Homogenization and Mechanical Dissipation in Thermoviscoelasticity

GILLES A. FRANCFORT & PIERRE M. SUQUET

Communicated by C. M. DAFERMOS

Introduction

When dealing with a composite material, *i.e.* a body with a great number of heterogeneities organized in a structured fashion, one is confronted with the problem of determining the macroscopic behavior of this body from the knowledge of its microstructure. The theory of homogenization considered in [1], [11] and in this paper focuses on an idealized class of composite materials with a periodic structure which makes it possible to determine exactly the macroscopic properties of the material under consideration.

In this paper we consider a specific class of such composites whose *macroscopic mechanical and thermal behavior* we investigate with the help of homogenization theory. The material is composed of homogeneous viscoelastic materials of Kelvin-Voigt type (see Section 1).

The study demonstrates that the classical theories of viscoelasticity with short range memory (KELVIN, MAXWELL, *etc.*) can be related to the theories of viscoelasticity with fading memory (see the articles by COLEMAN & NOLL in the volume compiled in C. TRUESDELL [16], for example), through homogenization theory. This indicates that materials with fading memory may arise from the composition of "crystals" of viscoelastic materials of Kelvin-Voigt type.

In Section 1 we outline the general theory of viscoelastic material of Kelvin-Voigt type and then perform the simplifications that will enable us to carry out the subsequent analysis.

In Section 2 we introduce, as is standard in homogenization theory, a reference cell which contains all the relevant information on the microstructure. We then describe the composite material as the composition of scaled versions of the cell by a small scaling parameter. We establish the existence and uniqueness of the solution in displacement and temperature of an initial boundary value problem for a bounded domain of $\mathbb{R}^3$ made of this composite material.

In Section 3 we examine the behavior of the *displacement field* found in Section 2 when the scaling parameter becomes small, *i.e.* when the inhomogeneities become dense in the domain. The displacement field is found to converge (weakly) to the displacement of a body, with the same configuration, made of a homogeneous material which is *no longer of Kelvin-Voigt type* but rather a material with fading memory. A result of this nature was first obtained by SANCHEZ-PALENCIA ([11], Ch. 6). Our analysis, however, gives more explicit results and enables us to treat the problem in the context of thermodynamics.

In Section 4 we prove a theorem of strong convergence for the strain rate field, without any assumption on the regularity of the displacement field of the homogenized problem (Theorem 4.1). For that purpose we use the method of TARTAR [15] (see also SUQUET [13]). This theorem enables us to homogenize the mechanical dissipation and so ultimately to homogenize the energy equation (Section 5). The expression of the homogenized mechanical dissipation *could not be obtained from a simple inspection of the balance equation for the homogenized material.*

Throughout the paper dx will stand for $dx_1\, dx_2\, dx_3$ and an overdot $\cdot$ will denote time differentiation. Einstein's summation convention will be used.

The symbols σ, e, T, $\tau = T - T_0$ will denote respectively the stress and strain tensors, the temperature field and the temperature increment field with respect to a uniform reference temperature T_0.

1. The Theory of Linear Thermoviscoelasticity

For the material under consideration the stress tensor at each point can be split into two parts,

$$\sigma_{ij} = \sigma^r_{ij} + \sigma^{ir}_{ij}, \tag{1.1}$$

given by constitutive relations

$$\sigma^r = \varrho \frac{\partial w^e}{\partial e}(e, \tau), \tag{1.2}$$

$$\sigma^{ir} = \frac{\partial D}{\partial \dot{e}}(\dot{e}), \tag{1.3}$$

where w^e stands for the free energy and D is the dissipation potential of the material given by

$$\begin{aligned} \varrho w_e &= \tfrac{1}{2}\, a_{ijkh}(e_{kh} - \alpha_{kh}\tau)(e_{ij} - \alpha_{ij}\tau) - \tfrac{1}{2}(\beta + a_{ijkh}\alpha_{ij}\alpha_{kh})\,\tau^2, \\ D(\dot{e}) &= \tfrac{1}{2}\, b_{ijkh}\dot{e}_{kh}\dot{e}_{ij}, \end{aligned} \tag{1.4}$$

where (a_{ijkh}), (b_{ijkh}), (α_{ij}) respectively denote the elasticity tensor, the viscosity tensor and the thermal expansion tensor and

$$\begin{aligned} a_{ijkh} &= a_{khij} = a_{jikh}, \\ b_{ijkh} &= b_{khij} = b_{jikh}. \end{aligned} \tag{1.5}$$

Therefore,

(1.6)
$$\sigma^r_{ij} = \varrho \frac{\partial w_e}{\partial e_{ij}} = a_{ijkh}(e_{kh} - \alpha_{kh}\tau),$$
$$\sigma^{ir}_{ij} = \frac{\partial D}{\partial \dot{e}_{ij}} = b_{ijkh}\dot{e}_{kh},$$

(1.7)
$$\sigma_{ij} = a_{ijkh}(e_{kh} - \alpha_{kh}\tau) + b_{ijkh}\dot{e}_{kh}.$$

As is well known, the mechanical dissipation is given by

(1.8)
$$d_1 = \sigma^{ir}_{ij}\dot{e}_{ij} = b_{ijkh}\dot{e}_{kh}\dot{e}_{ij}.$$

The stress, strain and displacement fields must satisfy the equations of motion and the compatibility conditions

(1.9)
$$\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = \varrho \ddot{u}_i,$$

(1.10)
$$e_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

The following expression for the entropy of the material is derived from the state laws

(1.11)
$$\varrho s = -\varrho \frac{\partial w^e}{\partial \tau} = \beta\tau + a_{ijkh}\alpha_{kh}e_{ij}.$$

In the absence of heat sources, the second law of thermodynamics reduces to

(1.12)
$$\varrho \dot{s} + \operatorname{div}\left(\frac{q}{T}\right) = \frac{d}{T} \geqq 0,$$

where d is the total dissipation and q is the thermal flux. The total dissipation is the sum of the mechanical dissipation d_1 and the thermal dissipation d_2 given by

(1.13)
$$d_2 = -\frac{q \cdot \operatorname{grad} T}{T}.$$

A simple computation leads to

(1.14)
$$\varrho T\dot{s} + \operatorname{div} q = d_1.$$

We adopt Fourier's law for the thermal diffusion,

(1.15)
$$q_i = -k_{ij}\frac{\partial T}{\partial x_j} = -k_{ij}\frac{\partial \tau}{\partial x_j},$$

since T_0 is assumed to be uniform. We assume that the conductivity tensor is symmetric

(1.16)
$$k_{ij} = k_{ji}.$$

With the help of (1.8), (1.11), equation (1.14) becomes

(1.17)
$$(T_0 + \tau)(\beta\dot{\tau} + a_{ijkh}\alpha_{kh}\dot{e}_{ij}) = \frac{\partial}{\partial x_i}\left(k_{ij}\frac{\partial \tau}{\partial x_j}\right) + b_{ijkh}\dot{e}_{ij}\dot{e}_{kh}.$$

The set of equations (1.7), (1.9), (1.10), (1.17) governs the *thermomechanical* processes of our body.

Here we will examine a simplified version of the above problem. We will assume that:

i) The thermal expansion tensor (α_{ij}) vanishes identically; the deformation induced by thermal dilatation is thus neglected as are temperature changes induced by deformation. This assumption is justified to the extent that the thermal dilatation properties of most commonly encountered materials ($\simeq 10^{-6}$ °C^{-1}) and the temperature increments τ under consideration ($\tau \simeq 20$ °C, order of magnitude of temperature changes for a road asphalt) induce strain changes that are very small compared to the total strain changes. This assumption uncouples the equations of motion from the energy equation and is necessary in order to carry out our analysis.

A study of the coupled system in the absence of viscous dissipation can be found in FRANCFORT [6].

ii) The term $\tau\dot{\tau}$ is neglected in equation (1.17) because the analysis is restricted to temperature increments τ that remain small compared to T_0. However the mechanical dissipation, which is of order 2 in $\dot{e}$, is not neglected in equation (1.17). The viscosity of classical fluids is generally small ($\simeq 10^{-3}$ kg m^{-1} s^{-1}) and the mechanical dissipation can then be neglected. On the contrary, the viscosity of the class of solid materials under consideration in this study is quite large ($\geqq 10^4$ or 10^5 kg m^{-1} s^{-1} for a typical asphalt), so that the resulting mechanical dissipation can modify the temperature field in a noticeable manner. This phenomenon is used to detect strained areas in dissipative structures. In these areas the large dissipation induces temperature changes which are visualized and measured by a thermographic device ([10]).

The main focus of our study will be to monitor the temperature changes induced by strain rates.

iii) The analysis is restricted to the quasistatic case: the inertia term $(\varrho\ddot{u}_i)$ drops out of the equation (1.9).

This last restriction can be relaxed at the expense of lengthening the presentation. The same methods apply and similar results are derived.

We record below the set of equations that govern the evolution of our material:

(1.18) $$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad (u \text{ displacement field}),$$

(1.19) $$\sigma_{ij} = a_{ijkh}e_{kh} + b_{ijkh}\dot{e}_{kh},$$

(1.20) $$\frac{\partial}{\partial x_j}\sigma_{ij} + f_i = 0,$$

(1.21) $$T_0\beta\dot{\tau} = \frac{\partial}{\partial x_i}\left(k_{ij}\frac{\partial \tau}{\partial x_j}\right) + b_{ijkh}\dot{e}_{kh}\dot{e}_{ij},$$

together with appropriate initial and boundary conditions.

2. The Composite Material

The inhomogeneous solid under consideration is composed of materials of the type described in Section 1. It occupies a bounded Lipschitz domain Ω of $\mathbb{R}^3$. We assume that this solid has a *periodic structure*, i.e. it is made of identical scaled versions of a reference cell $Y = \prod_{i=1}^{3} (0, Y_i)$. We call ε the scaling parameter.

The $a_{ijkh}(y)$'s, $b_{ijkh}(y)$'s and $k_{ij}(y)$'s are assumed to be in $L_\infty(Y)$. They exhibit all the symmetry properties described in Section 1 and are coercive: there is an $\alpha > 0$ such that for all ξ_i's in $\mathbb{R}^3$, and all symmetric ξ_{ij}'s in $\mathbb{R}^9$ we have

$$
\begin{aligned}
& a_{ijkh}(y)\, \xi_{ij}\xi_{kh} \geqq \alpha \xi_{ij}\xi_{ij} \quad \text{a.e. on } Y,\\
(2.1)\qquad & b_{ijkh}(y)\, \xi_{ij}\xi_{kh} \geqq \alpha \xi_{ij}\xi_{ij} \quad \text{a.e. on } Y,\\
& k_{ij}(y)\, \xi_i\xi_j \geqq \alpha \xi_i\xi_i \quad \text{a.e. on } Y.
\end{aligned}
$$

The coefficient $\beta(y)$ is positive and bounded away from 0. Finally we take α such that α^{-1} is a common upper bound to all the L_∞-norms of the coefficients.

We extend all the coefficients to $\mathbb{R}^3$ by εY periodicity and restrict them to Ω. We set

$$(2.2)\quad a^\varepsilon_{ijkh} = a_{ijkh}\left(\frac{x}{\varepsilon}\right), \quad b^\varepsilon_{ijkh} = b_{ijkh}\left(\frac{x}{\varepsilon}\right), \quad \beta^\varepsilon(x) = \beta\left(\frac{x}{\varepsilon}\right), \quad k^\varepsilon_{ij} = k_{ij}\left(\frac{x}{\varepsilon}\right).$$

Our system of equations becomes

$$(2.3)\qquad e^\varepsilon_{ij} = e_{ij}(u^\varepsilon) = \tfrac{1}{2}\left(\frac{\partial u^\varepsilon_i}{\partial x_j} + \frac{\partial u^\varepsilon_j}{\partial x_i}\right),$$

$$(2.4)\qquad \sigma^\varepsilon_{ij} = a^\varepsilon_{ijkh} e^\varepsilon_{kh} + b^\varepsilon_{ijkh}\dot{e}^\varepsilon_{kh},$$

$$(2.5)\qquad \frac{\partial}{\partial x_j}\sigma^\varepsilon_{ij} + f_i = 0,$$

$$(2.6)\qquad T_0\beta^\varepsilon\dot{\tau}^\varepsilon = \frac{\partial}{\partial x_i}\left(k^\varepsilon_{ij}\frac{\partial \tau^\varepsilon}{\partial x_j}\right) + b^\varepsilon_{ijkh}\dot{e}^\varepsilon_{kh}\dot{e}^\varepsilon_{ij},$$

u^ε_i and τ^ε being the displacement and the temperature increment fields for a given ε, *i.e.* for a given scaling.

