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OPTIMAL BOUNDS FOR CONDUCTION IN TWO-DIMENSIONAL, TWO-PHASE, ANISOTROPIC MEDIA

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Résumé. On considère les mélanges, en proportions non déterminées, de deux matériaux conducteurs, anisotropes, en dimension 2. On caractérise l'ensemble des matériaux effectifs obtenus par homogénéisation à partir de tels mélanges : c'est l'ensemble des tenseurs de conductivité dont les vecteurs propres sont quelconques et dont les valeurs propres appartiennent à un certain sous ensemble L de $[\mathbb{R}_+^*]^2$. La forme analytique de L dépend de l'ordre relatif des valeurs propres des tenseurs de conductivité des deux matériaux d'origine. Les démonstrations reposent sur des propriétés de la théorie de l'homogénéisation spécifiques à la dimension 2, et notamment sur un critère de stabilité : sous certaines conditions géométriques sur L, l'ensemble des tenseurs de conductivité dont les valeurs propres appartiennent à L est stable pour l'homogénéisation. Le détail des démonstrations sera donné dans un article à paraître.

Abstract. Two dimensional mixtures in arbitrary volume fraction of two anisotropic conducting materials are investigated. The set of all effective materials resulting from the homogenization of such mixtures is identified as the set of conductivity tensors whose eigenvalues belong to a definite region L of $[\mathbb{R}_+^*]^2$. The associated eigenvectors are arbitrary. The analytical determination of L depends on the ordering properties of the eigenvalues of the original conductivity tensors. The proofs are based on properties of the homogenization theory which are specific to the two dimensional case. In particular the following stability criterion holds : the set of all conductivity tensors whose eigenvalues belong to L is stable under the homogenization process whenever L meets several specific geometrical assumptions. The details of the proofs will appear in a forthcoming paper.

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INTRODUCTION

This paper is concerned with the determination of the set of all possible effective conductivities of a two-phase anisotropic material with arbitrary phase geometry. Since Hashin & Shtrikman's original bounds on the set of possible isotropic effective tensors of a two-phase material with isotropic phases due attention has been paid to the case of isotropic phases (*cf.* Hashin (1983), Tartar (1985), Kohn & Milton (1985), Francfort & Murat (1986), Ericksen, Kinderlehrer, Kohn & Lions (1986) and references therein).

The case of polycrystalline media has been considerably less investigated (*cf.* Schulgasser (1977)).

In a two-dimensional setting, Lurie & Cherkaev (1984) addressed the problem of characterizing the set of all anisotropic effective conductivity tensors of a two-phase material with anisotropically conducting phases in arbitrary volume fraction. In the present paper (which describes the results of Francfort & Murat (1987)), we revisit Lurie & Cherkaev's bounds and derive a complete characterization in the two-dimensional case.

We consider two homogeneous and anisotropic conducting materials. If they are positioned in a common reference configuration, there exists an orthonormal basis e_1, e_2 of $\mathbb{R}^2$ such that the conductivity tensors A_1 and A_2 of the two phases read as

$$\begin{cases} A_1 = \alpha_1 e_1 \otimes e_1 + \alpha_2 e_2 \otimes e_2, \\ A_2 = \beta_1 e_1 \otimes e_1 + \beta_2 e_2 \otimes e_2, \end{cases} \tag{1}$$

and we assume with no loss of generality that

$$\begin{cases} 0<\alpha_1 \leqslant \alpha_2 <+\infty, \\ 0<\beta_1 \leqslant \beta_2 <+\infty, \\ \alpha_1\alpha_2 \leqslant \beta_1\beta_2. \end{cases} \tag{2}$$

We seek the set of all possible anisotropic effective tensors corresponding to the mixture of the two phases with no restriction on the volume fractions.

When both materials are isotropic ($\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$), the result has been known since Tartar (1974). When the materials are anisotropic, the investigated set is shown to depend only on the eigenvalues of the effective conductivity tensor. Specifically it coincides with the set of all bounded measurable symmetric mappings on $\mathbb{R}^2$ whose eigenvalues lie in a compact subset L of $[\mathbb{R}_+^\star]^2$. The associated eigenvectors are arbitrary.

