light cones, etc... We do not intend to provide a review of those distinguishing traits and direct the interested reader to *e.g.* [4], [11], [15] among many contributions.

In this work, we turn our attention to homogenization, and, more precisely, to the simplest possible setting for homogenization, that where the oscillations of the coefficients are periodic. We thus consider throughout an elasticity tensor (Hooke's law) of the form

$$\mathbb{L}\in L^\infty_{\rm per}\big(Y_N;\mathscr{L}_s(\mathbb{R}_s^{N\times N})\big),$$

where $Y_N:=[0,1)^N$ is identified with the N-dimensional torus, and the tensor-valued function $\mathbb{L}$ defined in $\mathbb{R}^N$ is Y_N-periodic, namely,

$$\mathbb{L}(y+\kappa)=\mathbb{L}(y),\quad \text{a.e. in } \mathbb{R}^N,\ \forall\,\kappa\in\mathbb{Z}^N,$$

so that the rescaled function $\mathbb{L}(x/\varepsilon)$ is εY_N-periodic.

We then consider the Dirichlet boundary value problem

$$\begin{cases} -\operatorname{div}\big(\mathbb{L}(x/\varepsilon)\nabla u^\varepsilon\big) &= f \text{ in } \Omega\\ u^\varepsilon &= 0 \text{ on } \partial\Omega,\end{cases} \tag{1.6}$$

with $f\in H^{-1}(\Omega;\mathbb{R}^N)$. We could impose a very strong ellipticity condition on $\mathbb{L}$, namely

$$\alpha_{\rm vse}(\mathbb{L}):=\operatorname*{ess-inf}_{y\in Y_N}\left(\min\left\{\mathbb{L}(y)M\cdot M : M\in\mathbb{R}_s^{N\times N}, |M|=1\right\}\right)>0. \tag{1.7}$$

In such a setting, homogenization is straightforward as explained in Remarks 2.1, 2.3 below.

Instead, we will merely impose (strict) strong ellipticity, that is

$$\alpha_{\rm se}(\mathbb{L}):=\operatorname*{ess-inf}_{y\in Y_N}\left(\min\left\{\mathbb{L}(y)(a\otimes b)\cdot(a\otimes b) : a,b\in\mathbb{R}^N, |a|=|b|=1\right\}\right)>0, \tag{1.8}$$

[1] Note that, because of the symmetries of $\mathbb{L}$, replacing the symmetric matrix M in condition (1.4) by an arbitrary matrix would not change its definition.

and this **throughout the paper**.

Remark 1.1 (Strong ellipticity and isotropy)**.** Whenever $\mathbb{L}$ is isotropic, that is

$$\mathbb{L}(y)M = \lambda(y)\,\mathrm{tr}\,(M)\,I_N + 2\mu(y)\,M, \quad \text{for } y \in Y_N,\ M \in \mathbb{R}^{N\times N}_s,$$

then (1.8) reads as

$$\operatorname*{ess-inf}_{y\in Y_N}\left(\min\left\{\mu(y), \lambda(y)+2\mu(y)\right\}\right) > 0. \tag{1.9}$$

¶

The strong ellipticity condition (1.8) is the starting point of the study of homogenization performed in [6]. Under that condition, the authors investigate the Γ-convergence, for the weak topology of $H^1_0(\Omega;\mathbb{R}^N)$ on bounded sets (a metrizable topology), of the Dirichlet integral

$$\int_\Omega \mathbb{L}(x/\varepsilon)\nabla v\cdot\nabla v\,dx.$$

The results in [6] that will be of use to us are summarized in Theorem 2.7 of Section 2 below. That theorem asserts that, under conditions that will be detailed in that section, the Γ-limit is given through the expected homogenization formula (2.3) in spite of the lack of very strong ellipticity.

In [6], no examples are given of a setting for which the above mentioned result applies.

Our goal in this study is in part to remedy that situation by firstly establishing a reasonable list of conditions on a multi-phase periodic mixture of isotropic components so that the assumptions required for the application of Theorem 2.7 are met; this is the object of Theorem 2.9 which holds true whenever the mixture is either “laminate-like”, that is roughly one-directional, or else “inclusion like”, that is so that no “bad” phase (*i.e.*, one where very strong ellipticity does not hold) is in essence “surrounded by” another bad phase.

Then we exhibit essentially necessary and sufficient conditions for those conditions to be satisfied in the case of a one-dimensional mixture – one where the elasticity $\mathbb{L}(y)$ only depends on one variable, say y_1 – while producing a homogenized elasticity tensor that loses (strict) strong ellipticity. This is the object of Theorem 3.9.

To this effect, we will carefully revisit a laminate geometry introduced in [7, 8]. In those papers it is established that a certain homogenization scheme – labeled 1*-convergence (see Lemma 3.2) – applied to the lamination mixture of two phases, one of which is very strongly elliptic while the other is only (strictly) strongly elliptic, will lead to a limit behavior for which (strict) strong ellipticity, and even semi-strong ellipticity (the condition which consists in replacing positivity by non-negativity in (1.5)) fails. Remark 3.6 justifies the scheme introduced in those papers. Indeed we show that the 1*-limit coincides with the Γ-limit, thanks in part to an application of Theorem 3.9 in the specific setting at hand.

In turn, Theorem 3.9 demonstrates that the lamination process introduced in [7] is canonical : it is the only one among rank-one laminates that will result in a loss of (strict) strong ellipticity for the homogenized tensor.

In particular, no continuous dependence in y_1 can produce a loss of (strict) strong ellipticity; cf. Remark 3.10. This has a bearing on the usefulness of the Γ-convergence result of Theorem 2.7 whenever functional coercivity, *i.e.,* (1.3), is not satisfied. Indeed, that result is only helpful if H^1-bounds are secured for the sequence of minimizers associated with the investigated energy. A usually convenient way to ensure such a bound is to add a quadratic zeroth-order term of the form

$$\gamma \int_\Omega |u|^2 dx$$

to the energy and to use a Gärding type inequality to conclude to the coercivity of the resulting energy for an adequate value of the constant γ. But such a scheme is doomed from the very start in our context because it is known that Gärding's inequality is false when the coefficients are not continuous (see [17]).

So, in the end, the homogenization scheme produced by the Γ-convergence result is debatable at least in the lamination setting. Our contribution merely circumscribes the extent to which it could produce a loss of strict strong ellipticity, should the minimizing sequences associated with the periodic microstructure remain bounded in H^1 as the size ε of the microstructure goes to 0.

Finally, for the sake of completeness, we point the interested reader to [1] as the only paper which, to the extent of our knowledge, attempts to improve on the results of [6] by abandoning any notion of ellipticity, or even non-negativity for the elasticity of the microstructure. Unfortunately, that paper has no bearing on our work.

Notationwise:

- I_N is the unit matrix of $\mathbb{R}^{N\times N}$.
- $A\cdot B$ is the Frobenius inner product between two elements of $A, B \in \mathbb{R}^{N\times N}$, that is $A \cdot B := \operatorname{tr}(A^T B)$.
- If $A := \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2}$, the cofactor matrix of A is $\operatorname{cof} A := \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$.
- $H^1_{\rm per}(Y_N)$ is the space of the functions in $H^1_{\rm loc}(\mathbb{R}^N)$ which are Y_N-periodic; the spaces $L^p_{\rm per}(Y_N)$, $C^k_{\rm per}(Y_N)$, etc... are similarly defined.
- $\{\varepsilon\}$ denotes a sequence of positive numbers which converges to 0;
- If $\mathscr{I}^\varepsilon$ is an ε-indexed sequence of functionals with

  $$\mathscr{I}^\varepsilon : X \to \mathbb{R},$$

  (X reflexive Banach space), we will write that $\mathscr{I}^\varepsilon \overset{\Gamma(X)}{\rightharpoonup} \mathscr{I}^0$, with

  $$\mathscr{I}^0 : X \to \mathbb{R},$$

  if $\mathscr{I}^\varepsilon$ Γ-converges to $\mathscr{I}^0$ for the (metrizable) weak topology on bounded sets of X (see *e.g.* [2, Definition 7.1] for the appropriate definition).

2. The Γ-convergence viewpoint

As announced in the introduction, this section is devoted to a revisiting of some of the results obtained in [6]. For vector-valued (linear) problems, a successful application of Lax-Milgram's lemma to a Dirichlet problem of the type (1.6) hinges on the positivity of the following functional coercivity constant:

$$\Lambda(\mathbb{L}) := \inf \left\{ \int_{\mathbb{R}^N} \mathbb{L}(y)\nabla v \cdot \nabla v \, dy : v \in C^\infty_c(\mathbb{R}^N;\mathbb{R}^N), \ \int_{\mathbb{R}^N} |\nabla v|^2 \, dy = 1 \right\}. \tag{2.1}$$

As long as $\Lambda(\mathbb{L}) > 0$, existence and uniqueness of the solution to (1.6) is guaranteed by Lax-Milgram's lemma.

Remark 2.1 (Conductivity versus elasticity)**.** In the conductivity setting (for, say a symmetric $N \times N$- conductivity tensor $A(y)$), functional coercivity, that is

$$\inf\left\{\int_{\mathbb{R}^N} A(x)\nabla v\cdot\nabla v\,dx : v\in C_c^\infty(\mathbb{R}),\ \int_{\mathbb{R}^N}|\nabla v|^2 dx = 1\right\} > 0,$$

is equivalent to pointwise strong ellipticity, that is

$$\underset{y\in Y_N}{\text{ess-inf}}\left(\min\left\{A(y)\xi\cdot\xi : \xi\in\mathbb{R}^N\right\}\right) > 0,$$

as immediately seen by testing with a $v(x)$ of the form $\varphi(x)\cos(n\,\xi\cdot x)$, $\xi\in\mathbb{R}^N$, $\varphi\in C_c^\infty(\Omega)$, with $n\to\infty$. The vectorial analogue does not hold true and $\Lambda(\mathbb{L}) > 0$ only implies (strict) strong ellipticity, *i.e.* (1.8), while (1.7) is needed to pass from a pointwise condition to the positivity of $\Lambda(\mathbb{L})$.