For simplicity we limit ourselves to Dirichlet type boundary conditions, *i.e.*,

$$(2.7)\qquad u^\varepsilon_i = 0 \quad \text{on } \partial\Omega,$$

$$(2.8)\qquad \tau^\varepsilon = 0 \quad \text{on } \partial\Omega.$$

We impose as initial conditions

$$(2.9)\qquad u^\varepsilon_i(x, 0) = u^\varepsilon_{0i}(x),$$

$$(2.10)\qquad \tau^\varepsilon(x, 0) = \tau^\varepsilon_0(x).$$

We will show that for any fixed ε, the solution to (2.3)–(2.10) exists and is unique. More precisely, we define $\mathbb{H}$ as

$$\mathbb{H} = (H_0^1(\Omega))^3, \tag{2.11}$$

and we denote by $\| \|_{\mathbb{H}}$ the natural $\mathbb{H}$-norm.

The following theorem holds.

Theorem 2.1. *Assume that*

$$f \in L_2(0, T; L_2(\Omega)^3), \quad u_0^\varepsilon \in \mathbb{H}, \quad \tau_0^\varepsilon \in L_1(\Omega).$$

Then the system (2.3)–(2.10) *has a unique generalized solution* $(u^\varepsilon, \tau^\varepsilon)$ *in* $W^{1,2}(0, T; \mathbb{H}) \times C^0([0, T]; L_1(\Omega))$.

Remark 2.1. More specifically,

$$u^\varepsilon(t) = S^\varepsilon(t)\, u_0^\varepsilon + \int_0^t S^\varepsilon(t-\sigma)\, F^\varepsilon(\sigma)\, d\sigma, \tag{2.12}$$

$$\tau^\varepsilon(t) = \Sigma^\varepsilon(t)\, \tau_0^\varepsilon + \int_0^t \Sigma^\varepsilon(t-\sigma) \frac{1}{T_0 \beta^\varepsilon} d_1^\varepsilon(\sigma)\, d\sigma, \tag{2.13}$$

where $F^\varepsilon(\sigma)$ will be defined in (2.20), $d_1^\varepsilon = b_{ijkh}^\varepsilon \dot{e}_{kh}^\varepsilon \dot{e}_{ij}^\varepsilon$ and $S^\varepsilon(t)$ and $\Sigma^\varepsilon(t)$ are strongly continuous contraction semigroups on $\mathbb{H}$ and $L_1(\Omega)$, respectively satisfying

$$\begin{aligned} \|S^\varepsilon(t)\| &\leqq K, \\ \|\Sigma^\varepsilon(t)\| &\leqq K', \end{aligned} \tag{2.14}$$

where $\| \|$ stands for the appropriate operator norm and K and K' are independent of ε.

Remark 2.2. If there is no body force the solution $u^\varepsilon(t)$ given by (2.12) exists in all of $\mathbb{R}_+$; it belongs to $W^{p,\infty}(\mathbb{R}_+, \mathbb{H})$ for all p in $\mathbb{Z}_+$ and

$$u^\varepsilon(t) = S^\varepsilon(t)\, u_0^\varepsilon. \tag{2.15}$$

Proof of Theorem 2.1. This proof is attained in two steps: first we establish existence and uniqueness of a displacement field satisfying (2.3), (2.4), (2.5), (2.7), (2.9) (the temperature field does not enter any of these equations); then we prove the existence and uniqueness of a temperature field satisfying (2.6), (2.8), (2.10).

We consider the following inner product on $\mathbb{H}$:

$$(u, v)_{b^\varepsilon} = \int_\Omega b_{ijkh}^\varepsilon e_{kh}(u)\, e_{ij}(v)\, dx. \tag{2.16}$$

In view of the properties of the coefficients and by application of Korn's inequality (DUVAUT & LIONS [4], Ch. 3), this inner product induces a norm $\| \|_{b^\varepsilon}$ which is equivalent to the natural $\mathbb{H}$-norm. We have

$$C \|u\|_{\mathbb{H}}^2 \leqq \|u\|_{b^\varepsilon}^2 \leqq \alpha^{-1} \|u\|_{\mathbb{H}}^2, \tag{2.17}$$

where C is a constant independent of ε.

On $\mathbb{H}$ we define the operator A^ε through the Riesz representation theorem by

(2.18) $$(A^\varepsilon u, v)_{b^\varepsilon} = - \int_\Omega a^\varepsilon_{ijkh} e_{kh}(u)\, e_{ij}(v)\, dx.$$

A^ε is bounded, self-adjoint and negative. The Lumer-Phillips theorem (YOSIDA [17], Ch. 9) then implies that

(2.19) A^ε generates a strongly continuous semigroup of contractions $S^\varepsilon(t)$.

Finally we define $F^\varepsilon(t)$ in $\mathbb{H}$, for almost every t, by

(2.20) $$(F^\varepsilon(t), v)_{b^\varepsilon} = \int_\Omega f_i(t)\, v_i\, dx \quad \text{for any } v \text{ in } \mathbb{H}.$$

Since $f \in L_2(0, T; L_2(\Omega)^3)$, $F^\varepsilon(t) \in L_2(0, T; \mathbb{H})$. Equations (2.3), (2.4), (2.5), (2.7), (2.9) can then be rewritten as

(2.21) $$\begin{aligned} \dot{u}^\varepsilon &= A^\varepsilon u^\varepsilon + F^\varepsilon \quad \text{in } \mathbb{H}, \\ u^\varepsilon(0) &= u_0^\varepsilon. \end{aligned}$$

In view of (2.19), (2.21) has a unique solution u^ε in $C^0([0, T]; \mathbb{H})$ given by

(2.22) $$u^\varepsilon(t) = S^\varepsilon(t)\, u_0^\varepsilon + \int_0^t S^\varepsilon(t - \sigma)\, F^\varepsilon(\sigma)\, d\sigma, \quad 0 \leqq t \leqq T.$$

Since A^ε is bounded, $u^\varepsilon(t)$ has a time derivative in $\mathbb{H}$ which satisfies

(2.23) $$\dot{u}^\varepsilon(t) = S^\varepsilon(t)\, A^\varepsilon u_0^\varepsilon + \int_0^t S^\varepsilon(t - \sigma)\, A^\varepsilon F^\varepsilon(\sigma)\, d\sigma + F^\varepsilon(t).$$

Thus $\dot{u}^\varepsilon \in L_2(0, T; \mathbb{H})$ and

(2.24) $$u^\varepsilon \in W^{1,2}(0, T; \mathbb{H}).$$

This completes the first step in the proof.

In the energy equation which is now to be solved, the source term is the mechanical dissipation d_1^ε given by

$$d_1^\varepsilon = b^\varepsilon_{ijkh} e_{kh}(\dot{u}^\varepsilon)\, e_{ij}(\dot{u}^\varepsilon).$$

Since the strain rate tensor $e(\dot{u}^\varepsilon)$ is in L_2, it follows that d_1^ε lies in $L_1(0, T; L_1(\Omega))$. We thus have to solve a parabolic equation in an L_1 setting.

On $L_1(\Omega)$, we define $\| \ \|_{1,\varepsilon}$ by

(2.25) $$\|\tau\|_{1,\varepsilon} = \int_\Omega T_0 \beta^\varepsilon\, |\tau|\, dx, \quad \text{for any } \tau \text{ in } L_1(\Omega).$$

In view of the properties of the coefficient $\beta(y)$ and of the strict positivity of the reference temperature T_0, the norm $\| \ \|_{1,\varepsilon}$ is equivalent to the natural L_1-norm and

(2.26) $$C' \|\tau\|_{L_1(\Omega)} \leqq \|\tau\|_{1,\varepsilon} \leqq C \|\tau\|_{L_1(\Omega)},$$

where C and C' are independent of ε.

Next we need a lemma whose proof closely follows that of a related result in BRÉZIS & STRAUSS [2]:

Lemma 2.1. *Define on* $L_1(\Omega)$ *the operator* $B^\varepsilon = \frac{1}{T_0\beta^\varepsilon}\frac{\partial}{\partial x_i}\left(k_{ij}^\varepsilon\frac{\partial}{\partial x_j}\right)$ *with domain*

(2.27) $D(B^\varepsilon) = \{\tau \in L_1(\Omega)$ *such that* $B^\varepsilon\tau \in L_1(\Omega)$ *in the following sense: for any*

$$\omega \text{ in } H_0^1(\Omega) \cap L_\infty(\Omega) \text{ with } B^\varepsilon\omega \text{ in } L_\infty(\Omega)$$

$$\int_\Omega T_0\beta^\varepsilon(B^\varepsilon\tau)\,\omega\,dx = \int_\Omega T_0\beta^\varepsilon\tau(B^\varepsilon\omega)\,dx\}. \tag{2.28}$$

Then B^ε *generates on* $L_1(\Omega)$ *a strongly continuous semigroup of contractions in the norm* $\|\ \|_{1,\varepsilon}$.

Proof of Lemma 2.1. Let us first show that the operator B_2^ε defined on $L_2(\Omega)$ by

$$B_2^\varepsilon = \frac{1}{T_0\beta^\varepsilon}\frac{\partial}{\partial x_i}\left(k_{ij}^\varepsilon\frac{\partial}{\partial x_j}\right), \tag{2.29}$$

and with domain

$$D(B_2^\varepsilon) = \{\tau \in H_0^1(\Omega) \mid B_2^\varepsilon\tau \text{ is, as a distribution, in } L_2(\Omega)\} \tag{2.30}$$

generates in $L_2(\Omega)$ a strongly continuous semigroup of contractions in the equivalent L_2-norm

$$\|\tau\|_{2,\varepsilon} = \left(\int_\Omega T_0\beta^\varepsilon|\tau|^2\,dx\right)^{1/2}. \tag{2.31}$$

The domain $D(B_2^\varepsilon)$ is dense in $L_2(\Omega)$, although $C_0^\infty(\Omega)$ functions do not belong to it. It can be easily checked that B_2^ε is closed, that the range of $I - B_2^\varepsilon$ is $L_2(\Omega)$ and that B_2^ε is dissipative. The result then follows from application of the Lumer-Phillips theorem (YOSIDA [17], Ch. 9).

We define $\bar{B}_2^\varepsilon$ to be the closure of B_2 in $L_1(\Omega)$. It can be seen to be a linear one-to-one operator under the sole assumption that $\beta_{ij}(y) \in L_\infty(Y)$. We want to prove that $\bar{B}_2^\varepsilon$ generates in $L_1(\Omega)$ a contraction semigroup.