The region L is uniquely determined in terms of $\alpha_1, \alpha_2, \beta_1, \beta_2$ (*cf.* Definition 5). In fact it is the outermost region bounded by the eigenvalues of the effective tensors corresponding to rank-1 lamination of both phases with each other or with themselves. The direction of the rank-1 lamination which produces the boundary of that region strongly depends on the ordering properties of the eigenvalues of the original phases. Three cases have to be considered in the analysis, namely,

$\alpha_1 \leqslant \beta_1$ and $\alpha_2 \leqslant \beta_2$,
$\alpha_1 \leqslant \beta_1$ and $\alpha_2 > \beta_2$,
$\alpha_1 > \beta_1$ and $\alpha_2 \leqslant \beta_2$.

In the three cases parts of the boundary of L are achieved by lamination of the tensor A_1 (resp. A_2) with its image by a rotation of angle $\frac{\pi}{2}$ in the direction e_1 or e_2. The other parts of the boundary of L are obtained through a layering of the tensor A_1 with the tensor A_2 in the direction e_1 when $\alpha_1 \leqslant \beta_1$ and $\alpha_2 \leqslant \beta_2$ and in the direction e_2 otherwise. The reader is referred to Francfort & Murat (1987) for a complete exposition of this lamination process.

Our results agree with those of Lurie & Cherkaev (1984) in the case when $\alpha_1 \leqslant \beta_1$ and $\alpha_2 \leqslant \beta_2$ but disagree with theirs in the case when $\alpha_1 \leqslant \beta_1$ and $\alpha_2 > \beta_2$. The third case was not investigated in Lurie & Cherkaev (1984).

In the first Section of the paper the problem is formulated in the mathematical setting of H-convergence. The characterization of the possible effective tensors is given in the second Section.

It relies on a stability criterion pertaining to the form of the sets of conductivity tensors which remain stable under H-convergence. This stability criterion is the object of the third Section. The fourth and last Section is devoted to a sketch of its proof.

A complete and detailed exposition of the results presented here will be found in Francfort & Murat (1987). Our analysis is based on the theories of H-convergence and compensated compactness (*cf. e.g.* Murat (1977), (1978), Tartar (1977), (1979), (1985)).

1. SETTING OF THE PROBLEM

An arbitrary mixture of the two phases defined in (1) is obtained by considering the characteristic function $\chi(x)$ of the first phase in $\mathbb{R}^2$ and the orientation matrix $R(x)$ which quantifies the rotation of the conductivity tensor at the point x with respect to its reference configuration. In other words a conductivity tensor associated with a mixture of the two phases is a conductivity tensor of the form

$$A(x) = \chi(x)\ {}^tR(x)A_1R(x) + (1-\chi(x))\ {}^tR(x)A_2R(x),$$

where χ is the characteristic function of a measurable subset of $\mathbb{R}^2$, R is a measurable orthogonal matrix on $\mathbb{R}^2$, and A_1 and A_2 are the tensors defined in (1).

We consider a family A_ε of such mixtures, *i.e.* a sequence of χ_ε and R_ε such that

$$A_\varepsilon(x) = \chi_\varepsilon(x)\ {}^tR_\varepsilon(x)A_1R_\varepsilon(x)+(1-\chi_\varepsilon(x))\ {}^tR_\varepsilon(x)A_2R_\varepsilon(x), \tag{3}$$

where ε is a small parameter which may be viewed as the typical size of the heterogeneities in the mixture. We propose to investigate its macroscopic behaviour, with the help of the theory of H-convergence (Murat (1977), Tartar (1977)).

DEFINITION 1. *If* K *is a compact subset of* $[\mathbb{R}_+^*]^2$, $\mathcal{M}(K)$ *is the set of all symmetric tensors* A *with coefficients in* $L_\infty(\mathbb{R}^2)$ *whose eigenvalues* λ_1, λ_2 *satisfy*

$$(\lambda_1(x),\lambda_2(x))\in K,\ (\lambda_2(x),\lambda_1(x))\in K \text{ almost everywhere } \bullet$$