Actually, even very strong ellipticity, that is property (1.7), is not sufficient *per se* to ensure the positivity of $\Lambda(\mathbb{L})$. This has to be combined with a Korn type inequality which allows us to replace symmetrized gradients by full gradients. Such an inequality is satisfied in *e.g.* the case of Lipschitz domains and for general domains in the case at hand, *i.e.* for Dirichlet boundary conditions on $\partial\Omega$ (see [13, Ch.I, Sec.2]). ¶

Further, according to classical results in the theory of homogenization, under condition (1.7) the solution $u^\varepsilon\in H_0^1(\Omega;\mathbb{R}^N)$ of (1.6) satisfies

$$\left\{\begin{array}{ll} u^\varepsilon \rightharpoonup u, & \text{weakly in } H_0^1(\Omega;\mathbb{R}^N)\\ \mathbb{L}(x/\varepsilon)\nabla u^\varepsilon \rightharpoonup \mathbb{L}^0\nabla u, & \text{weakly in } L^2(\Omega;\mathbb{R}^{N\times N})\\ -\operatorname{div}\left(\mathbb{L}^0\nabla u\right) = f, & \end{array}\right. \tag{2.2}$$

with

$$\mathbb{L}^0 M\cdot M := \min\left\{\int_{Y_N}\mathbb{L}(y)(M+\nabla v)\cdot(M+\nabla v)\,dy : v\in H^1_{\text{per}}(Y_N;\mathbb{R}^N)\right\}. \tag{2.3}$$

Remark 2.2. It is stated in [14, Ch. 6, Section 11] that the first explicit homogenization result in the framework of linear elasticity – this under the assumption of very strong ellipticity (1.7) – is to be found in the work of G. Duvaut (unavailable reference cited therein). ¶

Remark 2.3 (Homogenization in the functionally coercive setting)**.** The additional remark that functional coercivity, that is $\Lambda(\mathbb{L}) > 0$, is actually the correct condition for performing homogenization and is stable under the homogenization process can be found in [5] independently of any assumption of periodicity.

In the periodic setting, the resulting homogenized tensor is still given by (2.3) and, since it is functionally coercive, it is also (strictly) strongly elliptic, that is, recalling (1.8),

$$\alpha_{\text{se}}(\mathbb{L}^0) > 0.$$

Further, if $v\in H^1(\Omega;\mathbb{R}^N)$ and $w^\varepsilon\in H^1(\Omega;\mathbb{R}^N)$ is the solution to

$$\left\{\begin{array}{l} -\operatorname{div}\left(\mathbb{L}(x/\varepsilon)\nabla w^\varepsilon\right) = -\operatorname{div}\left(\mathbb{L}^0\nabla v\right)\\ w^\varepsilon = v \text{ on } \partial\Omega, \end{array}\right. \tag{2.4}$$

then, multiplication of the equation by $w^\varepsilon - v$ results, thanks to functional coercivity, in an $H^1(\Omega;\mathbb{R}^N)$-bound on w^ε which in turn allows us to conclude that

$$\begin{cases} w^\varepsilon \rightharpoonup v, & \text{weakly in } H^1(\Omega;\mathbb{R}^N) \\ \mathbb{L}(x/\varepsilon)\nabla w^\varepsilon \rightharpoonup \mathbb{L}^0\nabla v, & \text{weakly in } L^2(\Omega;\mathbb{R}^{N\times N}), \end{cases} \tag{2.5}$$

as can be proved exactly as in [9, Theorem 1]. ¶

Set, for $v \in H^1(\Omega;\mathbb{R}^N)$,

$$\mathscr{I}^\varepsilon(v) := \int_\Omega \mathbb{L}(x/\varepsilon)\nabla v\cdot\nabla v\,dx, \quad \text{resp. } \mathscr{I}^0(v) := \int_\Omega \mathbb{L}^0\nabla v\cdot\nabla v\,dx, \tag{2.6}$$

We propose to quickly discuss the Γ-convergence properties of $\mathscr{I}^\varepsilon$ to $\mathscr{I}^0$. Under the condition of very strong ellipticity, Γ-convergence is known to hold true for both the weak $H^1_0(\Omega;\mathbb{R}^N)$ and the weak $H^1(\Omega;\mathbb{R}^N)$-topologies on bounded subsets of those spaces. In the lemma below, we generalize that result to the case where functional coercivity holds. Unfortunately, the result is more restrictive.

Lemma 2.4 (Γ-convergence – the functionally coercive case)**.** *Assume that the functional coercivity condition $\Lambda(\mathbb{L}) > 0$ holds true. Then,*

$$\mathscr{I}^\varepsilon \overset{\Gamma(H^1_0)}{\rightharpoonup} \mathscr{I}^0.$$

Further, if, a.e. in Y,

$$\mathbb{L}(y)M\cdot M \ge 0,\ \forall\, M \in \mathbb{R}^{N\times N}, \tag{2.7}$$

then

$$\mathscr{I}^\varepsilon \overset{\Gamma(H^1)}{\rightharpoonup} \mathscr{I}^0.$$

Proof. We only treat the more difficult case of $H^1(\Omega;\mathbb{R}^N)$ and comment on the easier case $H^1_0(\Omega;\mathbb{R}^N)$ at the end of the proof.

Consider $v^\varepsilon \rightharpoonup v$ in $H^1(\Omega;\mathbb{R}^N)$, and $w^\varepsilon \in H^1(\Omega;\mathbb{R}^N)$, solution to (2.4).

For $\varphi \in C^\infty_c(\Omega)$ with $0 \le \varphi \le 1$, one gets, thanks to (2.7), (2.5),

$$\liminf_{\varepsilon\to 0}\int_\Omega \mathbb{L}(x/\varepsilon)\nabla v^\varepsilon\cdot\nabla v^\varepsilon\,dx \ge \liminf_{\varepsilon\to 0}\int_\Omega \varphi\,\mathbb{L}(x/\varepsilon)\nabla v^\varepsilon\cdot\nabla v^\varepsilon\,dx \ge$$

$$\lim_{\varepsilon\to 0}\int_\Omega \varphi\left(2\,\mathbb{L}(x/\varepsilon)\nabla v^\varepsilon\cdot\nabla w^\varepsilon - \mathbb{L}(x/\varepsilon)\nabla w^\varepsilon\cdot\nabla w^\varepsilon\right)dx = \int_\Omega \varphi\,\mathbb{L}^0\nabla v\cdot\nabla v\,dx,$$

and then one lets $\varphi \nearrow 1$. The Γ-liminf inequality is established.

The sequence $\{w^\varepsilon\}$ is a good recovery sequence, as is immediately shown upon passing to the limit in

$$\int_\Omega \mathbb{L}(x/\varepsilon)\nabla w^\varepsilon\cdot\nabla(w^\varepsilon - v)dx = \int_\Omega \mathbb{L}^0\nabla v\cdot\nabla(w^\varepsilon - v)dx.$$

In the $H^1_0(\Omega;\mathbb{R}^N)$ case the same argument goes through without having to introduce the cut-off function φ and thus condition (2.7) is not needed. □

Remark 2.5 (About $\Gamma(H^1)$-convergence)**.** In the absence of pointwise non-negativity of the energy density $\mathbb{L}(y)M\cdot M$, it is not clear to us that that same result holds true under the sole condition $\Lambda(\mathbb{L}) > 0$. In truth, we do not even know if the Γ-limit of $\mathscr{I}^\varepsilon$ for the weak H^1-topology on H^1-bounded sets can be expressed as a local functional. ¶

Remark 2.6 (About the non-negativity of the energy density)**.** It is possible to consider a Hooke's law $\mathbb{L}(y)$ which satisfies $\Lambda(\mathbb{L}) > 0$, (2.7), and yet where α_{vse} defined in (1.7) is non-positive. An example is provided in Remark 3.7 below in the case $N = 2$. We are confident that a similar result is also valid when $N = 3$. ¶

The case $\Lambda(\mathbb{L}) < 0$ is easily disposed of. Indeed, it is immediate from the proof of [6, Prop. 3.2] that, in such a case,

$$\mathscr{I}^\varepsilon \overset{\Gamma(H_0^1)}{\rightharpoonup} \mathscr{I}^0 \equiv -\infty.$$

The case $\Lambda(\mathbb{L}) = 0$ is more delicate. Define

$$\Lambda_{\text{per}}(\mathbb{L}) := \inf\left\{ \int_{Y_N} \mathbb{L}(y)\nabla v \cdot \nabla v : v \in H^1_{\text{per}}(Y_N;\mathbb{R}^N),\ \int_{Y_N} |\nabla v|^2 dy = 1 \right\}. \tag{2.8}$$

Then the following result is found in [6, Theorem 3.4(i)].

Theorem 2.7 (Homogenization as a Γ-convergence result)**.** *If*

$$\Lambda_{\text{per}}(\mathbb{L}) > 0, \tag{2.9}$$

then, $\mathscr{I}^\varepsilon \overset{\Gamma(H_0^1)}{\rightharpoonup} \mathscr{I}^0$, *with* $\mathbb{L}^0$ *given by* (2.3)*.*

Remark 2.8. In strict parallel with Remark 2.5, we do not know whether the result of Theorem 2.7 still holds true when $H^1_0(\Omega;\mathbb{R}^N)$ is replaced by $H^1(\Omega;\mathbb{R}^N)$ in the Γ-convergence statement. ¶

No examples of applicability of the theorem were produced in [6], or, to the best of our knowledge, in any later study. We now demonstrate the existence of a large class of isotropic mixtures to which Theorem 2.7 applies. Our result is restricted to the two-dimensional case $N = 2$ with (1.9) satisfied so as to ensure (strict) strong ellipticity, although we suspect that similar results could be derived when $N = 3$.