Let λ be an element of $\mathbb{R}_+^*$ and g be an element of $L_2(\Omega)$; we set

$$\tau = (\lambda I - B_2^\varepsilon)^{-1}\,g. \tag{2.32}$$

Then τ belongs to $D(B_2^\varepsilon)$, hence to $H_0^1(\Omega)$, and for any ω in $H_0^1(\Omega)$

$$\lambda(\tau, \omega)_{2,\varepsilon} + \int_\Omega k_{ij}^\varepsilon\frac{\partial\tau}{\partial x_j}\frac{\partial\omega}{\partial x_i}\,dx = (g, \omega)_{2,\varepsilon}, \tag{2.33}$$

where $(\ ,\ )_{2,\varepsilon}$ denotes the inner product associated with $\|\ \|_{2,\varepsilon}$. We seek a test function that will permit us to obtain L_1-estimates on τ. We define φ_η by

$$\begin{aligned} \varphi_\eta(w) &= \operatorname{sgn}(w) && \text{if } |w| \geqq \eta, \\ \varphi_\eta(w) &= w/\eta && \text{if } |w| < \eta. \end{aligned} \tag{2.34}$$

Since φ_η is Lipschitz-continuous, $\varphi_\eta(t)$ is an element of $H_0^1(\Omega)$ and $\frac{\partial}{\partial x_i}\varphi_\eta(\tau) = \varphi'_\eta(\tau)\frac{\partial \tau}{\partial x_i}$ (STAMPACCHIA [12]). We take $\omega = \varphi_\eta(\tau)$ and we obtain

$$\lambda(\tau, \varphi_\eta(\tau))_{2,\varepsilon} + \int_\Omega k_{ij}^\varepsilon \frac{\partial \tau}{\partial x_j}\frac{\partial \tau}{\partial x_i}\varphi'_\eta(\tau)\,dx = (g, \varphi_\eta(\tau))_{2,\varepsilon}. \tag{2.35}$$

Because $\varphi'_\eta(\tau)$ is positive, and k_{ij}^ε is a positive matrix,

$$\lambda(\tau, \varphi_\eta(\tau))_{2,\varepsilon} \leqq \|\varphi_\eta\|_{L_\infty(\mathrm{R})}\|g\|_{1,\varepsilon} \leqq \|g\|_{1,\varepsilon}. \tag{2.36}$$

As η goes to zero, $\varphi_\eta(\tau)$ converges in $L_2(\Omega)$ to $\operatorname{sgn}\tau$ and so

$$\lambda\,\|\tau\|_{1,\varepsilon} \leqq \|g\|_{1,\varepsilon}. \tag{2.37}$$

Any element g of $L_1(\Omega)$ can be approximated by a sequence g_n of $L_2(\Omega)$ functions. We define τ_n by

$$\tau_n = (\lambda I - B_2^\varepsilon)^{-1}\, g_n. \tag{2.38}$$

Then, by (2.37), τ_n satisfies

$$\|\tau_n\|_{1,\varepsilon} \leqq \frac{1}{\lambda}\|g_n\|_{,1,\varepsilon} \qquad \|\tau_n - \tau_m\|_{1,\varepsilon} \leqq \frac{1}{\lambda}\|g_n - g_m\|_{1,\varepsilon}. \tag{2.39}$$

Thus τ_n converges in $L_1(\Omega)$ to $\tau = (\lambda I - \bar{B}_2^\varepsilon)^{-1}\, g$ and

$$\|\tau\|_{1,\varepsilon} \leqq \frac{1}{\lambda}\|g\|_{1,\varepsilon}, \tag{2.40}$$

i.e., λ belongs to the resolvent set of $\bar{B}_2^\varepsilon$ and $\left(I - \frac{1}{\lambda}\bar{B}_2^\varepsilon\right)^{-1}$ is a contraction. Thus applying the Hille-Yosida theorem (YOSIDA [17], Ch. 9), we conclude that $\bar{B}_2^\varepsilon$ generates a contraction semigroup. Furthermore, $\bar{B}_2^\varepsilon \subset B^\varepsilon$, since, if $\tau \in D(\bar{B}_2^\varepsilon)$, then for any ω in $H_0^1(\Omega) \cap L_\infty(\Omega)$ with $B^\varepsilon\omega$ in $L_\infty(\Omega)$ we have

$$\int_\Omega T_0\beta^\varepsilon \bar{B}_2^\varepsilon \tau\omega\,dx = \int_\Omega T_0\beta^\varepsilon\tau B^\varepsilon\omega\,dx. \tag{2.41}$$

The proof will be complete if we show that $\bar{B}_2^\varepsilon = B^\varepsilon$. Since $\lambda I - \bar{B}_2^\varepsilon$ is surjective and $\lambda I - \bar{B}_2^\varepsilon \subset \lambda I - B^\varepsilon$, it suffices to show that $\lambda I - B^\varepsilon$ is one-to-one. We take τ to be an element of $D(B^\varepsilon)$ that satisfies

$$(\lambda I - B^\varepsilon)\,\tau = 0. \tag{2.42}$$

Then for any ω in $H_0^1(\Omega) \cap L_\infty(\Omega)$, with $B^\varepsilon\omega$ in $L_\infty(\Omega)$, we have

$$\int_\Omega T_0\beta^\varepsilon\tau(\lambda I - B^\varepsilon)\,\omega\,dx = 0. \tag{2.43}$$

If ψ is an arbitrary element of $C_0^\infty(\Omega)$, the equation

$$(2.44)\qquad \begin{aligned} (\lambda I - B^\varepsilon)\,\omega &= \frac{1}{T_0\beta^\varepsilon}\psi \overset{\text{def}}{=} h, \\ \omega &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

has a unique solution ω in $H_0^1(\Omega)$. This solution belongs to $L^\infty(\Omega)$ by the maximum principle (STAMPACCHIA [12]) and $B^\varepsilon\omega = \lambda\omega - h$ belongs to $L^\infty(\Omega)$. Thus (2.43) implies

$$(2.45)\qquad \int_\Omega \tau\psi\,dx = 0 \quad \text{for every } \psi \text{ in } C_0^\infty(\Omega).$$

Hence $\tau = 0$ and $(\lambda I - B^\varepsilon)$ is one-to-one. This completes the proof of Lemma 2.1.

Completing the proof of Theorem 2.1 is now straightforward. Since $\dot{u}^\varepsilon$ is an element of $L_2(0, T; \mathbb{H})$ (see (2.24)), the mechanical dissipation $d_1^\varepsilon(t) = b_{ijkh}^\varepsilon \dot{e}_{kh}^\varepsilon \dot{e}_{ij}^\varepsilon$ is in $L_1(0, T; L_1(\Omega))$ and $(T_0\beta^\varepsilon)^{-1}\,d_1^\varepsilon(t)$ also belongs to that space. Lemma 1, together with Duhamel's principle, insures that the equation

$$(2.46)\qquad \begin{aligned} \dot{\tau}^\varepsilon &= B^\varepsilon\tau^\varepsilon + \frac{1}{T_0\beta^\varepsilon} d_1^\varepsilon, \\ \tau^\varepsilon(0) &= \tau_0^\varepsilon, \end{aligned}$$

has a unique generalized solution in $C_0([0, T]; L_1(\Omega))$ given by

$$(2.47)\qquad \tau^\varepsilon(t) = \Sigma^\varepsilon(t)_0^\varepsilon\,\tau + \int_0^t \Sigma^\varepsilon(t-\sigma)\,\frac{1}{T_0\beta^\varepsilon}\,d_1^\varepsilon(\sigma)\,d\sigma,$$

where $\Sigma^\varepsilon(t)$ is the contraction semigroup generated by B^ε. The proof of Theorem 2.1 is now complete.

The relations (2.17) and (2.26) and the contractive character of $S^\varepsilon(t)$ and $\Sigma^\varepsilon(t)$ in the norms $\|\ \|_{b_\varepsilon}$ and $\|\ \|_{1,\varepsilon}$ yield

$$(2.48)\qquad \begin{aligned} \|S^\varepsilon(t)\| &\leqq K, \\ \|\Sigma^\varepsilon(t)\| &\leqq K'. \end{aligned}$$

The proof of Theorem 2.1 and Remark 2.1 is now complete.

The remainder of the paper is devoted to the study of the behavior of u^ε and τ^ε (unique solution of the system (2.3)–(2.10)) when the scaling parameter ε goes to 0.

3. Homogenization of the Equations of Motion

This section details and completes the results announced in FRANCFORT, LEGUILLON, & SUQUET [5], and previous results by SANCHEZ-PALENCIA [11], who first pointed out that the homogenized constitutive relation indicates fading memory response. Our analysis enables us to give an explicit expression for the associated relaxation kernel and the strain rate tensor (see Section 4).

The equations under consideration are (2.3), (2.4), (2.5), (2.7), (2.9) which by Theorem 2.1 admit for any fixed ε a unique solution in $W^{1,2}(0, T; \mathbb{H})$.

As is customary in homogenization theory (see DUVAUT [3] or BENSOUSSAN, LIONS, & PAPANICOLAOU [1] for example) we consider the space $\mathbb{H}^1_{\text{per}}(Y)$ of all $\mathbb{R}^3$-valued functions whose components are periodic and lie in $\mathbb{H}^1(Y)$. We define the so-called correctors $\chi^a_{ij}(y)$ and $\chi^b_{ij}(y)$ as the unique (up to a constant) solution in $\mathbb{H}^1_{\text{per}}(Y)$ of the equation

$$a_Y(\chi^a_{ij}, v) = -a_Y(P_{ij}, v) \quad \text{for every } v \text{ in } \mathbb{H}^1_{\text{per}}(Y), \tag{3.1}$$

where

$$a_Y(\varphi, \psi) = \int_Y a_{ijkh}(y)\, \tilde{e}_{kh}(\varphi)\, \tilde{e}_{ij}(\psi)\, dy, \qquad \varphi, \psi \in \mathbb{H}^1(Y)^3.$$

$P_{ij}(y)$ is the vector whose k^{th} component is $y_j\, \delta_{ik}$, and $\tilde{e}_{ij}(.)$ is the symmetrized gradient operator in y, i.e.,

$$\tilde{e}_{ij}(\psi) = \tfrac{1}{2}\left(\frac{\partial\psi_i}{\partial y_j} + \frac{\partial\psi_j}{\partial y_i}\right), \tag{3.2}$$

for any ψ in $H^1(Y)^3$. The χ^b_{ij}'s are obtained in an identical manner after replacing a by b.

The homogenized coefficients a^{hom}_{ijkh} and b^{hom}_{ijkh} are then given by (SANCHEZ-PALENCIA [11] Ch. 6)

$$\begin{cases} a^{\text{hom}}_{ijkh} = \langle a_{pqkh}(y)\, \tilde{e}_{pq}(\chi^a_{ij} + P_{ij})\rangle, \\ b^{\text{hom}}_{ijkh} = \langle b_{pqkh}(y)\, \tilde{e}_{pq}(\chi^b_{ij} + P_{ij})\rangle, \end{cases} \tag{3.3}$$

where $\langle . \rangle$ denotes the averaging operator on Y, i.e.,

$$\langle . \rangle = \frac{1}{|Y|} \int_Y \cdot\, dy. \tag{3.4}$$

It can be shown that the coefficients a^{hom}_{ijkh} and b^{hom}_{ijkh} have the usual properties of symmetry and strong ellipticity (SANCHEZ-PALENCIA [11], Ch. 6).

We now define the "dissipative" corrector w_{ij} as the solution in $\mathbb{H}^1_{\text{per}}(Y)$ of the evolution problem

$$\begin{aligned} a_Y(w_{ij}, v) + b_Y(\dot{w}_{ij}, v) = 0 \quad &\text{for every } v \text{ in } \mathbb{H}^1_{\text{per}}(Y), \\ w_{ij}(0) = \chi^b_{ij} - \chi^a_{ij}. \end{aligned} \tag{3.5}$$

It can be easily shown that the problem (3.5) has a unique solution in $\mathbb{H}^1_{\text{per}}(Y)$ with the following regularity:

$$w_{ij} \in W^{p,\infty}(\mathbb{R}_+, \mathbb{H}^1_{\text{per}}(Y)) \quad \text{for all } p\text{'s in } \mathbb{Z}_+. \tag{3.6}$$

With the help of the dissipative correctors we define the time dependent coefficients

$$K_{ijkh}(t) = \langle a_{pqkh}(y)\, \tilde{e}_{pq}(w_{ij}(y, t)) + b_{pqkh}(y)\, \tilde{e}_{pq}(\dot{w}_{ij}(y, t))\rangle. \tag{3.7}$$

Finally for any strictly positive λ we set

$$c^{\lambda}_{ijkh}(y) = \frac{1}{\lambda} a_{ijkh}(y) + b_{ijkh}(y). \tag{3.8}$$

The homogenized coefficients $c^{\lambda,\text{hom}}_{ijkh}$ are given, as before, by

$$c^{\lambda,\text{hom}}_{ijkh} = \langle c^{\lambda}_{pqkh}(y)\, \tilde{e}_{pq}(\chi^{c^\lambda}_{ij} + P_{ij})\rangle, \tag{3.9}$$

where $\chi^{c^\lambda}_{kh}$ is the unique (up to a constant) solution in $\mathbb{H}^1_{\text{per}}(Y)$ of the problem,

$$c^{\lambda}_Y(\chi^{c^\lambda}_{kh}, v) = -c^{\lambda}_Y(P_{kh}, v), \quad \text{for every } v \text{ in } \mathbb{H}^1_{\text{per}}(Y). \tag{3.10}$$

In (3.10) c^λ_Y is defined as a_Y and b_Y in (3.1). The coefficients $c^{\lambda,\text{hom}}_{ijkh}$ have the usual properties of symmetry and coercivity.

From now on we denote by $\hat{f}(\lambda)$ the Laplace transform of a function $f(t)$.