DEFINITION 2. *Let* α, β *be two elements of* $\mathbb{R}_+^*$. *A sequence* A_ε *of elements of* $\mathcal{M}([\alpha,\beta]^2)$ *H-converges to a symmetric linear mapping* A_o *if and only if for any bounded domain* Ω of $\mathbb{R}^2$ *the relation*

$$q_o = A_o w_o \tag{4}$$

holds true for any sequence w_ε *in* $[L_2(\Omega)]^2$ *such that*

$$\begin{cases} w_\varepsilon \longrightarrow w_o, \\ q_\varepsilon = A_\varepsilon w_\varepsilon \longrightarrow q_o, \end{cases} \tag{5}$$

weakly in $[L_2(\Omega)]^2$ *as* ε *tends to zero while*

$$\begin{cases} \operatorname{curl} w_\varepsilon \overset{\text{def}}{=} \dfrac{\partial (w_\varepsilon)_1}{\partial x_2} - \dfrac{\partial (w_\varepsilon)_2}{\partial x_1}, \\ \operatorname{div} q_\varepsilon \overset{\text{def}}{=} \dfrac{\partial (w_\varepsilon)_1}{\partial x_1} + \dfrac{\partial (w_\varepsilon)_2}{\partial x_2}, \end{cases} \tag{6}$$

lie in a compact set of $H^{-1}_{loc}(\Omega)$ •

REMARK 1. The basic properties resulting from the above definition can be found in *e.g.* Murat (1977), Tartar (1977), (1985), Francfort & Murat (1986), (1987) •

The notion of H-limit is meaningful by virtue of the

THEOREM 1. *If* A_ε *is a family of elements of* $\mathcal{M}([\alpha,\beta]^2)$, *there exists a subsequence of* A_ε *which H-converges to an element* A_o *of* $\mathcal{M}([\alpha,\beta]^2)$ *and all possible H-limits belongs to* $\mathcal{M}([\alpha,\beta]^2)$ •

The above theorem was first proved by Spagnolo (67), then revisited by Tartar (*cf.* Murat (77), Tartar (77)). A similar proof may be found in *e.g.* Simon (79).

REMARK 2. The family A_ε considered in (3) belongs to $\mathcal{M}([\alpha,\beta]^2)$ with

$$\alpha = \min(\alpha_1, \beta_1) \ , \quad \beta = \max(\alpha_2, \beta_2),$$

and according to Theorem 1, a subsequence of A_ε H-converges to an element A_o of $\mathcal{M}([\alpha,\beta]^2)$ •

At the possible expense of extracting converging subsequences, we are left with the

DEFINITION 3. *Consider a sequence* A_ε *of the form* (3) *where* A_1, A_2 *are given by* (1), χ_ε *is a sequence of measurable characteristic functions and* R_ε *is a sequence of orthogonal matrices with measurable coefficients. Whenever* A_ε *H-converges to* A_o *as* ε *tends to zero, the tensor* A_o *is referred to as an effective tensor for the mixture of* A_1 *and* A_2 •

2. CHARACTERIZATION OF THE POSSIBLE EFFECTIVE TENSORS.

The complete characterization of all effective tensors for the mixture of A_1 and A_2 depends on the relative magnitudes of their eigenvalues.

DEFINITION 4. *The tensors* A_1 *and* A_2 *defined by* (1) *are said to be well ordered if and only if*

$$\alpha_1 \leqslant \beta_1 \quad \text{and} \quad \alpha_2 \leqslant \beta_2 .$$

Otherwise, i.e. *if*

$$\alpha_1 \leqslant \beta_1 \quad \text{and} \quad \alpha_2 > \beta_2$$

or if

$$\alpha_1 > \beta_1 \quad \text{and} \quad \alpha_2 \leqslant \beta_2$$

they are said to be badly ordered •

DEFINITION 5. *If* α_1, α_2, β_1, β_2 *satisfy* (2), *and if further* $\alpha_1\alpha_2 \neq \beta_1\beta_2$ *the sets* L_w *and* L_b *are defined as follows :*

L_w *is the set of all* $(\lambda_1,\lambda_2) \in [\mathbb{R}_+^\star]^2$ *such that*

$$\alpha_1\alpha_2 \leqslant \lambda_1\lambda_2 \leqslant \beta_1\beta_2, \tag{7}$$

$$\frac{(\beta_1-\alpha_1)\lambda_1\lambda_2 + (\beta_2-\alpha_2)\alpha_1\beta_1}{\beta_1\beta_2-\alpha_1\alpha_2} \leqslant \lambda_1,\lambda_2 \leqslant \frac{\lambda_1\lambda_2(\beta_1\beta_2-\alpha_1\alpha_2)}{(\beta_1-\alpha_1)\lambda_1\lambda_2+(\beta_2-\alpha_2)\alpha_1\beta_1}. \quad (8)$$