Specifically, we assume the existence of p phases Z_i, $i = 1, \ldots, p$, with Z_i open, connected and Lipschitz, satisfying

$$\forall\, i \neq j \in \{1, \ldots, p\},\ \ Z_i \cap Z_j = \emptyset \quad \text{and} \quad \bigcup_{i=1,\ldots,p} \bar{Z}_i = \bar{Y}_2,$$

such that

$$\begin{cases} \mathbb{L}(y)M = \lambda(y)\,\mathrm{tr}\,(M)\, I_N + 2\mu(y)M, & y \in Y_2,\ M \in \mathbb{R}^{2\times 2}_s \\ \lambda(y) = \lambda_i,\ \mu(y) = \mu_i, & \text{in } Z_i,\ i = 1, \ldots, p \\ \min\limits_{i=1,\ldots,p} \{\mu_i, \lambda_i + 2\mu_i\} > 0. & \end{cases} \tag{2.10}$$

We also assume the existence of $\gamma > 0$ such that

$$-\min_{i=1,\ldots,p}\{\lambda_i + \mu_i\} \le \gamma \le \min_{i=1,\ldots,p}\{\mu_i\}. \tag{2.11}$$

Define the following subset of indices:

$$\begin{cases} I := \big\{ i \in \{1, \ldots, p\} : \mu_i = \gamma \big\} \\ J := \big\{ j \in \{1, \ldots, p\} : \lambda_j + \mu_j = -\gamma \big\} \\ K := \{1, \ldots, p\} \setminus (I \cup J)\,, \end{cases} \tag{2.12}$$

and remark that, since $\lambda_i + 2\mu_i > 0$, those three sets are disjoint.

Then, the following theorem illustrated in Figure 1 below holds true:

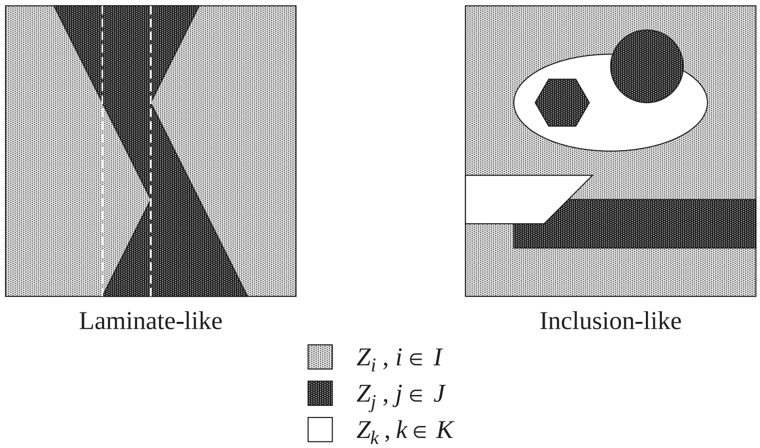

FIGURE 1. *Sets Z_i for a laminate-like or an inclusion-like micro-geometry.*

Theorem 2.9 (Applicability of Theorem 2.7 – the isotropic case)**.** *Let* (2.10), (2.11) *hold true. Then,* $\Lambda(\mathbb{L}) \geq 0$.

Assume further that the sets defined in (2.12) *meet one of the two following conditions:*

Case 1. *For each $j \in J$, there exists an interval $(a_j, b_j) \subset [0,1]$ such that*
$$\big[(a_j, b_j) \times \{0\}\big] \cup \big[(a_j, b_j) \times \{1\}\big] \subset \partial Z_j \quad \text{or}$$
$$\big[\{0\} \times (a_j, b_j)\big] \cup \big[\{1\} \times (a_j, b_j)\big] \subset \partial Z_j,$$
(we will refer to this setting as "laminate-like"); or

Case 2. *$K \neq \emptyset$ and, for each $j \in J$, there exists $k \in K$ with $\mathcal{H}^1(\partial Z_j \cap \partial Z_k) > 0$, (we will refer to this setting as "inclusion-like").*

Then, $\Lambda_{\rm per}(\mathbb{L}) > 0$.

Proof.

Step 1. $\Lambda(\mathbb{L}) \geq 0$: The proof that it is so is an adaptation of results presented in [7, Section 4].

Because, for any $v \in C_c^\infty(\mathbb{R}^2; \mathbb{R}^2)$,

$$\int_{Y_N} \det \nabla v(y)\, dy = 0,$$

the definition (2.1) of $\Lambda(\mathbb{L})$ equivalently reads as

$$\Lambda(\mathbb{L}) = \inf \left\{ \int_{\mathbb{R}^2} \big(\mathbb{L}(y_1)\nabla v \cdot \nabla v + 4\gamma \det \nabla v\big)\, dy : \right.$$
$$\left. v \in C_c^\infty(\mathbb{R}^2; \mathbb{R}^2),\ \int_{\mathbb{R}^2} |\nabla v|^2\, dy = 1 \right\}.$$

Thus,

$$\Lambda(\mathbb{L}) = \inf\left\{\int_{\mathbb{R}^2}\left[(\lambda(y)+2\mu(y))\left(\left(\frac{\partial v_1}{\partial y_1}\right)^2+\left(\frac{\partial v_2}{\partial y_2}\right)^2\right)+\right.\right. \tag{2.13}$$

$$\left.\left.(2\lambda(y)+4\gamma)\frac{\partial v_1}{\partial y_1}\frac{\partial v_2}{\partial y_2}+\mu(y)\left(\left(\frac{\partial v_1}{\partial y_2}\right)^2+\left(\frac{\partial v_2}{\partial y_1}\right)^2\right)+\big(2\mu(y)-4\gamma\big)\frac{\partial v_1}{\partial y_2}\frac{\partial v_2}{\partial y_1}\right]dy\right\}.$$

The above quantity is certainly non negative if it is pointwise non negative. Such is the case if we can choose γ satisfying

$$\text{ess-sup}\left\{\max\{0,-(\lambda+\mu)\}\right\}\le\gamma\le\text{ess-inf}\{\mu\}.$$

Hence the result in view of (2.11).

Step 2. $\Lambda_{\rm per}(\mathbb{L})>0$: Since the determinant is a null Lagrangian for elements of $H^1_{\rm per}(Y_2;\mathbb{R}^2)$, definition (2.8) equivalently reads as

$$\Lambda_{\rm per}(\mathbb{L}) = \inf\left\{\int_{Y_2}\big(\mathbb{L}(y)\nabla v\cdot\nabla v+4\gamma\det\nabla v\big)\,dy:\right.$$

$$\left. v\in H^1_{\rm per}(Y_2;\mathbb{R}^2),\ \int_{Y_2}|\nabla v|^2\,dy=1\right\},$$

or, as for $\Lambda(\mathbb{L})$,

$$\Lambda_{\rm per}(\mathbb{L}) = \inf\left\{\int_{Y_2}\left[P\left(y;\frac{\partial v_1}{\partial y_1},\frac{\partial v_2}{\partial y_2}\right)+Q\left(y;\frac{\partial v_1}{\partial y_2},\frac{\partial v_2}{\partial y_1}\right)\right]dy:\right. \tag{2.14}$$

$$\left. v\in H^1_{\rm per}(Y_2;\mathbb{R}^2),\ \int_{Y_2}|\nabla v|^2\,dy=1\right\},$$

where $P(y;\cdot)$ and $Q(y;\cdot)$ are the quadratic forms

$$\begin{cases} P(y;a,b):=\big(\lambda(y)+2\mu(y)\big)\,(a^2+b^2)+2\big(\lambda(y)+2\gamma\big)\,ab\\ Q(y;a,b):=\mu(y)\,(a^2+b^2)+2\big(\mu(y)-2\gamma\big)\,ab.\end{cases}$$

Upon considering the associated traces and determinants, it is easily concluded that there exists $\alpha>0$ such that,

$$P(y;a,b)\ge\alpha\,(a+b)^2,\quad Q(y;a,b)\ge\alpha\,(a-b)^2,\quad y\in Z_i,\ i\in I, \tag{2.15}$$

$$P(y;a,b)\ge\alpha\,(a-b)^2,\quad Q(y;a,b)\ge\alpha\,(a^2+b^2),\quad y\in Z_j,\ j\in J, \tag{2.16}$$

$$P(y;a,b)\ge\alpha\,(a^2+b^2),\quad Q(y;a,b)\ge\alpha\,(a^2+b^2),\quad y\in Z_k,\ k\in K. \tag{2.17}$$

Assume the existence of a sequence $\{v^n\}\subset H^1_{\rm per}(Y_2;\mathbb{R}^2)$, with $\int_{Y_2}v^n dy=0$, such that

$$\int_{Y_2}|\nabla v^n|^2dy=1, \tag{2.18}$$

while

$$\int_{Y_2}\mathbb{L}(y)\nabla v^n\cdot\nabla v^n\,dy\to 0.$$

Note that, by Poincaré-Wirtinger's inequality, it is immediate that

$$v^n \text{ is bounded in } L^2(Y_2;\mathbb{R}^2). \tag{2.19}$$

In the light of (2.14),

$$\int_{Y_2}\left[P\left(y;\frac{\partial v_1^n}{\partial y_1},\frac{\partial v_2^n}{\partial y_2}\right)+Q\left(y;\frac{\partial v_1^n}{\partial y_2},\frac{\partial v_2^n}{\partial y_1}\right)\right]dy\to 0. \tag{2.20}$$