In view of the definitions of w_{ij}, χ^a_{ij}, χ^b_{ij}, χ^c_{ij}, the following useful identity can be verified

$$\chi^{c^\lambda}_{ij}(y) = \lambda \hat{w}_{ij}(y, \lambda) + \chi^a_{ij}(y), \tag{3.11}$$

which yields

$$c^{\lambda,\text{hom}}_{ijkh} = \hat{K}_{ijkh}(\lambda) + b^{\text{hom}}_{ijkh} + \frac{1}{\lambda} a^{\text{hom}}_{ijkh}. \tag{3.12}$$

Using (3.12), we see easily that the coefficients $K_{ijkh}(t)$ have the same symmetries as a^{hom}_{ijkh} and b^{hom}_{ijkh}. It can also be shown, using (3.5), that the above coefficients decrease exponentially fast as time goes to $+\infty$.

In order to be in a position to examine the behavior of the solution of the equation under consideration when ε goes to zero, we have to specify a class of admissible if the divergence of the elastic part of the stress associated with u^ε_0 is in L_2, *i.e.*, initial states for the body. We will say that u^ε_0 is admissible if there is a g_0 in $L_2(\Omega)^3$ such that

$$\begin{aligned} \sigma^\varepsilon_{0ij} &= a^\varepsilon_{ijkh} e_{kh}(u^\varepsilon_0), \\ \frac{\partial}{\partial x_j}\sigma^\varepsilon_{0ij} &+ g_{0i} = 0. \end{aligned} \tag{3.13}$$

The standard theory of homogenization for elastic bodies (DUVAUT [3] or SANCHEZ-PALENCIA [10], Ch. 6) implies that u^ε_0 converges weakly in $\mathbb{H}$ to u^0_0, namely the unique solution in $\mathbb{H}$ of the problem

$$\begin{aligned} \sigma^0_{0ij} &= a^{\text{hom}}_{ijkh} e_{kh}(u^0_0), \\ \frac{\partial}{\partial x_j}\sigma^0_{0ij} &+ g_{0i} = 0. \end{aligned} \tag{3.14}$$

Theorem 3.1. *Suppose that the initial state of the body in motion is admissible* (cf. (3.13)), *and that f is in $L_2(0, T; L_2(\Omega)^3)$. Then the solution u^ε of* (2.3), (2.4), (2.5), (2.7), (2.9) *converges weakly in* $W^{1,2}(0, T; \mathbb{H})$ *to* u^0, *namely the unique solution in* $W^{1,2}(0, T; \mathbb{H})$ *of the problem*

$$\sigma^0_{ij}(t) = a^{\text{hom}}_{ijkh} e_{kh}(u^0(t)) + b^{\text{hom}}_{ijkh} e_{kh}(\dot{u}^0(t)) + \int_0^t K_{ijkh}(t-s)\, e_{kh}(\dot{u}^0(s))\, ds, \tag{3.15}$$

$$\frac{\partial}{\partial x_j} \sigma^0_{ij} + f_i = 0, \tag{3.16}$$

$$u^0 = 0 \quad \text{on } \partial\Omega, \tag{3.17}$$

$$u^0(x, 0) = u^0_0. \tag{3.18}$$

Moreover, if the body force f *is in* $W^{q,2}(0, T; L_2(\Omega)^3)$, u^ε *converges to* u^0 *in* $W^{q+1,2}(0, T; \mathbb{H})$ *and if* $f = 0$, u^ε *converges to* u^0 *in* $W^{p,\infty}(\mathbb{R}_+, \mathbb{H})$ *weak-* (for any* p *in* $\mathbb{Z}^+$*).*

Remark 3.1. The stress field σ^ε associated to u^ε converges weakly in $L_2(0, T; L_2(\Omega)^9)$ to the homogenized stress field σ^0.

Proof of Theorem 3.1. The proof of Theorem 3.1 is attained in two steps. The first step considers the case of zero body force. The second applies the integral representation (2.12) of the solution (Duhamel's principle).

We assume that $f = 0$ and establish an elementary *a priori* estimate on the solution u^ε. Since,

$$u^\varepsilon_0 \text{ is bounded in } \mathbb{H}, \tag{3.19}$$

we have, for any p in $\mathbb{Z}_+$,

$$\left\| \frac{d^p}{dt^p} S^\varepsilon(t)\, u^\varepsilon_0 \right\|_{\mathbb{H}} \leq C, \text{ (constant independent of } \varepsilon). \tag{3.20}$$

We then extract from $u^\varepsilon(x, t) = S^\varepsilon(t)\, u^\varepsilon_0$ a subsequence, still denoted $u^\varepsilon(x, t)$, such that

$$u^\varepsilon(x, t) \text{ converges weak-* in } W^{p,\infty}(\mathbb{R}_+, \mathbb{H})\ (p \in \mathbb{Z}_+) \text{ to } u^0(x, t), \text{ an element of } W^{p,\infty}(\mathbb{R}_+, \mathbb{H})\ (p \in \mathbb{Z}_+). \tag{3.21}$$

Identifying u^ε with its convergent subsequence will be eventually justified since we will show that the limit u^0 is independent of the specific subsequence chosen.

The type of convergence obtained implies that the Laplace transform in time $\hat{u}^\varepsilon$ of $u^\varepsilon(x, t)$ (which is well defined as an element of $\mathbb{H}$ since $u^\varepsilon \in W^{p,\infty}(\mathbb{R}_+, \mathbb{H})$) satisfies

$$\hat{u}^\varepsilon(\lambda) \text{ converges weakly in } \mathbb{H} \text{ to } \hat{u}^0(\lambda). \tag{3.22}$$

Our goal is to identify $\hat{u}^0(\lambda)$. Because $S^\varepsilon(t)$ is a contraction, (Yosida [17], Ch. 9) it is a direct consequence of (2.14) in Remark 2.1. that the right half of the complex plane belongs to the resolvent set of A^ε, for every ε. Furthermore, the resolvent of the generator of a semigroup applied to a vector is equal to the Laplace Transform of the semigroup acting on that same vector (YOSIDA [17], Ch. 9). Thus we have, for any λ in $\mathbb{R}_+^*$,

$$(\lambda I - A^\varepsilon)\, \hat{u}^\varepsilon(\lambda) = u_0^\varepsilon. \tag{3.23}$$

Recalling (2.16) and (2.18), we conclude that $\hat{u}^\varepsilon(\lambda)$ is the unique solution in $\mathbb{H}$ of

$$\begin{aligned} &\hat{\sigma}_{ij}^\varepsilon = (a_{ijkh}^\varepsilon + \lambda b_{ijkh}^\varepsilon)\, e_{kh}(\hat{u}^\varepsilon(\lambda)) - b_{ijkh}^\varepsilon e_{kh}(u_0^\varepsilon), \\ &\frac{\partial}{\partial x_j} \hat{\sigma}_{ij}^\varepsilon = 0, \\ &\hat{u}^\varepsilon(\lambda) = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.24}$$

The system (3.24), with the help of (3.8) and (3.13) can be rewritten as follows:

$$\theta^\varepsilon = \lambda \hat{u}^\varepsilon(\lambda) - u_0^\varepsilon, \tag{3.25}$$

$$\begin{aligned} &\xi_{ij}^\varepsilon = c_{ijkh}^\lambda \left(\frac{x}{\varepsilon}\right) e_{kh}(\theta^\varepsilon), \\ &\frac{\partial}{\partial x_j} \xi_{ij}^\varepsilon = \frac{1}{\lambda} g_{0i} \overset{\text{def}}{=} h_i, \\ &\theta^\varepsilon = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.26}$$

The case of interest reduces to a problem of homogenization for an elastic body with c_{ijkh}^λ as elastic coefficients. As mentioned before (*cf.* (3.13)),

$$\theta^\varepsilon \text{ converges weakly in } \mathbb{H} \text{ to } \theta^0, \tag{3.27}$$

namely the unique solution in $\mathbb{H}$ of the problem

$$\begin{aligned} &\xi_{ij}^0 = c_{ijkh}^{\lambda,\text{hom}} e_{kh}(\theta^0), \\ &\frac{\partial}{\partial x_j} \xi_{ij}^0 = h_i, \\ &\theta^0 = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.28}$$

where $c_{ijkh}^{\lambda,\text{hom}}$ has been defined by (3.9). Note that we have

$$\theta^0 = \lambda \hat{u}^0(\lambda) - u_0^0. \tag{3.29}$$

The uniqueness of $\hat{u}^0(\lambda)$ results from the ellipticity properties of the $c_{ijkh}^{\lambda,\text{hom}}$'s.

Using (3.12), (3.14), we conclude that $\hat{u}^0(\lambda)$ satisfies

$$\begin{aligned} &\hat{\sigma}_{ij}^0 = (\hat{K}_{ijkh}(\lambda) + b_{ijkh}^{\text{hom}})\, e_{kh}(\lambda \hat{u}^0 - u_0^0) + a_{ijkh}^{\text{hom}} e_{kh}(\hat{u}^0), \\ &\frac{\partial}{\partial x_j} \hat{\sigma}_{ij}^0 = 0, \\ &\hat{u}^0(\lambda) = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.30}$$

We have proved so far that $\hat{u}^\varepsilon(\lambda)$ converges weakly in $\mathbb{H}$ to $\hat{u}^0(\lambda)$, the unique solution in $\mathbb{H}$ of (3.30).

The uniqueness of $\hat{u}^0(\lambda)$ implies the uniqueness of $u^0(x, t)$, since the Laplace transformation is one to one. Furthermore, the $\hat{\sigma}^0_{ij}$'s are the Laplace transforms of the σ^0_{ij}'s defined by (3.15). In view of (3.30), these σ^0_{ij}'s satisfy

$$\frac{\partial}{\partial x_j}\sigma^0_{ij} = 0. \tag{3.31}$$

The proof of the first step is now complete.

If f is an arbitrary element of $W^{q,2}(0, T; L_2(\Omega)^3)$, $q \geqq 0$, its representative F^ε (*cf.* (2.20)) lies in $W^{q,2}(0, T; \mathbb{H})$ and is bounded there independently of ε. Equation (2.21) can then be differentiated q times with respect to time.

The uniform boundedness in ε of $\|A^\varepsilon\|$ defined in (2.18) together with the uniform boundedness of F^ε in $W^{q,2}(0, T; \mathbb{H})$ yield the following *a priori* estimate for u^ε given by (2.12):

$$u^\varepsilon \text{ is bounded in } W^{q+1,2}(0, T; \mathbb{H}). \tag{3.32}$$

We extract a weakly convergent subsequence of u^ε and we call u^0 the limit in $W^{q+1,2}(0, T; \mathbb{H})$, which is computed in $L_2(\Omega)^3$ for every t in $[0, T]$. The limit of the first term on the right-hand side of (2.12) is easily identified with the help of our analysis above since for every t in $[0, T]$,

$$\lim_{\varepsilon\to 0} S^\varepsilon(t)\, u_0^\varepsilon = \boldsymbol{u}^0, \quad \text{in } \mathbb{H} \text{ weak (and } L_2(\Omega)^3 \text{ strong)}, \tag{3.33}$$

where $\boldsymbol{u}^0$ is the unique solution in $W^{p,\infty}(\mathbb{R}_+, \mathbb{H})$ of (3.15)–(3.18) with $f = 0$.

It remains to consider the limit of $\int_0^t S^\varepsilon(t - \sigma)\, F^\varepsilon(\sigma)\, d\sigma$. For almost every σ in $(0, T)$, $F^\varepsilon(\sigma)$ satisfies

$$\frac{\partial}{\partial x_j}(b^\varepsilon_{ijkh} e_{kh}(F^\varepsilon(\sigma))) + f_i(\sigma) = 0. \tag{3.34}$$

The standard theorem on homogenization in elastostatics mentioned above ensures that, for almost every σ in $(0, T)$, $F^\varepsilon(\sigma)$ converges weakly in $\mathbb{H}$ to $F^0(\sigma)$, namely the unique solution in $\mathbb{H}$ of the problem

$$\frac{\partial}{\partial x_j}(b^{\text{hom}}_{ijkh} e_{kh}(F^0(\sigma))) + f_i(\sigma) = 0. \tag{3.35}$$

The argument of the first step can be easily reproduced in the case of an initial state satisfying (3.34). If we let $u^0(t, \sigma)$ denote the unique solution of (3.15)–(3.18) with $F^0(\sigma)$ as initial condition, for every z in $\mathbb{R}_+$, and almost every σ in $(0, T)$, $u^0(z, \sigma)$ is the weak limit in $\mathbb{H}$ and the strong limit in $L_2(\Omega)^3$ of $S^\varepsilon(z)\, F^\varepsilon(\sigma)$.