L_b *is the set of all* $(\lambda_1,\lambda_2)\in[\mathbb{R}_+^\star]^2$ *such that*

$$\alpha_1\alpha_2 \leqslant \lambda_1\lambda_2 \leqslant \beta_1\beta_2, \quad (9)$$

$$\frac{\lambda_1\lambda_2(\beta_1\beta_2-\alpha_1\alpha_2)}{(\beta_2-\alpha_2)\lambda_1\lambda_2+(\beta_1-\alpha_1)\alpha_2\beta_2} \leqslant \lambda_1,\lambda_2 \leqslant \frac{(\beta_2-\alpha_2)\lambda_1\lambda_2+(\beta_1-\alpha_1)\alpha_2\beta_2}{\beta_1\beta_2-\alpha_1\alpha_2}. \quad (10)$$

If $\alpha_1\alpha_2=\beta_1\beta_2$,

$$L_w = L_b = \{(\lambda_1,\lambda_2)\in[\mathbb{R}_+^\star]^2 \mid \lambda_1\lambda_2=\alpha_1\alpha_2=\beta_1\beta_2, \min(\alpha_1,\beta_1)\leqslant\lambda_1,\lambda_2\leqslant\max(\alpha_2,\beta_2)\} \bullet$$

REMARK 3. The set L_w and L_b are compacts subsets of $[\mathbb{R}_+^\star]^2$. Their boundaries are hyperbolic segments in the (λ_1,λ_2) variables (see Figures 1 to 3 and Remark 5 below for a geometrical representation of L_w and L_b in an other set of variables) •

The set of all effective tensors for the mixture of A_1 and A_2 is characterized with the help of Definitions 4 and 5. Specifically we obtain the following

THEOREM 2. *In the context of Definition* 1 *to* 5, *a symmetric tensor* $A_o(x)$ *with coefficients in* $L_\infty(\mathbb{R}^2)$ *is an effective tensor for the mixture of* A_1 *and* A_2 *if and only if it belongs to* $\mathcal{M}(L_w)$ *when* A_1 *and* A_2 *are well-ordered and to* $\mathcal{M}(L_b)$ *when they are not* •