Then, in view of (2.17), for $k\in K$,

$$\int_{Z_k}\left[P\left(y;\frac{\partial v_1^n}{\partial y_1},\frac{\partial v_2^n}{\partial y_2}\right)+Q\left(y;\frac{\partial v_1^n}{\partial y_2},\frac{\partial v_2^n}{\partial y_1}\right)\right]dy\geq\alpha\int_{Z_k}|\nabla v^n|^2\,dy,$$

so that, by virtue of (2.20),

$$\int_{Z_k}|\nabla v^n|^2\,dy\to 0.$$

Summing over $k\in K$ we get

$$\lim_{n\to\infty}\sum_{k\in K}\int_{Z_k}\sum_{r,q=1,2}\left(\frac{\partial v_r^n}{\partial y_q}\right)^2dy=0. \tag{2.21}$$

In turn, in view of (2.16), for $j\in J$,

$$\int_{Z_j}\left[P\left(y;\frac{\partial v_1^n}{\partial y_1},\frac{\partial v_2^n}{\partial y_2}\right)+Q\left(y;\frac{\partial v_1^n}{\partial y_2},\frac{\partial v_2^n}{\partial y_1}\right)\right]dy\geq$$

$$\alpha\int_{Z_j}\left[\left(\frac{\partial v_1^n}{\partial y_2}\right)^2+\left(\frac{\partial v_2^n}{\partial y_1}\right)^2+\left(\frac{\partial v_1^n}{\partial y_1}-\frac{\partial v_2^n}{\partial y_2}\right)^2\right]dy,$$

so that, by virtue of (2.20),

$$\lim_{n\to\infty}\int_{Z_j}\left[\left(\frac{\partial v_1^n}{\partial y_2}\right)^2+\left(\frac{\partial v_2^n}{\partial y_1}\right)^2+\left(\frac{\partial v_1^n}{\partial y_1}-\frac{\partial v_2^n}{\partial y_2}\right)^2\right]dy=0. \tag{2.22}$$

Now, according to (2.18),

$$\frac{\partial v_1^n}{\partial y_1}\ \text{ is bounded } L^2(Z_j), \tag{2.23}$$

while, in view of (2.22),

$$\frac{\partial}{\partial y_2}\left(\frac{\partial v_1^n}{\partial y_1}\right)=\frac{\partial}{\partial y_1}\left(\frac{\partial v_1^n}{\partial y_2}\right)\to 0\ \text{ strongly in } H^{-1}(Z_j), \tag{2.24}$$

and, further,

$$\frac{\partial^2 v_1^n}{\partial y_1^2}=\frac{\partial}{\partial y_1}\left(\frac{\partial v_2^n}{\partial y_2}\right)+r_n=\frac{\partial}{\partial y_2}\left(\frac{\partial v_2^n}{\partial y_1}\right)+r_n,$$

with $r_n\to 0$ in $H^{-1}(Z_j)$. Again using (2.22), we conclude that

$$\frac{\partial^2 v_1^n}{\partial y_1^2}\to 0\ \text{ strongly in } H^{-1}(Z_j). \tag{2.25}$$

Next, by Korn's theorem applied to the Lipschitz domain Z_j (see *e.g.* [12]) the two norms $\|\nabla\cdot\|_{H^{-1}(Z_j;\mathbb{R}^2)}+\|\cdot\|_{H^{-1}(Z_j)}$ and $\|\cdot\|_{L^2(Z_j)}$ are equivalent on $L^2(Z_j)$, so that recalling (2.23), (2.24), (2.25),

$$\frac{\partial v_1^n}{\partial y_1}\ \text{ converges strongly in } L^2(Z_j).$$

Using once again (2.24), (2.25) combined with the connectedness of Z_j, we obtain that, for some constant d_j,

$$\frac{\partial v_1^n}{\partial y_1} \to d_j \text{ strongly in } L^2(Z_j).$$

This combined with (2.22) leads to

$$\frac{\partial v_2^n}{\partial y_2} \to d_j \text{ strongly in } L^2(Z_j).$$

Thus, we conclude that

$$\nabla v^n \to d_j\, I_2 \text{ strongly in } L^2(Z_j;\mathbb{R}^{2\times 2}),$$

hence, in view of (2.19),

$$v^n \to v := d_j\, y + V_j \text{ strongly in } H^1(Z_j;\mathbb{R}^2),$$

for some constant vector V_j.

We now appeal to the assumptions in the statement of the theorem. In case 1, we observe that, by periodicity, $v = d_j\, y + V_j$ should take the same values on *e.g.* $(a_j, b_j) \times \{0\}$ and on $(a_j, b_j) \times \{1\}$. But this is impossible unless $d_j = 0$. In case 2, we observe that v should take a constant value on $\partial Z_j \cap \partial Z_k$ for some $k \in K$ that satisfies the assumption. Indeed, v^n converges strongly to $1_{Z_j}\, v + 1_{Z_k}\, c_k$ in $H^1(Z_j \cup Z_k)$ for some constant c_k, this thanks to (2.21) and to the connectedness of Z_k. Thus, due to the regularity of the open sets Z_j and Z_k, the trace of v agrees with the constant c_k a.e. on $\partial Z_j \cap \partial Z_k$. But $d_j\, y + V_j$ cannot remain constant on a set of non-zero $\mathcal{H}^1$-measure unless $d_j = 0$. Hence, ∇v^n converges strongly to 0 in $H^1(Z_j;\mathbb{R}^2)$. Therefore, we conclude that

$$\lim_{n\to\infty} \sum_{j\in J} \int_{Z_j} \sum_{r,q=1,2} \left(\frac{\partial v_r^n}{\partial y_q}\right)^2 dy \to 0. \tag{2.26}$$

Finally, in view of (2.15), for $i \in I$,

$$\int_{Z_i} \left[P\left(y; \frac{\partial v_1^n}{\partial y_1}, \frac{\partial v_2^n}{\partial y_2}\right) + Q\left(y; \frac{\partial v_1^n}{\partial y_2}, \frac{\partial v_2^n}{\partial y_1}\right)\right] dy \geq$$

$$\alpha \int_{Z_i} \left[\left(\frac{\partial v_1^n}{\partial y_2} - \frac{\partial v_2^n}{\partial y_1}\right)^2 + \left(\frac{\partial v_1^n}{\partial y_1} + \frac{\partial v_2^n}{\partial y_2}\right)^2\right] dy,$$

so that, by virtue of (2.20),

$$\int_{Z_i} \left[\left(\frac{\partial v_1^n}{\partial y_1} + \frac{\partial v_2^n}{\partial y_2}\right)^2 + \left(\frac{\partial v_1^n}{\partial y_2} - \frac{\partial v_2^n}{\partial y_1}\right)^2\right] dy \to 0. \tag{2.27}$$

But, recalling (2.21), (2.26), combined with

$$\int_{Y_2} \det \nabla v^n \, dy = 0,$$

we obtain that

$$\lim_{n\to\infty} \sum_{i\in I} \int_{Z_i} \det \nabla v^n \, dy = 0.$$

Hence, upon subtracting twice this quantity to the sum over $i \in I$ of (2.27),

$$\lim_{n\to\infty} \sum_{i\in I} \int_{Z_i} \sum_{r,q=1,2} \left(\frac{\partial v_r^n}{\partial y_q}\right)^2 dy \to 0. \tag{2.28}$$

Recalling (2.21), (2.26), (2.28) results in a contradiction with assumption (2.18). □

Remark 2.10 (A case where $\Lambda(\mathbb{L}) > 0$)**.** Note that if $\gamma > 0$ can be chosen such that the inequalities in (2.11) are strict, then $\Lambda(\mathbb{L}) > 0$. This is because one can argue as in the case where the phase is in K in the proof above at the expense of replacing $H^1_{\text{per}}(Y_2;\mathbb{R}^2)$ by $H^1(\mathbb{R}^2;\mathbb{R}^2)$; in particular, the integrand in (2.13) is bounded from below by $c\,|\nabla v|^2$ for some constant $c > 0$. ¶

Remark 2.11. As a special case of a general algebraic result on quadratic forms (see, *e.g.*, [16, (CKKS), P. 222]) and references therein), the strong ellipticity of $\mathbb{L}(y)$ in dimension two implies the existence of a function $\gamma(y)$ in $L^\infty(Y)$ such that

$$\mathbb{L}(y)M \cdot M + 4\,\gamma(y)\,\det M \geq 0, \quad \text{for a.e. } y \in Y,\ \forall\, M \in \mathbb{R}^{2\times 2}.$$

In the proof of the previous theorem, we require the stronger condition (2.11), which amounts to assuming that γ is independent of y. ¶

Our goal in Section 3 below is to demonstrate that a careful revisiting of the layering construction proposed in [7, 8] delivers a L^∞-mapping $\mathbb{L}(y_1)$ which satisfies precisely the assumptions of Theorem 2.7. We further demonstrate the uniqueness (in a sense that will be further elaborated upon) of such a construction.

3. The canonical character of Gutiérrez's lamination

3.1. **Gutiérrez's laminate.** In [7], the Dirichlet problem (1.6) is investigated in dimensions 2 or 3 under two additional assumptions. First, the microstructure exhibits a laminate geometry, that is $\mathbb{L}(y) := \mathbb{L}(y_1)$, for $y_1 \in Y_1$.