Thus for every t in $[0, T]$ and almost every σ in $(0, t)$ we have

$$\lim_{\varepsilon\to 0} S^\varepsilon(t - \sigma)\, F^\varepsilon(\sigma) = u^0(t - \sigma, \sigma) \text{ in } \mathbb{H} \text{ weak (and } L_2(\Omega)^3 \text{ strong)}. \tag{3.36}$$

But in view of (2.14) in Remark 2.1,

(3.37) $\|S^\varepsilon(t-.)\,F^\varepsilon(.)\|_{L_2(\Omega)^3}$ is bounded in $L_2(0,t)$ for every t in $[0,T]$.

Relations (3.36), (3.37), together with Lebesgue's dominated convergence theorem, show that for every t in $[0,T]$

$$\int_0^t S^\varepsilon(t-\sigma)\,F^\varepsilon(\sigma)\,d\sigma \xrightarrow{\varepsilon\to 0} \int_0^t u^0(t-\sigma,\sigma)\,d\sigma \text{ strongly in } L_2(\Omega)^3. \tag{3.38}$$

We have thus proved that

$$u^0(t) = \boldsymbol{u}^0(t) + \int_0^t u^0(t-\sigma,\sigma)\,d\sigma. \tag{3.39}$$

It remains to verify that the stress field $\sigma^0(t)$ associated with u^0 by (3.15) satisfies the equilibrium equations (3.16). This follows from a simple computation and completes the proof.

Under suitable regularity properties of the domain, the homogenized displacement field u^0 has additional regularity properties due to elliptic regularity. We define $\mathbb{H}^2(\Omega) \overset{\text{def}}{=} (H^2(\Omega)^3)$ and state

Theorem 3.2. *If the boundary $\partial\Omega$ of Ω is sufficiently "smooth" and the body force $f(x,t) \in W^{q,2}(0,T;L_2(\Omega)^3)$, $q \geqq 0$, then the solution u^0 of* (3.15)–(3.18) *has the following regularity*:

$$u^0 \in W^{q+1,2}(0,T;\mathbb{H}^2(\Omega)\cap\mathbb{H}). \tag{3.40}$$

Proof of Theorem 3.2. The proof rests on a simple fixed-point argument in $C^0([0,T];\mathbb{H}^2(\Omega)\cap\mathbb{H})$. We first deduce from elliptic regularity that since g_0 is in $L_2(\Omega)^3$, the initial displacement field u_0^0 is in $\mathbb{H}^2(\Omega)\cap\mathbb{H}$. If f is in $L_2(0,T;$ $L_2(\Omega)^3)$ the mapping T defined by

$$Tu = u_0^0 - \int_0^t B^{\mathrm{hom}-1}\Big\{(a_{ijkh}^{\mathrm{hom}} + K_{ijkh}(0))\frac{\partial}{\partial x_j}(e_{kh}(u(s))) + \left[\int_0^s \dot{K}_{ijkh}(s-\sigma)\frac{\partial}{\partial x}(e_{kh}(u(\sigma)))\,d\sigma\right] - K_{ijkh}(s)\frac{\partial}{\partial x_j}(e_{kh}(u_0^0(s))) + f(s)\Big\}\,ds, \tag{3.41}$$

where

$$B^{\mathrm{hom}} = b_{ijkh}^{\mathrm{hom}}\frac{\partial}{\partial x_j}(e_{kh}(.)), \tag{3.42}$$

can be shown to be a Lipschitz map from $C^0([0,T],\mathbb{H}^2(\Omega)\cap\mathbb{H})$ into itself. Specifically for every t in $[0,T]$

$$\|T^n u - T^n v\|_{\mathbb{H}^2(\Omega)\cap\mathbb{H}} \leqq C_T^n\frac{t^n}{n!}\|u-v\|_{C^0([0,T];\mathbb{H}^2(\Omega)\cap\mathbb{H})} \tag{3.43}$$

where $C_T = C(1 + T\,\|K\|_{W^{1,\infty}(0,T)})$.

An iterate of T is a strict contraction and Banach's fixed point theorem is applicable. The fixed point obtained in this manner is easily identified as the solution of the system (3.15)–(3.18).

4. Homogenized Mechanical Dissipation

This section is devoted to determining the limit of the mechanical dissipation

$$d_1^\varepsilon = b_{ijkh}^\varepsilon e_{kh}(\dot{u}^\varepsilon)\, e_{ij}(\dot{u}^\varepsilon). \tag{4.1}$$

d_1^ε is the product of 3 weakly converging terms and a sense of convergence stronger than what was obtained in Theorem 3.1 is necessary in order to pass to the limit. It should be noted at this point that the limit of (4.1) is needed in order to homogenize the energy equation (see Section 5 below).

In the first part of this section we establish a result of strong convergence *for admissible initial states* (*cf.* (3.13)). We then determine the limit of (4.1).

We denote $L_p(\Omega)^9$ by $\mathbb{L}_p(\Omega)$ for all $p \geqq 1$. Letting Ψ_{ij} be an arbitrary element of $W^{1,2}(0, T; \mathbb{L}_2(\Omega))$ with $\Psi_{ij}(t = 0) = 0$, we set

$$E_{ij}(\Psi, y, t) = \tilde{e}_{ij}(\chi_{kh}^a(y) + P_{kh}(y))\, \Psi_{kh}(x, t) + \int_0^t \tilde{e}_{ij}(w_{kh}(y, t-s))\, \dot{\Psi}_{kh}(x, s)\, ds. \tag{4.2}$$

Integrating by parts in time the second term in the expression for $E(\Psi)$, using the definition (3.5) of the w_{ij}'s and the fact that the initial value of Ψ vanishes, yields

$$E_{ij}(\Psi, y, t) = \tilde{e}_{ij}(\chi_{kh}^b(y) + P_{kh}(y))\, \Psi_{kh}(x, t) + \int_0^t \tilde{e}_{ij}(\dot{w}_{kh}(y, t-s))\, \Psi_{kh}(x, s)\, ds. \tag{4.3}$$

We set

$$E_{ij}^\varepsilon(\Psi, t) = E_{ij}\left(\Psi, \frac{x}{\varepsilon}, t\right). \tag{4.4}$$

Then the following result, whose proof is inspired by TARTAR [15] (see also SUQUET [13]) holds.

Theorem 4.1. *Suppose that the initial state u_0^ε is in the admissible class* (3.13) *and that the body force f lies in $W^{1,2}(0, T; L_2(\Omega)^3)$. Then $e(\dot{u}^\varepsilon) - \dot{E}^\varepsilon(e(u^0))$ converges strongly to* 0 *in $L_2(0, T; \mathbb{L}_1(\Omega))$. Furthermore, if either*

$$\chi_{ij}^b, \chi_{ij}^a \in W^{1,\infty}(Y)^3, \quad w_{ij} \in W^{1,\infty}(0, T; W^{1,\infty}(Y)^3) \tag{4.5}$$

or

$$\chi_{ij}^b, \chi_{ij}^a \in W^{1,r'}(Y)^3, \quad w_{ij} \in W^{1,\infty}(0, T; W^{1,r'}(Y)^3) \tag{4.5$'$}$$

and

$$e(u^0) \in W^{1,2}(0, T; \mathbb{L}_r(\Omega))$$

with

$$\frac{1}{r}+\frac{1}{r'}=\tfrac{1}{2}, \quad +\infty > r' > 2,$$

then $e(\dot{u}^\varepsilon) - \dot{E}^\varepsilon(e(u^0))$ *converges strongly to zero in* $L_2(0, T; \mathbb{L}_2(\Omega))$.

Remark 4.1. If the boundary of Ω is smooth enough, $e(u^0)$ lies in $W^{2,2}(0, T; \mathbb{H}^1(\Omega))$. (see Theorem 3.2), and hence, by Sobolev's embedding theorem, in $W^{2,2}(0, T; \mathbb{L}_6(\Omega))$. Thus (4.5)′ is satisfied as long as

$$\chi_{ij}^b, \chi_{ij}^a \in W^{1,3}(Y^3), \; w_{ij} \in W^{1,\infty}(0, T; W^{1,3}(Y)^3). \tag{4.5$''$}$$

Proof of Theorem 4.1. The proof of this theorem is attained in three steps of which the first is the most substantial. In the *two first steps we assume that* $u_0^\varepsilon = 0$.

Let Φ be an arbitrary element of $(C_0^\infty((0, T]\times\Omega)^9)$. We proceed to show that $E^\varepsilon(\Phi)$ belongs to $C^\infty([0, T], L_2(\Omega))$, and that it satisfies

$$E^\varepsilon(\Phi) \xrightarrow[\varepsilon\to 0]{\text{weak}*} \Phi \text{ in } W^{p,\infty}(0, T, \mathbb{L}_2(\Omega)), \quad p \in \mathbb{Z}_+. \tag{4.6}$$

It is known (see SANCHEZ-PALENCIA [11], Ch. 5) that, since $\tilde{e}_{ij}(\chi_{kh}^a + P_{kh})|_{y=\frac{x}{\varepsilon}}$ is a periodic element of $L_2^{\text{loc}}(\mathbb{R}^3)$, it converges weakly to its Y-average in $L_2^{\text{loc}}(\mathbb{R}^3)$. Thus

$$\tilde{e}_{ij}(\chi_{kh}^a + P_{kh})|_{y=\frac{x}{\varepsilon}} \xrightarrow[\varepsilon\to 0]{\text{weak-}*} T_{ij|kh} \text{ in } L_2(\Omega), \tag{4.7}$$

where $T_{ij|kh} = \frac{1}{2}(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk})$.

In a similar fashion, by virtue of the regularity (3.6) of w_{kh} we obtain

$$\tilde{e}_{ij}(w_{kh}(y, t-s))|_{y=\frac{x}{\varepsilon}} \xrightarrow[\varepsilon\to 0]{\text{weak-}*} 0 \text{ in } W^{p,\infty}(\mathbb{R}_+, L_2(\Omega)), \tag{4.8}$$

for all p's in $\mathbb{Z}_+$. Furthermore,

$$\int_0^t \tilde{e}_{ij}(w_{kh}(y, t-s))|_{y=\frac{x}{\varepsilon}} \dot{\Phi}_{kh}(s)\, ds \in C^\infty([0, T]; L_2(\Omega)) \tag{4.9}$$

and this integral is bounded in $W^{p,\infty}(0, T; L_2(\Omega))$ independently of ε, for any p in $\mathbb{Z}_+$. Hence $E^\varepsilon(\Phi)$ is bounded in $W^{p,\infty}(0, T; \mathbb{L}_2(\Omega))$, $p \in \mathbb{Z}_+$ and, in view of (4.7) and (4.8), it converges weak-* to Φ in this space.

We also define

$$\Sigma_{ij}(\Phi, y, t) = a_{ijkh}(y)\, E_{kh}(\Phi) + b_{ijkh}(y)\, \dot{E}_{kh}(\Phi), \tag{4.10}$$

and

$$\Sigma_{ij}^\varepsilon(\Phi, t) = \Sigma_{ij}\left(\Phi, \frac{x}{\varepsilon}, t\right). \tag{4.11}$$

Then an argument of the type used above implies

(4.12) $$\Sigma^\varepsilon(\Phi) \xrightarrow[\varepsilon\to 0]{\text{weak-*}} \Sigma^0(\Phi) \quad \text{in } W^{p,\infty}(\mathbb{R}_+, \mathbb{L}_2(\Omega)), \quad p \in \mathbb{Z}_+,$$

where

(4.13) $$\Sigma^0_{ij}(\Phi)(x,t) = a^{\text{hom}}_{ijkh}\Phi_{kh}(x,t) + \int_0^t K_{ijkh}(t-s)\,\dot{\Phi}_{kh}(x,s)\,ds + b^{\text{hom}}_{ijkh}\dot{\Phi}_{kh}(x,t).$$

These results will help us to prove the following

Lemma 4.1.

(4.14) $$\overline{\lim_{\varepsilon\to 0}} \int_0^T \int_\Omega |e(\dot{u}^\varepsilon) - \dot{E}^\varepsilon(\Phi)|^2\,(t)\,dx\,dt \leqq C_T \|e(u^0) - \Phi\|^2_{W^{1,2}(0,T;\mathbb{L}_2(\Omega))}$$

where C_T is a constant depending only on T.