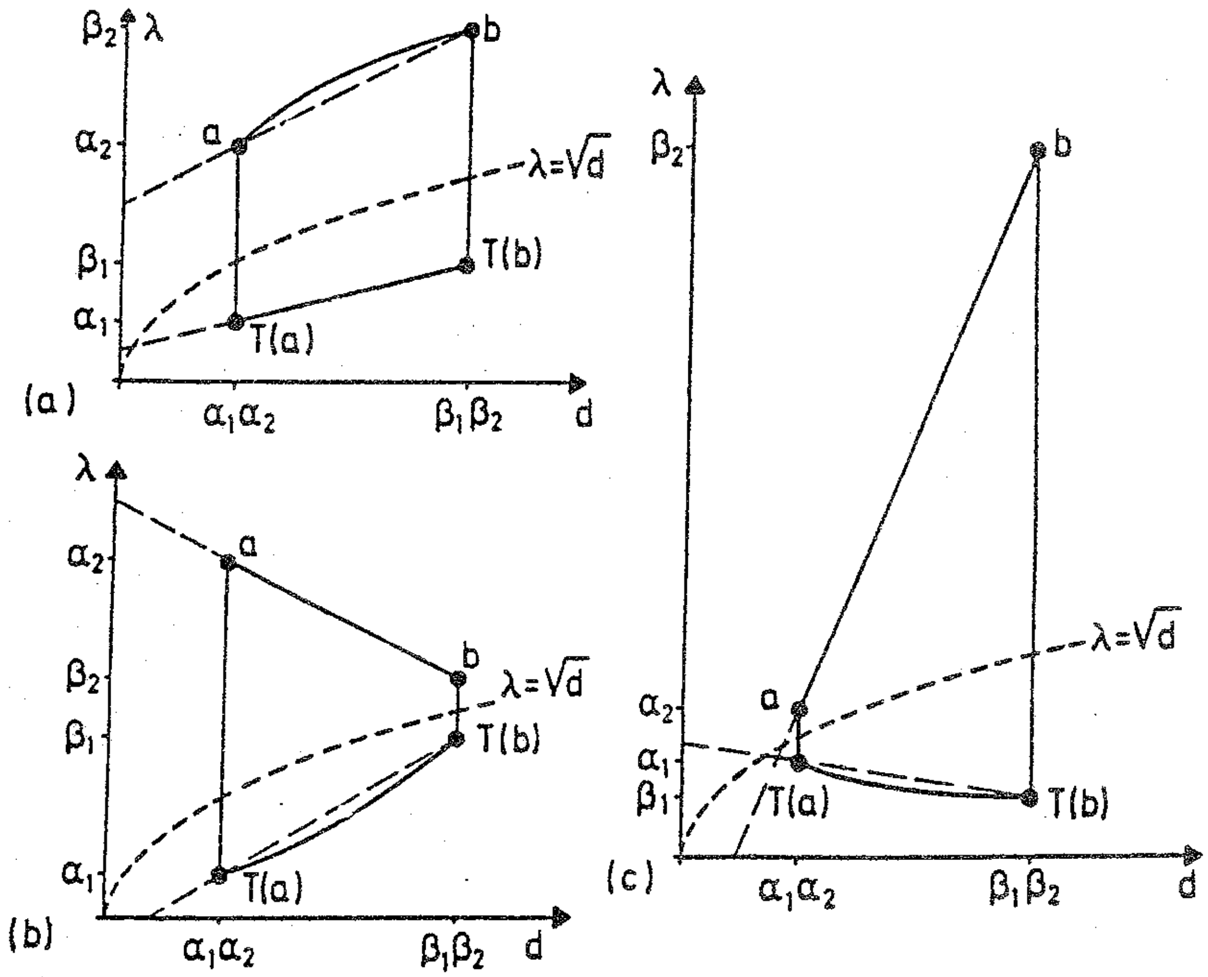
In the case where both A_1 and A_2 are isotropic ($\alpha_1=\alpha_2$, $\beta_1=\beta_2$) Theorem 2 is the result of Tartar (1974).

REMARK 4. Theorem 2 says that all elements of $\mathcal{M}(L_w)$ (respectively $\mathcal{M}(L_b)$) can be achieved as H-limits of a sequence of A_ε satisfying (3) (*cf.* Definition 3) *and* that they are the only ones. Note the absence of any restrictions on the eigenvectors of the possible effective tensors $A_o(x)$.

The proof of the achievability of all elements of $\mathcal{M}(L_w)$ and $\mathcal{M}(L_b)$ is performed through explicit construction using multiple layering.

It will not be given here and the reader is referred to Tartar (1985), Francfort & Murat (1987). In the present paper we focus our attention on the "only if" part of Theorem 2 •

Figure 1. (d-λ) representation of L_w in the well ordered case and L_b in the badly ordered cases.



REMARK 5. Various geometrical representations of the sets L_w and L_b are presented in Francfort & Murat (1987). Figures 1(a), 1(b), 1(c) correspond to the so-called (d-λ)-representation of the sets L_w in the well-ordered case or L_b in the badly ordered case. Each point (λ_1,λ_2) of L_w (or L_b) is mapped onto two points p,T(p) whose coordinates are $(\lambda_1\lambda_2,\ \max(\lambda_1,\lambda_2))$ and $(\lambda_1\lambda_2,\ \min(\lambda_1,\lambda_2))$. Straight vertical line segments correspond to effective tensors with equal determinants. In the well ordered case the boundaries of the set L_w (for which inequalities (7), (8) become equalities) become the two vertical straight line segments ([a,T(a)],

$[b,T(b)])$ together with the concave hyperbolic segment $\widehat{ab}$ and the straight line segment $[T(a),T(b)]$ (*cf.* Fig. 1(a)). In the badly ordered cases the boundaries of the sets L_b (for which inequalities (9), (10) become equalities) are the two vertical line segments $([a,T(a)], [b,T(b)])$ together with the straight line segment $[a,b]$ and the convex hyperbolic segment $\widehat{T(a)T(b)}$ (*cf.* Fig. 1(b)-1(c)). When $\alpha_1\alpha_2 = \beta_1\beta_2$, the set $L_w (= L_b)$ reduces to a vertical straight line segment in its (d,λ)-representation.