We define, for any $\mathbb{L} \in \mathscr{L}_s(\mathbb{R}^{N\times N}_s)$, the element $A[\mathbb{L}] \in \mathbb{R}^{N\times N}_s$ as

$$A[\mathbb{L}](y_1)\xi := \big[\mathbb{L}(y_1)\,(\xi \otimes e_1)\big]\, e_1, \quad y_1 \in Y_1,\ \xi \in \mathbb{R}^N, \tag{3.1}$$

so that $A_{ij}[\mathbb{L}] = \mathbb{L}_{1i1j}$.

Remark 3.1 (Invertibility)**.** The matrix $A[\mathbb{L}](y_1)$ is symmetric and definite positive whenever $\mathbb{L}$ satisfies the strong ellipticity assumption (1.8). This was assumed to hold true throughout this work. ¶

For such laminate geometries, the following periodic homogenization result is an adaptation to the linear elasticity setting of a well-known result originally called 1*-convergence by F. Murat and L. Tartar (see *e.g.* [3, Ch. 3, Sect. 4]):

Lemma 3.2 (1*-convergence)**.** *Consider problem* (1.6) *and assume that it admits a solution u^ε and that the resulting sequence $\{u^\varepsilon\}_\varepsilon$ is a bounded sequence in $H^1_0(\Omega;\mathbb{R}^N)$. Then there exists $u^0 \in H^1_0(\Omega;\mathbb{R}^N)$ such that* (2.2) *holds true upon* *replacing* $\mathbb{L}^0$ *by* $\mathbb{L}^\ell$ *defined through the following formulae*

$$\begin{cases} A^{-1}[\mathbb{L}^\ell] = \displaystyle\int_{Y_1} A^{-1}[\mathbb{L}](t)\,dt \\ A^{-1}_{im}[\mathbb{L}^\ell]\,\mathbb{L}^\ell_{1mkl} = \displaystyle\int_{Y_1} \left\{A^{-1}_{im}[\mathbb{L}](t)\,\mathbb{L}_{1mkl}(t)\right\} dt \\ \mathbb{L}^\ell_{ijkl} - \mathbb{L}^\ell_{ij1m} A^{-1}_{mn}[\mathbb{L}^\ell]\,\mathbb{L}^\ell_{1nkl} = \displaystyle\int_{Y_1} \left\{\mathbb{L}_{ijkl}(t) - \mathbb{L}_{ij1m}(t) A^{-1}_{mn}[\mathbb{L}](t)\,\mathbb{L}_{1nkl}(t)\right\} dt. \end{cases}$$

Proof. We merely sketch the proof. First we note that a straightforward application of the div-curl lemma [10] yields that if

$$a^\varepsilon(x_1) \rightharpoonup a^0(x_1) \quad \text{weakly-}* \text{ in } L^\infty(\Omega),$$

while

$$\begin{cases} u^\varepsilon \rightharpoonup u, \text{ weakly in } H^1(\Omega;\mathbb{R}^N) \\ \sigma^\varepsilon \rightharpoonup \sigma, \text{ weakly in } L^2(\Omega;\mathbb{R}^{N\times N}) \\ \operatorname{div}\sigma^\varepsilon \in \text{ compact set of } H^{-1}(\Omega;\mathbb{R}^N), \end{cases}$$

then, for $1 \le i \le N$,

$$\begin{cases} a^\varepsilon \dfrac{\partial u^\varepsilon_i}{\partial x_j} \rightharpoonup a^0 \dfrac{\partial u_i}{\partial x_j} & \text{weakly in } L^2(\Omega), \text{ for } 1 \le i \le N,\ 2 \le j \le N \\ a^\varepsilon \sigma^\varepsilon_{1j} \rightharpoonup a^0 \sigma_{1j} & \text{weakly in } L^2(\Omega), \text{ for } 1 \le j \le N. \end{cases}$$

Then it suffices to write the constitutive equation

$$\sigma^\varepsilon := \mathbb{L}(x_1/\varepsilon)\nabla u^\varepsilon,$$

as

$$\mathscr{B}^\varepsilon = \mathbf{M}^\varepsilon(x_1/\varepsilon)\,\mathscr{G}^\varepsilon$$

with

$$\mathscr{B}^\varepsilon := \begin{pmatrix} \dfrac{\partial u^\varepsilon_j}{\partial x_1}, & j \ge 1 \\ \sigma^\varepsilon_{ij}, & i,j \ge 2 \end{pmatrix}, \quad \mathscr{G}^\varepsilon := \begin{pmatrix} \sigma^\varepsilon_{1j}, & j \ge 1 \\ \dfrac{\partial u^\varepsilon_i}{\partial x_j}, & i \ge 1, j \ge 2 \end{pmatrix}$$

and to pass to the limit in ε using the previous convergence results. The algebra is left to the reader. Remark that the resulting matrix $\mathbf{M}^\varepsilon$ is not a square matrix. □

Remark 3.3 (Isotropic laminates)**.** If $\mathbb{L}(y_1)$ is isotropic (see Remark 1.1), then, whenever e_2 is a unit vector perpendicular to the lamination direction, an easy computation based on the result of Lemma 3.2 leads to

$$\mathbb{L}^\ell(e_2 \otimes e_2)\cdot(e_2 \otimes e_2) = \left(\int_{Y_1} \frac{ds}{\lambda(s)+2\mu(s)}\right)^{-1} \left(\int_{Y_1} \frac{\lambda(s)}{\lambda(s)+2\mu(s)}\,ds\right)^2 + 4\left(\int_{Y_1} \frac{\mu(s)(\lambda(s)+\mu(s))}{\lambda(s)+2\mu(s)}\,ds\right).$$

(See also [7] for a more direct derivation.) ¶

The second assumption used in [7] is that the microstructure is a two-phase material and that both phases $\mathbb{L}^1$ and $\mathbb{L}^2$ are isotropic (see Remark 1.1). Specifically, for a characteristic function $\chi \in L^\infty_{\mathrm{per}}(Y_N;\{0,1\})$, we define

$$\mathbb{L}(y_1)M = \lambda(y_1)\,\mathrm{tr}\,(M)\,I_N + 2\mu(y_1)M,\ y_1 \in Y_1,\ M \in \mathbb{R}^{N\times N}_s, \tag{3.2}$$

with

$$
\left\{
\begin{array}{l}
\lambda(y_1) := \chi(y_1)\,\lambda_1 + \big(1-\chi(y_1)\big)\,\lambda_2 \\
\mu(y_1) := \chi(y_1)\,\mu_1 + \big(1-\chi(y_1)\big)\,\mu_2 \\
\int_{Y_1} \chi(t)\,dt = \theta,
\end{array}
\right.
\tag{3.3}
$$

and we assume further that, if $N = 2$,

$$
\left\{
\begin{array}{l}
\mu_2 > 0,\ \lambda_2 + 2\mu_2 > 0,\ \lambda_2 + \mu_2 < 0 \\
\mu_1 = -\left(\lambda_2 + \mu_2\right),\ \lambda_1 + \mu_1 > 0,
\end{array}
\right.
\tag{3.4}
$$

while, if $N = 3$,

$$
\left\{
\begin{array}{l}
\mu_2 > 0,\ \lambda_2 + 2\mu_2 > 0,\ 2\lambda_2 + 3\mu_2 < 0 \\
\mu_1 = -\left(\lambda_2 + \mu_2\right),\ 2\lambda_1 + 3\mu_1 > 0.
\end{array}
\right.
\tag{3.5}
$$

Then, using Lemma 3.2 and Remark 3.3, the following result is obtained in [7, Proposition 1, Appendices A, B] for $N = 2$ or $N = 3$.

Proposition 3.4 (Loss of (strict) strong ellipticity)**.** *Take $\theta = 1/2$ and assume that* (3.2)-(3.5) *hold true. Then, the tensor $\mathbb{L}^\ell$ associated with $\mathbb{L}(y_1)$ through Lemma 3.2 satisfies*

$$
\mathbb{L}^\ell\,(e_2 \otimes e_2)\cdot(e_2 \otimes e_2) = 0,
$$

where e_2 is a unit vector perpendicular to the lamination direction.

We propose below to revisit these results within the framework developed in [6] and expanded upon in Section 2.

As a first step in that direction, we first establish that, provided that $\Lambda_{\rm per}(\mathbb{L}) > 0$, the tensor $\mathbb{L}^\ell$ obtained in Lemma 3.2 is also the tensor $\mathbb{L}^0$ defined through (2.3).

Lemma 3.5 (Identification of the homogenized tensor for a laminate)**.** *When $\mathbb{L}(y) := \mathbb{L}(y_1)$, $y_1 \in Y_1$, with $\Lambda_{\rm per}(\mathbb{L}) > 0$, the homogenized tensor $\mathbb{L}^0$ given by* (2.3) *can also be expressed by the following formula:*

$$
\mathbb{L}^0 M = \int_{Y_1} \mathbb{L}(t)\big(M + v_M'(t)\otimes e_1\big)\,dt, \quad \text{for } M \in \mathbb{R}^{N\times N},
\tag{3.6}
$$

where $v_M \in H^1_{\rm per}(Y_1;\mathbb{R}^N)$ is given by

(3.7)

$$
\left\{
\begin{array}{l}
v_M'(y_1) = A^{-1}[\mathbb{L}](y_1)\Big\{-\left(\mathbb{L}(y_1)\,M\right)e_1 + \\
\qquad\qquad \left(\int_{Y_1} A^{-1}[\mathbb{L}](t)\,dt\right)^{-1} \int_{Y_1} A^{-1}[\mathbb{L}](t)\,\big((\mathbb{L}(t)\,M)e_1\big)\,dt\Big\} \\
\displaystyle\int_{Y_1} v_M(t)\,dt = 0.
\end{array}
\right.
$$

Further, $\mathbb{L}^0 = \mathbb{L}^\ell$ given by Lemma 3.2.