Proof of Lemma 4.1. By the ellipticity properties of the coefficients $a_{ijkh}(y)$ and $b_{ijkh}(y)$, and since the initial state is undeformed ($u^\varepsilon_0 = 0$), the following string of inequalities holds:

(4.15) $$\begin{aligned} 0 &\leqq \int_0^T \|e(\dot{u}^\varepsilon(t)) - \dot{E}^\varepsilon(\Phi(t))\|^2_{\mathbb{L}_2(\Omega)}\,dt \\ &\leqq \tfrac{1}{2}\|e(u^\varepsilon(T)) - E^\varepsilon(\Phi(T))\|^2_{\mathbb{L}_2(\Omega)} + \int_0^T \|e(\dot{u}^\varepsilon(t) - \dot{E}^\varepsilon(\Phi(t))\|^2_{\mathbb{L}_2(\Omega)}\,dt \\ &\leqq \frac{\alpha^{-1}}{2}\int_\Omega a^\varepsilon_{ijkh}(e_{kh}(u^\varepsilon(T)) - E^\varepsilon_{kh}(\Phi,T))\,(e_{ij}(u^\varepsilon(T)) - E^\varepsilon_{ij}(\Phi,T))\,dx \\ &\quad + \alpha^{-1}\int_0^T\int_\Omega b^\varepsilon_{ijkh}(e_{kh}(\dot{u}^\varepsilon(s)) - \dot{E}^\varepsilon_{kh}(\Phi,s))\,(e_{ij}(\dot{u}^\varepsilon(s)) - \dot{E}^\varepsilon_{ij}(\Phi,s))\,dx\,ds \\ &\leqq \alpha^{-1}\int_0^T\int_\Omega (\sigma_{ij}(u^\varepsilon(s)) - \Sigma^\varepsilon_{ij}(\Phi(s))\,(e_{ij}(\dot{u}^\varepsilon(s)) - \dot{E}^\varepsilon_{ij}(\Phi(s)))\,dx\,ds, \end{aligned}$$

where $\sigma(u^\varepsilon)$ is the stress field associated with u^ε (see (2.4)).

By the symmetry properties of the σ_{ij}'s and Σ_{ij}'s the integrand in the last term of (4.15) can be rewritten as

(4.16) $$I^\varepsilon(x,s) \stackrel{\text{def}}{=} (\sigma_{ij}(u^\varepsilon(s)) - \Sigma^\varepsilon_{ij}(\Phi,s))\left(\frac{\partial \dot{u}^\varepsilon_i}{\partial x_j}(s) - \dot{F}^\varepsilon_{ij}(\Phi,s)\right),$$

where

(4.17) $$F^\varepsilon_{ij}(\Phi,s) = F_{ij}\left(\Phi, \frac{x}{\varepsilon}, s\right),$$

with

$$F_{ij}(\Phi, y, s) = \frac{\partial}{\partial y_j}(\chi^a_{kh,i} + P_{kh,i})(y)\,\Phi_{kh}(x, s) + \int_0^s \frac{\partial}{\partial y_j}(w_{kh,i}(y, s-\sigma))\,\dot{\Phi}_{kh}(x, \sigma)\,d\sigma. \tag{4.18}$$

Since f lies in $W^{1,2}(0, T; L_2(\Omega)^3)$, Theorem 3.1 implies that u^ε is bounded in $W^{2,2}(0, T; \mathbb{H})$.

In view of the definitions of $\sigma(u^\varepsilon)$, $\Sigma^\varepsilon(\Phi)$, the relations (4.6), (4.12), and the boundedness of u^ε in $W^{2,2}(0, T; \mathbb{H})$,

$$\begin{aligned} &\sigma(u^\varepsilon) \text{ and } \Sigma^\varepsilon(\Phi) \text{ are bounded in } L_2(0, T; \mathbb{L}_2(\Omega)),\\ &\operatorname{div}(\sigma(u^\varepsilon)) \text{ and } \operatorname{div}(\Sigma^\varepsilon(\Phi)) \text{ are bounded in } L_2(0, T; L_2(\Omega)^3),\\ &\frac{\partial \dot{u}^\varepsilon_i}{\partial x_j} \text{ and } \dot{F}^\varepsilon_{ij}(\Phi) \text{ are bounded in } W^{1,2}(0, T; L_2(\Omega)),\\ &\operatorname{curl}\left(\frac{\partial \dot{u}^\varepsilon_i}{\partial x_j}\right) \text{ and } \operatorname{curl}(\dot{F}^\varepsilon_{ij}(\Phi)) \text{ are bounded in } L_2(0, T; \mathbb{L}_2(\Omega)). \end{aligned} \tag{4.19}$$

With the help of (4.19) we are in a position to apply a theorem on compensated compactness (TARTAR [14], Theorem 11) which permits us to pass to the limit in (4.16) in spite of the presence of a product of weakly convergent expressions. This theorem yields

$$I^\varepsilon \rightharpoonup I^0 = (\sigma_{ij}(u^0) - \Sigma^0_{ij}(\Phi))\,(e_{ij}(\dot{u}^0) - \dot{\Phi}_{ij}), \qquad \varepsilon \to 0, \tag{4.2}$$

in $(\mathscr{D}'((0, T) \times \Omega)^9)$, where u^0 is the homogenized displacement field and $\sigma_{ij}(u^0)$ the corresponding homogenized stress field (see (3.15)). Once more we have made use of the symmetry properties of the $\sigma_{ij}(u^0)$'s and $\Sigma^0_{ij}(\Phi)$'s. The type of convergence obtained in (4.20) is local; it does not allow us to pass to the limit in $\int_0^T \int_\Omega I^\varepsilon\,dxdt$ which is the quantity to estimate in (4.15).

This difficulty is circumvented in the following manner. A straightforward integration by parts shows that

$$\lim_{\varepsilon \to 0} \int_0^T \int_\Omega \sigma_{ij}(u^\varepsilon)\,e_{ij}(\dot{u}^\varepsilon)\,dx\,dt = \int_0^T \int_\Omega \sigma_{ij}(u^0)\,e_{ij}(\dot{u}^0)\,dx\,dt. \tag{4.21}$$

We define J^ε by

$$J^\varepsilon \overset{\text{def}}{=} I^\varepsilon - \sigma_{ij}(u^\varepsilon)\,e_{ij}(\dot{u}^\varepsilon). \tag{4.22}$$

Setting $\Phi = 0$ in (4.20) implies

$$\sigma_{ij}(u^\varepsilon)\,e_{ij}(\dot{u}^\varepsilon) \xrightharpoonup{\mathscr{D}'((0,T)\times\Omega)^9} \sigma_{ij}(u^0)\,e_{ij}(\dot{u}^0), \tag{4.23}$$

and thus

$$J^\varepsilon \xrightharpoonup{\mathscr{D}'((0,T)\times\Omega)^9} J^0 \overset{\text{def}}{=} I^0 - \sigma_{ij}(u^0)\,e_{ij}(\dot{u}^0). \tag{4.24}$$

Letting φ and ψ be elements of $C_0^\infty(\Omega)$ and $C_0^\infty(0, T)$, we have

(4.25) $$\lim_{\varepsilon\to 0} \int_0^T \int_\Omega J^\varepsilon(x, t)\,\varphi(x)\,\psi(t)\,dx\,dt = \int_0^T \int_\Omega J^0(x, t)\,\varphi(x)\,\psi(t)\,dx\,dt.$$

The J^ε's are *uniformly* compactly supported in Ω, since Φ has compact support in Ω. Taking φ to be equal to 1 on the support of J^ε in (4.25) yields

(4.26) $$\lim_{\varepsilon\to 0} \int_0^T \int_\Omega J^\varepsilon(x, t)\,\psi(t)\,dx\,dt = \int_0^T \int_\Omega J^0(x, t)\,\psi(t)\,dx\,dt.$$

It also results from (4.6) and (4.12) that

(4.27) $\int_\Omega J^\varepsilon(x, t)\,dx$ is bounded in $L_2(0, T)$ independently of ε.

Since $C_0^\infty(0, T)$ is dense in $L_2(0, T)$, we obtain

(4.28) $$\lim_{\varepsilon\to 0} \int_0^T \int_\Omega J^\varepsilon(x, t)\,dx\,dt = \int_0^T \int_\Omega J^0(x,t)\,dx\,dt.$$

Relations (4.21), (4.28) yield

(4.29) $$\lim_{\varepsilon\to 0} \int_0^T \int_\Omega I^\varepsilon(x, t)\,dx\,dt = \int_0^T \int_\Omega I^0(x, t)\,dx\,dt,$$

and thus

(4.30) $$\overline{\lim_{\varepsilon\to 0}} \int_0^T \|e(\dot u^\varepsilon(t)) - \dot E^\varepsilon(\Phi(t))\|^2_{\mathrm{L}_2(\Omega)}\,dt$$
$$\leqq \alpha^{-1} \int_0^T \int_\Omega (\sigma_{ij}(u^0)(s)) - \Sigma_{ij}^0(\Phi(s)))\,(e_{ij}(\dot u^0(s)) - \dot\Phi_{ij}(s))\,dx\,ds.$$

Since

(4.31) $$(\sigma_{ij}(u^0) - \Sigma_{ij}^0(\Phi))\,(s) = a_{ijkh}^{\text{hom}}(e_{kh}(u^0) - \Phi_{kh})\,(s)$$
$$+ b_{ijkh}^{\text{hom}}(e_{kh}(\dot u^0) - \dot\Phi_{kh})\,(s) + \int_0^s K_{ijkh}(s - z)\,(e_{kh}(\dot u^0) - \dot\Phi_{kh})\,(z)\,dz,$$

and the kernel $K_{ijkh}(t)$ is bounded in time

(4.32) $$\alpha^{-1} \int_0^T \int_\Omega (\sigma_{ij}(u^0) - \Sigma_{ij}^0(\Phi))\,(s)\,(e_{ij}(\dot u^0) - \dot\Phi_{ij})\,(s)\,dx\,ds$$
$$\leqq C_T\left[\int_0^T \|e(\dot u^0) - \dot\Phi\|^2_{\mathrm{L}_2(\Omega)}\,ds + \|(e(u^0) - \Phi))\,(T)\|^2_{\mathrm{L}_2(\Omega)}\right],$$

where C_T is a constant depending on T. This last inequality together with (4.30) completes the proof of Lemma 4.1, since the injection of $W^{1,2}(0, T; \mathrm{L}_2(\Omega))$ into $C^0([0, T]; \mathrm{L}_2(\Omega))$ is continuous.

We now return to the proof of Theorem 4.1. We recall that we are assuming $u_0^\varepsilon = 0$ and thus $u_0^0 = 0$. The homogenized strain tensor $e(u^0)$ can then be approximated in $W^{1,2}(0, T; \mathrm{L}_2(\Omega))$ by a sequence of elements Φ^n in $C_0^\infty((0, T] \times \Omega)^9$.

The regularity property (3.6) together with (4.4) yields

$$(4.33)\qquad \|\dot{E}^{\varepsilon}(e(u^0)) - \dot{E}^{\varepsilon}(\Phi^n)\|_{\mathbb{L}_2(0T;\mathbb{L}_1(\Omega))}$$
$$\leqq (\|\chi^a_{kh}\|_{\mathbb{H}^1_{per}(Y)} + 1 + T\| \, \|w_{kh}\|_{W^{1,\infty}(\mathbb{R}_+,\mathbb{H}^1_{per}(Y))}) \, \|e_{kh}(u^0) - \Phi^n_{kh}\|_{W^{1,2}(0,T;\mathbb{L}_2(\Omega))}$$
$$\leqq C''_T \, \|e(u^0) - \Phi^n\|_{W^{1,2}(0,T;\mathbb{L}_2(\Omega))},$$

where C''_T is a constant depending on T. Inequalities (4.14) applied to Φ^n and (4.33) imply the first part of Theorem 4.1 in the case $u_0^\varepsilon = 0$.

The second part of Theorem 4.1 results from a simple modification of (4.33), which consists in replacing the $\mathbb{H}^1_{per}(Y)$ norms by $W^{1,\infty}_{per}(Y)$ norms, when (4.5) holds, or by $W^{1,r'}(Y)$ norms, when (4.5)′ holds.

We now consider an initial displacement field u_0^ε satisfying (3.13). The solution u^ε of (2.3), (2.4), (2.5), (2.7), (2.9) can be written as

$$(4.34)\qquad u^\varepsilon = u_0^\varepsilon + v^\varepsilon,$$

where v^ε satisfies the same equations as u^ε but with $v_0^\varepsilon = 0$ and f replaced by $f + g_0$. The two first steps of the proof apply to v^ε and yield

(4.35) $e(\dot{v}^\varepsilon) - \dot{E}^\varepsilon(e(v^0))$ converges to zero strongly in $L_2(0, T; \mathbb{L}_1(\Omega))$ for in $L_2(0, T; \mathbb{L}_2(\Omega))$ if the additional assumptions required are met.

A direct application of Theorem 3.1 enables us to write the homogenized field u^0 associated to u^ε as

$$(4.36)\qquad u^0(t) = u_0^0 + v^0(t),$$

where u_0^0 and v^0 are the homogenized fields corresponding to u_0^ε and v^ε.

Since u_0^ε and u_0^0 are independent of the time variable we obtain

$$(4.37)\qquad e(\dot{u}^\varepsilon) = e(\dot{v}^\varepsilon) \quad \text{and} \quad e(\dot{u}^0) = e(\dot{v}^0).$$

Finally a simple inspection of (4.2) implies, by virtue of (4.36),

$$(4.38)\qquad \dot{E}^\varepsilon(e(v^0)) = \dot{E}^\varepsilon(e(u^0)).$$

Then (4.35) reads

(4.39) $e(\dot{u}^\varepsilon) - \dot{E}^\varepsilon(e(u^0))$ converges to zero strongly in $L_2(0, T; \mathbb{L}_1(\Omega))$ or $L_2(0, T; \mathbb{L}_2(\Omega))$,

and this completes the proof of Theorem 4.1.

Theorem 4.1 allows to determine the limit of the mechanical dissipation (4.1). We define

$$(4.40)\qquad d_1^0 = b^{hom}_{ijkh} e_{kh}(\dot{u}^0)\, e_{ij}(\dot{u}^0)$$
$$+ \left\langle b_{ijmn}(y) \left\{ \int_0^t \tilde{e}_{mn}(\dot{w}_{kh}(y, t-s))\, e_{kh}(\dot{u}^0(s))\, ds \right.\right.$$
$$\left.\left. \times \int_0^t \tilde{e}_{ij}(\dot{w}_{pq}(y, t-s))\, e_{pq}(\dot{u}^0(s))\, ds \right\} \right\rangle$$

and

(4.41) $$D_1^\varepsilon = b_{ijkh}^\varepsilon \dot{E}_{kh}^\varepsilon(e(u^0))\, \dot{E}_{ij}^\varepsilon(e(u^0))\,.$$

Theorem 4.2. *We assume that the initial state* u_0^ε *is in the admissible class* (3.13), *that* $f \in W^{1,2}(0, T; L_2(\Omega)^3)$, *and that the following hypothesis is satisfied:*

(4.5)''' $$\chi_{ij}^b, \chi_{ij}^a \in W^{1,r'}(Y)^3, \quad w_{ij} \in W^{1,\infty}(0, T;\, W^{1,r'}(Y))^3,$$
$$e(u^0) \in W^{2,2}(0, T;\, \mathbb{L}_r(\Omega)), \quad \text{with} \quad \frac{1}{r} + \frac{1}{r'} = \tfrac{1}{4},\ +\infty \geqq r' > 4.$$

Then

(4.42) $$\lim_{\varepsilon\to 0} (d_1^\varepsilon - D_1^\varepsilon) = 0 \quad \text{in } L_1(0, T; L_1(\Omega)) \quad \text{strong}$$

and

(4.43) $$\lim_{\varepsilon\to 0} (D_1^\varepsilon - d_1^0) = 0 \quad \text{in } W^{1,2}(0, T; L_2(\Omega)) \quad \text{weak}.$$

Remark 4.2. Theorem 4.2 shows that the *homogenized dissipation, i.e.* the weak limit of d_1^ε, *is* d_1^0, given by (4.40). This cannot be deduced from a simple inspection of the homogenized constitutive law (3.15).

Remark 4.3. If the boundary of Ω is smooth enough, $e(u^0)$ lies in $W^{2,2}(0, T;\, \mathbb{L}_6(\Omega))$ (recall Remark 5.1). Thus (4.5)''' is satisfied as long as

(4.5)'''' $$\chi_{ij}^b, \chi_{ij}^a \in W^{1,12}(Y)^3, \quad w_{ij} \in W^{1,\infty}(0, T;\, W^{1,12}(Y)^3)\,.$$

Proof of Theorem 4.2. A direct application of Theorem 4.1 implies that

(4.44) $e(\dot{u}^\varepsilon) - \dot{E}^\varepsilon(e(u^0))$ converges strongly to 0 in $L_2(0, T;\, \mathbb{L}_2(\Omega))$.

We have

(4.45) $$\int_0^T \|d_1^\varepsilon - D_1^\varepsilon\|_{L_1(\Omega)}\, dt$$
$$\leqq \int_0^T \|b_{ijkh}^\varepsilon e_{kh}(\dot{u}^\varepsilon)\, (e_{ij}(\dot{u}^\varepsilon) - \dot{E}_{ij}^\varepsilon(e(u^0)))\|_{L_1(\Omega)}\, dt$$
$$+ \int_0^T \|b_{ijkh}^\varepsilon \dot{E}_{ij}^\varepsilon(e(u^0))\, (e_{kh}(\dot{u}^\varepsilon) - \dot{E}_{kh}^\varepsilon(e(u^0)))\|_{L_1(\Omega)}\, dt.$$

Since $b_{ijkh}^\varepsilon e_{kh}(\dot{u}^\varepsilon(x, t))$ and $b_{ijkh}^\varepsilon \dot{E}_{ij}^\varepsilon(e(u^0), t)$ are clearly bounded in $L_2(0, T;\, \mathbb{L}_2(\Omega))$ we conclude, with the help of (4.44), that

(4.46) $$\lim_{\varepsilon\to 0} d_1^\varepsilon - D_1^\varepsilon = 0 \quad \text{in } L_1(0, T; L_1(\Omega)) \text{ strongly}.$$

The regularity hypothesis (4.5)''' implies that

(4.47) $e(\dot{u}^0)$ and $\dot{E}^\varepsilon(e(u^0))$ lie in $W^{1,2}(0, T;\, \mathbb{L}_4(\Omega))$.

Thus

(4.48) $$D_1^\varepsilon \text{ and } d_1^0 \text{ lie in } W^{1,2}(0, T; L_2(\Omega)).$$

It remains to verify that

(4.49) $$D_1^\varepsilon \xrightarrow{\varepsilon\to 0} d_1^0 \text{ weakly in } W^{1,2}(0, T; L_2(\Omega)).$$

The proof of (4.49) follows by a straightforward albeit lengthy argument, similar to the one that led to (4.6).

The proof of Theorem 4.2 is now complete.

5. Homogenization of the Energy Equation

We define k_{ij}^{hom} to be the usual homogenized thermal conductivity tensor associated with the tensor $k_{ij}(y)$ (see BENSSOUSSAN, LIONS, & PAPANICOLAOU [1], Ch. 1, for example), namely,

(5.1) $$k_{ij}^{\text{hom}} = \left\langle k_{ip}(y) \frac{\partial}{\partial y_p} (\chi_j^k(y) + y_j) \right\rangle,$$

where χ_j^k is the solution in $H^1_{\text{per}}(Y)$ of the problem

(5.2) $$\int_Y k_{ip}(y) \frac{\partial}{\partial y_p} (\chi_j^k(y) + y_j) \frac{\partial \zeta}{\partial y_i} dy = 0 \quad \text{for any } \zeta \text{ in } H^1_{\text{per}}(Y).$$

We also set

(5.3) $$\bar{\beta} = \langle \beta(y) \rangle.$$

Then the exact analogue of Lemma 2.1 applies to the operator B^0 defined by

(5.4) $$B^0 = \frac{1}{T_0 \bar{\beta}} k_{ij}^{\text{hom}} \frac{\partial^2}{\partial x_i \, \partial x_j},$$

with domain

(5.5) $$D(B^0) = \{\tau \in L_1(\Omega) \text{ such that } B^0\tau \in L_1(\Omega)$$

and such that, for any ω in $H_0^1(\Omega) \cap L_\infty(\Omega)$ with $B^0\omega$ in $L_\infty(\Omega)$,

$$\int_\Omega (B^0\tau) \, \omega \, dx = \int_\Omega \tau(B^0\omega) \, dx\}.$$

Hence B^0 generates in $L_1(\Omega)$ a strongly continuous contraction semigroup $\Sigma^0(t)$ under the norm

(5.6) $$\|\omega\|_{1,0} = T_0 \bar{\beta} \, \|\omega\|_{L_1(\Omega)}.$$

Remark 5.1. B^0 is the L_1-closure of its L_2 restriction B_2^0 with domain

(5.7) $$D(B_2^0) = \{\tau \in H_0^1(\Omega) \mid B_2^0\tau \text{ as a distribution lies in } L_2(\Omega)\}.$$

Remark 5.2. Since the infinitesimal generators B_2^ε (see (2.29))and B_2^0 are dissipative and self-adjoint, the restriction to $L_2(\Omega)$ of the semigroups $\Sigma^\varepsilon(t)$ and $\Sigma^0(t)$ are analytic contraction semigroups on $L_2(\Omega)$ (KATO [9], Ch. 7).

We are now in a position to homogenize the energy equation.

We assume that the hypotheses of Theorem 4.2 hold and that the initial increment of temperature τ_0^ε is independent of ε, *i.e.*

$$\tau_0^\varepsilon = \tau_0^0, \quad \tau_0^0 \text{ in } L_1(\Omega). \tag{5.8}$$

We denote by $\tau^\varepsilon(t)$ the temperature increment field which is the unique solution in $C^0([0, T]; L_1(\Omega))$ of (2.6), (2.8), (2.10). We also define

$$\underset{\sim}{\tau}_0^0(t) = \Sigma^0(t)\,\tau_0^0, \tag{5.9}$$

$$\bar{\tau}^\varepsilon(t) = \int_0^t \Sigma^\varepsilon(t-\sigma)\,\frac{1}{T_0\beta^\varepsilon}\,D_1^\varepsilon(\sigma)\,d\sigma, \tag{5.10}$$

where D_1^ε is given by (4.41).

We finally set

$$\bar{\tau}^0(t) = \int_0^t \Sigma^0(t-\sigma)\,\frac{1}{T_0\bar{\beta}}\,d_1^0(\sigma)\,d\sigma, \tag{5.11}$$

$$\tau^0(t) = \underset{\sim}{\tau}_0^0(t) + \bar{\tau}^0(t), \tag{5.12}$$

where d_1^0 is given by (4.40). The temperature increment field τ^0 is the unique generalized solution in $C^0([0, T]; L_1(\Omega))$ of the problem

$$T_0\bar{\beta}\dot{\tau}^0 = k_{ij}^{\text{hom}}\,\frac{\partial^2\tau^0}{\partial x_i\,\partial x_j} + d_1^0, \tag{5.13}$$

$$\tau^0(x, 0) = \tau_0^0(x), \tag{5.14}$$

$$\tau^0 = 0 \quad \text{on } \partial\Omega. \tag{5.15}$$

Theorem 5.1. *Let us assume that the hypotheses of Theorem 4.2 are satisfied, together with* (5.8). *Then*

$$\lim_{\varepsilon\to 0} \tau^\varepsilon - \tau^0 = 0 \text{ in } C^0([0, T]; L_1(\Omega)) \text{ strong}. \tag{5.16}$$

Specifically,

$$\lim_{\varepsilon\to 0} \tau^\varepsilon - (\underset{\sim}{\tau}_0^0 + \bar{\tau}^\varepsilon) = 0 \text{ in } C^0([0, T]; L_1(\Omega)) \text{ strong}, \tag{5.17}$$

$$\lim_{\varepsilon\to 0} \bar{\tau}^\varepsilon - \bar{\tau}^0 = 0 \text{ in } W^{1,2}(0, T; H_0^1(\Omega)) \text{ weak}. \tag{5.18}$$

Remark 5.3. If the boundary of Ω is smooth enough, the hypotheses of Theorem 4.2 are satisfied so long as (4.5)'''' holds (see Remark 4.3).