Proof. Since $\Lambda_{\rm per}(\mathbb{L}) > 0$, the bilinear form

$$
\int_{Y_N} \mathbb{L}(y_1)\nabla v\cdot\nabla v\,dy
$$

is coercive on $H^1_{\rm per}(Y_N;\mathbb{R}^N)$, so that, for any $M\in\mathbb{R}^{N\times N}$, there exists a unique solution $w_M\in H^1_{\rm per}(Y_N;\mathbb{R}^N)$ to

$$\left\{\begin{array}{l} -\operatorname{div}\big[\mathbb{L}(y_1)(M+\nabla w_M)\big]=0 \quad \text{in } \mathbb{R}^N \\ \displaystyle\int_{Y_N} w_M\,dy=0. \end{array}\right. \tag{3.8}$$

Because of uniqueness a straightforward verification shows that

$$w_M(y)\equiv v_M(y_1)$$

defined by (3.7). Moreover, it is the unique minimizer in (2.3) with zero mean. Therefore, the tensor given by formula (3.6) is also that given by (2.3).

It remains to establish that it coincides with the tensor $\mathbb{L}^\ell$ given through Lemma 3.2. Setting

$$u^\varepsilon(x):=Mx+\varepsilon\,v'_M(x_1/\varepsilon)$$

we have

$$u^\varepsilon\rightharpoonup Mx \ \text{ weakly in } H^1(\Omega;\mathbb{R}^N),$$

and

$$\sigma^\varepsilon:=\mathbb{L}(x_1/\varepsilon)\nabla u^\varepsilon\rightharpoonup\sigma \ \text{ weakly in } L^2(\Omega;\mathbb{R}^{N\times N}),$$

which further satisfy

$$\operatorname{div}\sigma^\varepsilon=0.$$

Then, on the one hand,

$$\sigma:=\lim_{\text{weak-}L^2}\sigma^\varepsilon=\mathbb{L}^\ell M,$$

with $\mathbb{L}^\ell$ given by Lemma 3.2.

On the other hand, since $\sigma^\varepsilon=\mathbb{L}(x_1/\varepsilon)\big(M+v'_M(x_1/\varepsilon)\big)$ is Y_1-periodic, it converges weakly to its mean in $L^1(\Omega;\mathbb{R}^{N\times N})$ which is precisely (3.6). The arbitrariness of M yields the result. □

Remark 3.6 (Example of a loss of (strict) strong ellipticity *via* homogenization)**.** It is immediate that, under assumptions (3.2)-(3.5), assumptions (2.10), (2.11) are in turn satisfied with $\gamma=\mu_1$, as well as those of Case 1 in Theorem 2.9, so that the laminate construction whose strong ellipticity was shown to degenerate in Proposition 3.4 satisfies

$$\Lambda(\mathbb{L})\geq 0 \quad\text{and}\quad \Lambda_{\rm per}(\mathbb{L})>0.$$

Further, since in such a case Lemma 3.5 actually permits to identify the homogenized tensor as $\mathbb{L}^\ell$, and since the strong ellipticity of $\mathbb{L}^\ell$ is not strict, $\Lambda(\mathbb{L})$ cannot be strictly positive, because it would give rise to a (strictly) strongly elliptic tensor $\mathbb{L}^0=\mathbb{L}^\ell$ (see Remark 2.3). We thus conclude that

$$\Lambda(\mathbb{L})=0 \quad\text{and}\quad \Lambda_{\rm per}(\mathbb{L})>0.$$

The latter had not been remarked. This provides, to our knowledge, the first example of a periodic composite for which *bona fide* homogenization results in a loss of (strict) strong ellipticity. ¶

Remark 3.7. If, in the two-dimensional case, we consider

$$\begin{cases} \mu_2 > 0,\ \lambda_2 + 2\mu_2 > 0,\ \lambda_2 + \mu_2 = 0 \\ \mu_1 > 0,\ \lambda_1 + \mu_1 > 0, \end{cases}$$

in lieu of (3.4), we can consider $\gamma > 0$ such that

$$\gamma < \min\{\mu_1, \mu_2\}.$$

Then Remark 2.10 will apply and establish that $\Lambda(\mathbb{L}) > 0$, although $\mathbb{L}(y)$ cannot satisfy the very strong ellipticity condition (1.7) since $\mathbb{L}(y)I_2 \cdot I_2 = 0$ for a.e. y in $\{\chi = 0\}$. Note however that the energy density $\mathbb{L}(y)M \cdot M$ is pointwise non-negative for every $M \in \mathbb{R}^2 \times \mathbb{R}^2$.

According to [7, Proposition 2], the resulting homogenized tensor $\mathbb{L}^0 = \mathbb{L}^\ell$ is very strongly elliptic. ¶

In the next subsection, we demonstrate that the scenario put forth in [7] is in essence unique within the framework of periodic lamination in (linearized) elasticity if loss of ellipticity is the goal.

3.2. **Loss of strong ellipticity for the homogenized tensor.** The first paragraph addresses the case of a general laminate, while the second paragraph specializes the results to the isotropic setting.

3.2.1. *A general framework.* Let $\mathbb{L} \in L^\infty_{\rm per}\big(Y_1; \mathscr{L}_s(\mathbb{R}^{N\times N}_s)\big)$ be a Y_1-periodic tensor-valued function which is uniformly strongly elliptic in Y_1, namely $\alpha_{\rm se}(\mathbb{L}) > 0$ (see (1.8)).

Define, for a.e. $y_1 \in Y_1$, the y_1-dependent inner product

$$(\xi, \eta) \in \mathbb{R}^2 \mapsto \mathbb{L}(y_1)\,(\xi \otimes e_1)\,.\,(\eta \otimes e_1)\,.$$

It is indeed an inner product because $\alpha_{\rm se}(\mathbb{L}) > 0$. The matrix-valued function defined by

$$(3.9)\quad L(y_1) = \begin{pmatrix} \ell_1(y_1) & \ell_{21}(y_1) \\ \ell_{21}(y_1) & \ell_2(y_1) \end{pmatrix} := \begin{pmatrix} \mathbb{L}(y_1)(e_1 \otimes e_1)\cdot(e_1 \otimes e_1) & \mathbb{L}(y_1)(e_1 \otimes e_1)\cdot(e_2 \otimes e_1) \\ \mathbb{L}(y_1)(e_1 \otimes e_1)\cdot(e_2 \otimes e_1) & \mathbb{L}(y_1)(e_2 \otimes e_1)\cdot(e_2 \otimes e_1) \end{pmatrix}$$

is then symmetric positive definite.

The following result holds true:

Lemma 3.8 (Loss of strong ellipticity for a general laminate)**.** *Assume that, for some constant* γ*,*

$$(3.10)\qquad \mathbb{L}(y_1)A \cdot A + 4\gamma \det A \geq 0, \qquad \text{a.e. in } Y_1,\ \forall\, A \in \mathbb{R}^{2\times 2}.$$

Then, for any rank-one matrix M *in* $\mathbb{R}^{2\times 2}$ *and a.e. in* Y_1*,*

(3.11)

$$\begin{aligned} \mathbb{L}(y_1)M \cdot M \quad &\geq Q(M) := \frac{1}{\ell_1}\left[\mathbb{L}(y_1)M \cdot (e_1 \otimes e_1) + 2\gamma\,\mathrm{cof}M \cdot (e_1 \otimes e_1)\right]^2 \\ + \frac{\ell_1(y_1)}{\det(L(y_1))}&\Big[\mathbb{L}(y_1)M \cdot (e_2 \otimes e_1) + 2\gamma\,\mathrm{cof}M \cdot (e_2 \otimes e_1) - \\ &\frac{\ell_{21}(y_1)}{\ell_1(y_1)}\big(\mathbb{L}(y_1)M \cdot (e_1 \otimes e_1) + 2\gamma\,\mathrm{cof}M \cdot (e_1 \otimes e_1)\big)\Big]^2. \end{aligned}$$

Moreover, the homogenized tensor $\mathbb{L}^0$ is not (strictly) strongly elliptic if, and only if there exists a rank-one matrix M such that

$$\mathbb{L}(y_1)M \cdot M = Q(M), \quad \text{a.e. in } Y_1, \tag{3.12}$$

together with

$$\begin{cases} \displaystyle\int_{Y_1} \frac{\ell_2(t)}{\det(L(t))}\big(\mathbb{L}(t)M \cdot (e_1 \otimes e_1) + 2\gamma \operatorname{cof} M \cdot (e_1 \otimes e_1)\big)\,dt \\ \qquad\qquad \displaystyle = \int_{Y_1} \frac{\ell_{21}(t)}{\det(L(t))}\big(\mathbb{L}(t)M \cdot (e_2 \otimes e_1) + 2\gamma \operatorname{cof} M \cdot (e_2 \otimes e_1)\big)\,dt \\ \displaystyle\int_{Y_1} \frac{\ell_{21}(t)}{\det(L(t))}\big(\mathbb{L}(t)M \cdot (e_1 \otimes e_1) + 2\gamma \operatorname{cof} M \cdot (e_1 \otimes e_1)\big)\,dt \\ \qquad\qquad \displaystyle = \int_{Y_1} \frac{\ell_1(t)}{\det(L(t))}\big(\mathbb{L}(t)M \cdot (e_2 \otimes e_1) + 2\gamma \operatorname{cof} M \cdot (e_2 \otimes e_1)\big)\,dt. \end{cases} \tag{3.13}$$

In (3.13) *above, the matrix M can be substituted with its transpose M^T.*

Proof. Throughout, we will omit the y_1-dependence in (in)equalities that hold a.e. in Y_1.