Remark 5.4. The convergence (5.18) has a meaning since both $\bar{\tau}^\varepsilon$ and $\bar{\tau}^0$ lie in $W^{1,2}(0, T; H_0^1(\Omega))$. Indeed, the hypotheses of Theorem 4.2 imply that

$$D_1^\varepsilon \text{ and } d_1^0 \text{ lie in } W^{1,2}(0, T; L_2(\Omega)). \tag{5.19}$$

With the help of (5.19) and by virtue of the analytic character of $\Sigma^\varepsilon(t)$ and $\Sigma^0(t)$ (see Remark 5.2), we conclude that

(5.20) $$\bar{\tau}^\varepsilon \text{ and } \bar{\tau}^0 \text{ are in } W^{1,2}(0, T; H_0^1(\Omega)),$$

and that they satisfy

(5.21) $$T_0\beta^\varepsilon\dot{\bar{\tau}}^\varepsilon = \frac{\partial}{\partial x_i}\left(k_{ij}^\varepsilon \frac{\partial\bar{\tau}^\varepsilon}{\partial x_j}\right) + D_1^\varepsilon, \qquad \bar{\tau}^\varepsilon(0) = 0,$$

for all t in $[0, T]$, and

(5.22) $$T_0\beta^\varepsilon\ddot{\bar{\tau}}^\varepsilon = \frac{\partial}{\partial x_i}\left(k_{ij}^\varepsilon \frac{\partial\dot{\bar{\tau}}^\varepsilon}{\partial x_j}\right) + \dot{D}_1^\varepsilon, \qquad \dot{\bar{\tau}}^\varepsilon(0) = \frac{1}{T_0\beta^\varepsilon} D_1^\varepsilon(0),$$

for almost all t in $[0, T]$. Furthermore, $\dot{\bar{\tau}}^\varepsilon$ can be expressed as

(5.23) $$\dot{\bar{\tau}}^\varepsilon(t) = \Sigma^\varepsilon(t)\frac{1}{T_0\beta^\varepsilon} D_1^\varepsilon(0) + \int_0^t \Sigma^\varepsilon(\sigma)\frac{1}{T_0\beta^\varepsilon}\dot{D}_1^\varepsilon(t-\sigma)\,d\sigma.$$

The analogues of (5.21)–(5.23) hold for $\bar{\tau}^0$ after replacing β^ε by $\bar{\beta}$, k_{ij}^ε by k_{ij}^{hom}, Σ^ε by Σ^0 and D_1^ε by d_1^0.

Proof of Theorem 5.1. The proof of Theorem 5.1 is attained in two steps. The first step deals with the homogenization of contraction semigroups in $L_1(\Omega)$. The second step consists in applying Duhamel's principle to the homogeneous problem, with the mechanical dissipation as a forcing term.

Lemma 5.1. *As ε tends to zero, $\Sigma^\varepsilon(t)\,\tau_0^0$ converges strongly in $C_0([0, T], L_1(\Omega))$ to $\tau_0^0(t) = \Sigma^0(t)\,\tau_0^0$, where $\Sigma^0(t)$ is the contraction semigroup generated by B^0.*

Proof of Lemma 5.1. Since $\Sigma^\varepsilon(t)$ is contractive, all λ in the right half complex plane belong to the resolvent set of B^ε (YOSIDA [17], Ch. 9). Setting, for any f in $L_1(\Omega)$,

(5.24) $$(\lambda I - B^\varepsilon)^{-1} f = \omega^\varepsilon,$$

we obtain

(5.25) $$\|\omega^\varepsilon\|_{L_1(\Omega)} \leqq M/\lambda\,\|f\|_{L_1(\Omega)},$$

by virtue of (2.26). Let us emphasize that M (which is equal to C/C', where C, C' are the constants appearing in (2.26)) is independent of ε. We consider a sequence of $L_2(\Omega)$ functions f_n converging to f in $L_1(\Omega)$. Then,

(5.26) $$\omega_n^\varepsilon = (\lambda I - B_2^\varepsilon)^{-1} f_n$$

converges to ω^ε in $L_1(\Omega)$ and

(5.27) $$\|\omega_n^\varepsilon - \omega^\varepsilon\|_{L_1(\Omega)} \leqq M/\lambda\,\|f_n - f\|_{L_1(\Omega)}.$$

In the same way,

(5.28) $$\omega_n^0 = (\lambda I - B_2^0)^{-1} f_n$$

converges to ω^0 in $L_1(\Omega)$, where ω^0 is given by

(5.29) $$\omega^0 = (\lambda I - B^0)^{-1} f.$$

We also have

(5.30) $$\|\omega_n^0 - \omega^0\|_{L_1(\Omega)} \leqq M/\lambda \, \|f_n - f\|_{L_1(\Omega)}.$$

The homogenization theory for elliptic second order problems in L_2 (see BENSOUSSAN, LIONS, & PAPANICOLAOU [1], Ch. 1) implies that

(5.31) $$\omega_n^\varepsilon \xrightarrow{\varepsilon\to 0} \omega_n^0 \quad \text{weakly in } H_0^1(\Omega).$$

Applying Rellich's theorem to (5.31) yields

(5.32) $$\omega_n^\varepsilon \xrightarrow{\varepsilon\to 0} \omega_n^0 \quad \text{strongly in } L_2(\Omega).$$

Now, in view of (5.27), (5.30) the following inequalities hold:

(5.33) $$\|\omega^\varepsilon - \omega^0\|_{L_1(\Omega)} \leqq \|\omega^\varepsilon - \omega_n^\varepsilon\|_{L_1(\Omega)} + \|\omega_n^\varepsilon - \omega_n^0\|_{L_1(\Omega)}$$
$$+ \|\omega_n^0 - \omega^0\|_{L_1(\Omega)} \leqq \frac{2M}{\lambda} \|f - f_n\|_{L_1(\Omega)} + \operatorname{vol}(\Omega)^{1/2} \|\omega_n^\varepsilon - \omega_n^0\|_{L_2(\Omega)}.$$

Thus, using (5.32), we finally show that

(5.34) $$\omega^\varepsilon \to \omega^0 \text{ strongly in } L_1(\Omega), \varepsilon \to 0,$$

(5.35) $$(\lambda I - B^\varepsilon)^{-1} f \xrightarrow{\varepsilon\to 0} (\lambda I - B^0)^{-1} f \text{ strongly in } L_1(\Omega).$$

The assertion of the lemma is then a direct application of the Trotter-Kato theorem (KATO [9], Ch. 9).

Returning to the proof of Theorem 5.1, we proceed to show (5.17). As we saw earlier (see (2.13)) the solution τ^ε of (2.6), (2.8), (2.10) can be expressed as

(5.36) $$\tau^\varepsilon(t) = \Sigma^\varepsilon(t)\, \tau_0^0 + \int_0^t \Sigma^\varepsilon(t-\sigma) \frac{1}{T_0 \beta^\varepsilon} d_1^\varepsilon(\sigma)\, d\sigma.$$

Recalling (5.9) and (5.10) we obtain

(5.37) $$\|\tau^\varepsilon(t) - (\underset{\sim}{\tau}_0^0 + \tilde{\tau}^\varepsilon(t))\|_{L_1(\Omega)} \leqq \|(\Sigma^\varepsilon(t) - \Sigma^0(t))\, \tau_0^0\|_{L_1(\Omega)}$$
$$+ \int_0^t \left\| \Sigma^\varepsilon(t-\sigma) \frac{1}{T_0 \beta^\varepsilon} (d_1^\varepsilon - D_1^\varepsilon)(\sigma) \right\|_{L_1(\Omega)} d\sigma.$$

In view of Lemma 5,1. the first term on the right-hand side of the inequality (5.37) tends to zero uniformly on $[0, T]$. Since Σ^ε is contracting and $\beta(y)$ is bounded away from zero, the second term is majorized by

(5.38) $$M \int_0^T \|d_1^\varepsilon - D_1^\varepsilon\|_{L_1(\Omega)}\, dt,$$

and this tends to zero by direct application of Theorem 4.2. The convergence (5.17) has thus been established.

Next we establish (5.18). From the hypotheses of Theorem 4.2 it follows that

$$D_1^\varepsilon \text{ is bounded in } W^{1,2}(0, T; L^2(\Omega)). \tag{5.39}$$

Then taking $\bar{\tau}^\varepsilon$ and $\dot{\bar{\tau}}^\varepsilon$ as test functions in (5.21) and (5.22) and integrating over $]0, T[\times \Omega$ shows that

$$\bar{\tau}^\varepsilon \text{ is bounded in } W^{1,2}(0, T; H_0^1(\Omega)). \tag{5.40}$$

We extract a weakly convergent subsequence of $\bar{\tau}_1^\varepsilon$ in $W^{1,2}(0, T; H_0^1(\Omega))$ and we denote its limit by $\bar{\tau}^0$. Since D_1^ε converges weakly to d_1^0 in $W^{1,2}(0, T; L_2(\Omega))$ it converges strongly to d_1^0 in $L_2(0, T; H^{-1}(\Omega))$. The derivation of (5.18) now reduce to a classical homogenization problem. It suffices to pass to the limit in (5.21), as performed in BENSOUSSAN, LIONS, & PAPANICOLAOU [1], Ch. 2, Remark 1.6.

The convergence (5.18) has now been proved. The convergence (5.16) is an immediate consequence of (5.17) and (5.18).

The proof of Theorem 5.1 is complete.

Acknowledgement. The authors are grateful to D. LEGUILLON, NGUYEN QUOC SON and L. TARTAR for their help at various stages of this study. They are also indebted to F. MURAT for numerous remarks which improved the original draft.

References

1. A. BENSOUSSAN, J.-L. LIONS, & G. PAPANICOLAOU, *Asymptotic Analysis for Periodic Structures*, North Holland, Amsterdam, 1978.
2. H. BRÉZIS, & A. STRAUSS, "Semi Linear Second Order Elliptic Equations in L_1", Journal of Math. Soc. of Japan, **25**, 1973, p. 565–590.
3. G. DUVAUT, "Analyse Fonctionnelle et Mécanique des Milieux Continus", in *Th. and Appl. Mech.*, Ed. W. KOITER, North Holland, Amsterdam, 1976, p. 119–132.
4. G. DUVAUT & J. L. LIONS, *Variational Inequalities in Mechanics and Physics*, Springer Verlag, Berlin-Heidelberg-New York, 1976.
5. G. FRANCFORT, D. LEGUILLON, & P. SUQUET, "Homogénisation de Milieux Viscoélastiques Linéaires de Kelvin-Voigt", C. R. Acad. Sc. Paris I, **296**, p. 287–290, 1983.
6. G. FRANCFORT, "Homogenization and Linear Thermoelasticity" S.I.A.M. J. of Math. Anal., **14**, 1983, p. 696–708.
7. P. GERMAIN, *Mécanique des Milieux Continus*, Masson, Paris, 1973.
8. P. GERMAIN, NGUYEN QUOC SON, & P. SUQUET, "Continuum Thermodynamics", J. Appl. Mech., **105**, 1983, p. 1010–1020.
9. T. KATO, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin-Heidelberg-New York, 1966.
10. B. NAYROLES, R. BOUC, M. CAUMON, J. C. CHEZEAUX, & E. GIACOMMETTI, "Téléthermographie infrarouge et Mécanique des Structures", I.J.E.S., **19**, 1981, p. 929–947.
11. E. SANCHEZ-PALENCIA, *Non Homogeneous Media and Vibration Theory*, Springer-Verlag Monograph in Physics, **127**, Berlin-Heidelberg-New York, 1980.
12. G. STAMPACCHIA, "Le problème de Dirichlet pour les Equations Elliptiques du

Second Ordre à Coefficients Discontinus", Ann. Inst. Fourier, Grenoble, **15**, 1, 1968, p. 189–258.
13. P. SUQUET, "Une méthode duale en homogénéisation ..." J. Méca. Th. Appl., N° spécial, 1982, p. 79–98.
14. L. TARTAR, "Compensated Compactness and applications to P.D.E." in *Non-Linear Mech. and Analysis, Heriot-Watt Symposium, Volume IV,* Ed. R. J. KNOPS, Pitman Resarch Notes in Mathematics, **39**, 1979 p. 136–212.
15. L. TARTAR, Cours Peccot Collège de France 1977, partially written in F. MURAT, "*H*-convergence", Séminaire d'Analyse fonctionnelle 1977–1978, Alger (Multigraphed).
16. C. TRUESDELL, *Continuum Mechanics* **3**, Foundations of Elasticity Theory, Gordon and Breach, New York, 1965.
17. K. YOSIDA, *Functional Analysis,* Springer Verlag, Berlin-Heidelberg-New York, Sixth Edition, 1980.

Laboratoire Central des Ponts
et Chaussées
58 boulevard Lefèbvre
Paris 15

&

Laboratoire Générale des Milieux Continus
Université Montpellier II
Montpellier

(Received January 30, 1986)