Let M be a rank-one matrix of $\mathbb{R}^{2\times 2}$, so that, in particular, $\det M = 0$. First, definition (2.3) of the homogenized tensor $\mathbb{L}^0$, the quasi-affinity of the determinant and (3.10) imply that

$$\begin{aligned} &\mathbb{L}^0 M \cdot M = \min_{\varphi \in H^1_{\text{per}}(Y_2;\mathbb{R}^2)} \left\{ \int_{Y_2} \mathbb{L}(M + \nabla\varphi) \cdot (M + \nabla\varphi)\,dy \right\} \\ &= \min_{\varphi \in H^1_{\text{per}}(Y_2;\mathbb{R}^2)} \left\{ \int_{Y_2} \big[\mathbb{L}(M + \nabla\varphi) \cdot (M + \nabla\varphi) + 4\gamma \det(M + \nabla\varphi)\big] dy \right\} \geq 0. \end{aligned} \tag{3.14}$$

Take $\varphi = \varphi(y_1) = (\varphi_1, \varphi_2) \in C^1_{\text{per}}(Y_1;\mathbb{R}^2)$. Then

$$\nabla\varphi = \varphi' \otimes e_1 = \varphi_1'(e_1 \otimes e_1) + \varphi_2'(e_2 \otimes e_1)$$

is a rank-one (or the null) matrix. Because of (3.10) and since

$$\det(A + B) = \det A + \det B + \operatorname{cof} A \cdot B, \quad \forall A, B \in \mathbb{R}^{2\times 2},$$

we have

$$\begin{aligned} &0 \leq \mathbb{L}(M + \nabla\varphi) \cdot (M + \nabla\varphi) + 4\gamma \det(M + \nabla\varphi) \\ &= \mathbb{L}\big(M + \varphi_1'(e_1 \otimes e_1) + \varphi_2'(e_2 \otimes e_1)\big) \cdot \big(M + \varphi_1'(e_1 \otimes e_1) + \varphi_2'(e_2 \otimes e_1)\big) + \\ &\qquad\qquad 4\gamma \operatorname{cof} M \cdot \big(\varphi_1'(e_1 \otimes e_1) + \varphi_2'(e_2 \otimes e_1)\big), \end{aligned}$$

for any Lebesgue point of $\mathbb{L}$ in Y_1. Using (3.9) it follows that

$$\begin{aligned} &0 \leq \ell_1(\varphi_1')^2 + 2\ell_{21}\varphi_1'\varphi_2' + \ell_2(\varphi_2')^2 + 2\big(\mathbb{L}M \cdot (e_1 \otimes e_1) + 2\gamma \operatorname{cof} M \cdot (e_1 \otimes e_1)\big)\varphi_1' \\ &+ 2\big(\mathbb{L}M \cdot (e_2 \otimes e_1) + 2\gamma \operatorname{cof} M \cdot (e_2 \otimes e_1)\big)\varphi_2' + \mathbb{L}M \cdot M. \end{aligned}$$

If attempting to rewrite the expression in the right hand-side of the above inequality as a sum of squares, one is led to

(3.15)

$$
\begin{aligned}
&0 \le \mathbb{L}\,(M+\nabla\varphi)\cdot(M+\nabla\varphi) + 4\gamma\,\det(M+\nabla\varphi)\\
&= \mathbb{L}M\cdot M - Q(M) + \ell_1\left[\varphi_1' + \frac{\ell_{21}}{\ell_1}\,\varphi_2' + \frac{1}{\ell_1}\big(\mathbb{L}M\cdot(e_1\otimes e_1) + 2\gamma\,\mathrm{cof}M\cdot(e_1\otimes e_1)\big)\right]^2\\
&+ \frac{\det(L)}{\ell_1}\left[\varphi_2' - \frac{\ell_{21}}{\det(L)}\big(\mathbb{L}M\cdot(e_1\otimes e_1) + 2\gamma\,\mathrm{cof}M\cdot(e_1\otimes e_1)\big) + \right.\\
&\qquad\qquad\left.\frac{\ell_1}{\det(L)}\big(\mathbb{L}M\cdot(e_2\otimes e_1) + 2\gamma\,\mathrm{cof}M\cdot(e_2\otimes e_1)\big)\right]^2,
\end{aligned}
$$

where $Q(M)$ was defined in (3.11).

Because the derivatives φ_1' and φ_2' can be chosen arbitrarily, the two square brackets in (3.15) can be equated to 0 at any Lebesgue point $y_1 \in Y_1$ of $\mathbb{L}$, which yields (3.11) a.e. in Y_1. Note that, by a density argument, inequality (3.15) also holds a.e. in Y_1 for any $\varphi \in H^1_{\mathrm{per}}(Y_1;\mathbb{R}^2)$.

Now assume that $\mathbb{L}^0$ is not (strictly) strongly elliptic, so that there exists a rank-one matrix M such that $\mathbb{L}^0M\cdot M = 0$. The minimizer in (3.7) satisfies

$$
\begin{aligned}
0 = \mathbb{L}^0M\cdot M &= \int_{Y_1}\mathbb{L}(t)\big(M + v_M'(t)\otimes e_1\big)\cdot\big(M + v_M'(t)\otimes e_1\big)\,dt\\
&= \int_{Y_1}\big[\mathbb{L}(t)\big(M+\nabla v_M(t)\big)\cdot\big(M+\nabla v_M(t)\big) + 4\gamma\,\det\big(M+\nabla v_M(t)\big)\big]\,dt.
\end{aligned}
$$

In view of (3.15), the integrand in the expression above must be pointwise null, which implies equality (3.12) as well as

$$
\left\{
\begin{aligned}
&(v_M')_1 + \frac{\ell_{21}}{\ell_1}(v_M')_2 + \frac{1}{\ell_1}\big(\mathbb{L}M\cdot(e_1\otimes e_1) + 2\gamma\,\mathrm{cof}M\cdot(e_1\otimes e_1)\big) = 0\\
&(v_M')_2 - \frac{\ell_{21}}{\det(L)}\big(\mathbb{L}M\cdot(e_1\otimes e_1) + 2\gamma\,\mathrm{cof}M\cdot(e_1\otimes e_1)\big) +\\
&\qquad\qquad \frac{\ell_1}{\det(L)}\big(\mathbb{L}M\cdot(e_2\otimes e_1) + 2\gamma\,\mathrm{cof}M\cdot(e_2\otimes e_1)\big) = 0.
\end{aligned}
\right. \tag{3.16}
$$

Due to the Y_1 periodicity of v_M, integrating the second equality of (3.16) over Y_1 we get the second equality of (3.13). Finally, replacing $(v_M')_2$ in the first equality of (3.16) and integrating over Y_1 we obtain the first equality of (3.13).

Conversely, assume that equalities (3.12) and (3.13) hold. Successive consideration of the second equation of (3.13) then of the first one yields the existence of two functions φ_2 and φ_1 in $W^{1,\infty}_{\mathrm{per}}(Y_1)$ such that, a.e. in Y_1,

$$
\left\{
\begin{aligned}
&\varphi_2' - \frac{\ell_{21}}{\det(L)}\big(\mathbb{L}M\cdot(e_1\otimes e_1) + 2\gamma\,\mathrm{cof}M\cdot(e_1\otimes e_1)\big) +\\
&\qquad\qquad \frac{\ell_1}{\det(L)}\big(\mathbb{L}M\cdot(e_2\otimes e_1) + 2\gamma\,\mathrm{cof}M\cdot(e_2\otimes e_1)\big) = 0\\
&\varphi_1' + \frac{\ell_{21}}{\ell_1}\,\varphi_2' + \frac{1}{\ell_1}\big(\mathbb{L}M\cdot(e_1\otimes e_1) + 2\gamma\,\mathrm{cof}M\cdot(e_1\otimes e_1)\big) = 0.
\end{aligned}
\right.
$$

This combined with (3.12) yields equality in (3.15), which, by (3.14), implies that

$$
0 = \int_{Y_1}\big[\mathbb{L}\,(M+\nabla\varphi)\cdot(M+\nabla\varphi) + 4\gamma\,\det(M+\nabla\varphi)\big]\,dy_1 \ge \mathbb{L}^0M\cdot M \ge 0.
$$

Therefore, $\mathbb{L}^0$ is not (strictly) strongly elliptic.

Finally, since $\mathbb{L}^0 M \cdot M = \mathbb{L}^0 M^T \cdot M^T$, conditions (3.12) and (3.13) are equivalent to the corresponding equalities obtained upon replacing M by M^T. □

3.2.2. *The isotropic case.* In this paragraph, we specialize the result of the previous paragraph to the isotropic setting (see Remark 1.1).

Condition (3.10) is easily seen to become

$$-\lambda(y_1) - \mu(y_1) \le \gamma \le \mu(y_1), \quad \text{for a.e. } y_1 \in Y_1. \tag{3.17}$$

Then,

Theorem 3.9 (Loss of strong ellipticity in the isotropic case)**.** *Assume that condition* (3.17) *holds true. The resulting homogenized tensor* $\mathbb{L}^0$ *is not (strictly) strongly elliptic if, and only if the following conditions are satisfied*

$$\gamma > 0, \quad \mathcal{L}^1(\{\mu = \gamma\}) = 1/2, \quad \textit{and} \quad \lambda + \mu + \gamma = 0 \ \textit{ a.e. in } \{\mu \ne \gamma\}. \tag{3.18}$$

Further, in such a case,

$$\left\{M \in \mathbb{R}^{2\times 2} : \det M = 0 \textit{ and } \mathbb{L}^0 M \cdot M = 0\right\} = \mathbb{R}\,(e_2 \otimes e_2)\,. \tag{3.19}$$

Remark 3.10 (Canonical character of Gutiérrez's laminate)**.** In particular, Theorem 3.9 shows that the example in [7] of loss of (strict) strong ellipticity for $\mathbb{L}^0$ is actually unique in the class of two-phase laminates.

Furthermore, (3.19) asserts that, even in the more general setting of (3.18), the matrix $e_2 \otimes e_2$ remains the sole rank-one direction in which $\mathbb{L}^0$ loses strong ellipticity. Also note that the last condition in (3.18) combined with (1.9) implies that the functions λ, μ cannot both be continuous, which *a posteriori* justifies the use of a two-phase laminate in the example of [7]. See also our previous remarks in the introduction. ¶

Proof. Assume that $\mathbb{L}^0$ is not (strictly) strongly elliptic. Then, consider a rank-one matrix $M := \xi \otimes \eta$ such that $\mathbb{L}^0 M \cdot M = 0$.

When $\mathbb{L}$ is isotropic the matrix L of (3.9) is

$$L = \begin{pmatrix} \lambda + 2\mu & 0 \\ 0 & \mu \end{pmatrix}.$$

Also,

$$\begin{gathered} \operatorname{cof} M \cdot (e_1 \otimes e_1) = \xi_2 \eta_2, \quad \operatorname{cof} M \cdot (e_2 \otimes e_1) = -\xi_1 \eta_2, \\ \mathbb{L} M \cdot (e_1 \otimes e_1) = (\lambda + 2\mu)\, \xi_1 \eta_1 + \lambda\, \xi_2 \eta_2, \quad \mathbb{L} M \cdot (e_2 \otimes e_1) = \mu\, (\xi_1 \eta_2 + \xi_2 \eta_1)\,, \\ \mathbb{L} M \cdot M = (\lambda + \mu)\, (\xi \cdot \eta)^2 + \mu\, |\xi|^2 |\eta|^2. \end{gathered}$$

By Lemma 3.8 the equality $\mathbb{L}^0 M \cdot M = 0$ is equivalent to conditions (3.12) and (3.13) which, in the light of the previous equalities, read as

$$\begin{aligned} &\frac{1}{\lambda + 2\mu} \left[\, (\lambda + 2\mu)\, \xi_1 \eta_1 + (\lambda + 2\gamma)\, \xi_2 \eta_2 \right]^2 + \frac{1}{\mu} \left[\mu\, (\xi_1 \eta_2 + \xi_2 \eta_1) - 2\gamma\, \xi_1 \eta_2 \right]^2 \\ &= (\lambda + \mu)\, (\xi \cdot \eta)^2 + \mu\, |\xi|^2 |\mu|^2, \quad \text{a.e. in } Y_1, \end{aligned} \tag{3.20}$$

together with

$$\xi_1 \eta_1 + \xi_2 \eta_2 \int_{Y_1} \frac{\lambda + 2\gamma}{\lambda + 2\mu}(t)\, dt = \xi_1 \eta_2 + \xi_2 \eta_1 - 2\, \xi_1 \eta_2 \int_{Y_1} \frac{\gamma}{\mu}(t)\, dt = 0. \tag{3.21}$$

Equation (3.20) actually reduces to

$$\eta_2^2\left[\frac{4\gamma\,\mu-4\gamma^2}{\mu}\,\xi_1^2+\frac{(\lambda+2\mu)^2-(\lambda+2\gamma)^2}{\lambda+2\mu}\,\xi_2^2\right]=0,\quad \text{a.e. in } Y_1,$$

which is equivalent to

$$\eta_2=0\quad\text{or}\quad(\mu-\gamma)\left(\frac{\gamma}{\mu}\,\xi_1^2+\frac{\lambda+\mu+\gamma}{\lambda+2\mu}\,\xi_2^2\right)=0,\ \text{ a.e. in } Y_1. \tag{3.22}$$

According to the last part of Lemma 3.8 conditions (3.21) and (3.22) are also equivalent to the corresponding equalities obtained upon permutation of ξ and η.

From (3.21) and $\xi,\eta\neq 0$, we easily deduce that $\eta_2\neq 0$.

If $\gamma<0$, then by (3.17) $\lambda+\mu\geq-\gamma>0$ a.e. in Y_1, which implies that $\mathbb{L}$ is uniformly very strongly elliptic and thus that $\mathbb{L}^0$ is (strictly) strongly elliptic. Therefore, since $\mathbb{L}^0$ is assumed to lose (strict) strong ellipticity, $\gamma\geq 0$. Moreover, if $\gamma=0$, then by (3.22) combined with $\mu>0$, we obtain that $\lambda+\mu=0$ a.e. in Y_1, or $\xi_2=0$. Inserting this into (3.21) easily yields a contradiction if $\xi_2=0$. Thus $\lambda+\mu=0$ a.e. in Y_1. But then

$$\xi_1\eta_1-\xi_2\eta_2=\xi_1\eta_2+\xi_2\eta_1=0,$$

which contradicts the fact that $\xi,\eta\neq 0$. Therefore, we can assume that $\gamma>0$.

Next, if $\mu=\gamma$ a.e. in Y_1, then (3.21) gives

$$\xi_1\eta_1+\xi_2\eta_2=\xi_2\eta_1-\xi_1\eta_2=0,$$

again in contradiction with $\xi,\eta\neq 0$. Hence, the set $\{\mu\neq\gamma\}$ has a positive Lebesgue measure and (3.22), together with $\gamma>0$ and (3.17), implies that

$$\gamma\,\xi_1^2=(\lambda+\mu+\gamma)\,\xi_2^2=0,\quad \text{a.e. in } \{\mu\neq\gamma\}.$$

Hence, $\xi_1=0$, $\xi_2\neq 0$, and the third condition of (3.18). Putting $\xi_1=0$ in the second inequality of (3.21) we also have $\eta_1=0$, $\eta_2\neq 0$. Using (3.21) once again, we obtain that

$$0=\int_{Y_1}\frac{\lambda+2\gamma}{\lambda+2\mu}(t)\,dt=\int_{\{\mu=\gamma\}}\frac{\lambda+2\gamma}{\lambda+2\mu}(t)\,dt+\int_{\{\mu\neq\gamma\}}\frac{\lambda+2\gamma}{\lambda+2\mu}(t)\,dt=\mathcal{L}^1(\{\mu=\gamma\}-\mathcal{L}^1(\{\mu\neq\gamma\}),$$

which yields the second condition of (3.18). Finally, the equalities $\xi_1=\eta_1=0$ imply (3.19).

Therefore, the lack of (strict) strong ellipticity of $\mathbb{L}^0$, or equivalently, conditions (3.21) and (3.22) satisfied by some rank-one matrix $\xi\otimes\eta$, imply (3.18) and (3.19).

The converse is obvious, which completes the proof. $\square$

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References

[1] Wojciech Barański. On infinitesimal stability and homogenization of linearly elastic periodic composites. *Math. Models Methods Appl. Sci.*, 10(8):1251–1262, 2000.

[2] Andrea Braides and Anneliese Defranceschi. *Homogenization of multiple integrals*, volume 12 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1998.

[3] A. Cherkaev and R.V. Kohn, editors. *Topics in the mathematical modelling of composite materials*. Birkhäuser Boston Inc., Boston, MA, 1997.

[4] Gaetano Fichera. Il teorema del massimo modulo per l'equazione dell'elastostatica tridimensionale. *Arch. Rational Mech. Anal.*, 7:373–387, 1961.

[5] Gilles A. Francfort. Homogenisation of a class of fourth order equations with application to incompressible elasticity. *Proc. Roy. Soc. Edinburgh Sect. A*, 120(1-2):25–46, 1992.

[6] G. Geymonat, S. Müller, and N. Triantafyllidis. Homogenization of non-linearly elastic materials, microscopic bifurcation and macroscopic loss of rank-one convexity. *Arch. Rational Mech. Anal.*, 122(3):231–290, 1993.

[7] Sergio Gutiérrez. Laminations in linearized elasticity: the isotropic non-very strongly elliptic case. *J. Elasticity*, 53(3):215–256, 1998/99.

[8] Sergio Gutiérrez. Laminations in planar anisotropic linear elasticity. *Quart. J. Mech. Appl. Math.*, 57(4):571–582, 2004.

[9] F. Murat and L. Tartar. H-convergence. In A. Cherkaev and R. V. Kohn, editors, *Topics in the mathematical modelling of composite materials*, chapter 3, pages 21–44. Birkhäuser, Boston, 1997.

[10] François Murat. Compacité par compensation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 5(3):489–507, 1978.

[11] J. Nečas and M. Štípl. A paradox in the theory of linear elasticity. *Apl. Mat.*, 21(6):431–433, 1976.

[12] Jindřich Nečas. Sur les normes équivalentes dans $W^k(\Omega)$ et sur la coercivité des formes formellement positives. *Séminaire de mathématiques supérieures, Équations aux derivées partielles*, 19:101–108, 1966.

[13] O. A. Oleĭnik, A. S. Shamaev, and G. A. Yosifian. *Mathematical problems in elasticity and homogenization*, volume 26 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 1992.

[14] Enrique Sánchez-Palencia. *Nonhomogeneous Media and Vibration Theory*, volume 127 of *Lecture Notes in Physics*. Springer-Verlag, Berlin-New York, 1980.

[15] Luc Tartar. Homogenization and hyperbolicity. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 25(3-4):785–805 (1998), 1997. Dedicated to Ennio De Giorgi.

[16] Frank Uhlig. A recurring theorem about pairs of quadratic forms and extensions: a survey. *Linear Algebra Appl.*, 25:219–237, 1979.

[17] Kewei W. Zhang. A counterexample in the theory of coerciveness for elliptic systems. *J. Partial Differential Equations*, 2(3):79–82, 1989.

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