The (d,λ)-representation is convenient when addressing the proof of Theorem 2. It is also at the root of a characterization of the sets of all effective tensors for the mixture of more than two anisotropic conducting materials (*cf.* Francfort & Milton (1987)) •

3. A STABILITY CRITERION UNDER H-CONVERGENCE

The proof of the "only if part" of Theorem 2 is an easy consequence of a stability criterion under H-convergence of sets of the form $\mathcal{M}(K)$ (*cf.* Remark 8 below). Specifically, our notion of stability is to be understood as the following

DEFINITION 6. *In the context of Definition* 2 *a subset* $\mathcal{N}$ *of* $\mathcal{M}([\alpha,\beta]^2)$ *is* H-*stable if and only if the* H-*limit* A_0 *of any* H-*converging sequence* A_ε *of elements of* $\mathcal{N}$ *also belongs to* $\mathcal{N}$ •

Our stability criterion is the object of the following

THEOREM 3. *Let* γ *and* δ *be two strictly positive real numbers with*

$$\alpha^2 \leq \gamma \leq \delta \leq \beta^2.$$

Let φ *and* ψ *be two real-valued functions defined on* $[\gamma,\delta]$ *with the following properties :*

$$\begin{cases} \varphi \text{ and } \psi \text{ are } C^1\text{-functions with values in } \mathbb{R}_+^\star, \\ \varphi \text{ is concave,} \\ \psi \text{ is convex,} \\ \varphi(d)\psi(d) = d \text{ for any } d \text{ in } [\gamma,\delta]. \end{cases} \tag{11}$$

Define $K(\gamma,\delta,\varphi,\psi)$ *as the set of all* (λ_1,λ_2) *in* $[\alpha,\beta]^2$ *such that*

$$\gamma \leqslant \lambda_1\lambda_2 \leqslant \delta,$$
$$\psi(\lambda_1\lambda_2) \leqslant \lambda_1, \lambda_2 \leqslant \varphi(\lambda_1\lambda_2).$$

Then $\mathcal{M}(K(\gamma,\delta,\varphi,\psi))$ *is H-stable* •

REMARK 6. The last assumption of (11) is natural. Indeed whenever the boundary of a given set can be parametrized in the form $\lambda_2 = \varphi(\lambda_1\lambda_2)$, it can be equally parametrized in the form $\lambda_1 = \psi(\lambda_1\lambda_2)$, with $\psi(d) = \dfrac{d}{\varphi(d)}$ •

REMARK 7. Conditions (11) essentially characterize the sets of the form $\mathcal{M}(K)$ which are stable under H-convergence. The following converse of Theorem 3 holds true. Define

$$K = \left\{(\lambda_1,\lambda_2) \in [\alpha,\beta]^2 \text{ such that } \psi(\lambda_1\lambda_2) \leqslant \lambda_1, \lambda_2 \leqslant \varphi(\lambda_1\lambda_2)\right\},$$

where $\varphi \leqslant \psi$ are two C^1-function from $\mathbb{R}_+^\star$ into itself such that $\varphi(d)\psi(d) = d$ for any d in $\mathbb{R}_+^\star$ with

$$\min_{(\lambda_1,\lambda_2)\in K} \lambda_1\lambda_2 \leqslant d \leqslant \max_{(\lambda_1,\lambda_2)\in K} \lambda_1\lambda_2. \qquad (12)$$

If K is stable under H-convergence, φ is concave and ψ is convex on the interval defined by inequalities (12).

The proof of this last result can be found in Francfort & Murat (1987) •

REMARK 8. The proof of the "only if" part of Theorem 2 is now straightforward. Since

$$\{(\alpha_1,\alpha_2)\} \cup \{(\beta_1,\beta_2)\} \subset L_w \cap L_b$$

(*cf.* Definition 5) all sequences A_ε satisfying (3) (*cf.* Definition 3) belong to $\mathcal{M}(L_w) \cap \mathcal{M}(L_b)$ and the "only if" part will result from the H-stability of $\mathcal{M}(L_w)$ when A_1 and A_2 are well-ordered and of $\mathcal{M}(L_b)$ when A_1 and A_2 are badly ordered (*cf.* Definition 4).

Let us set

$$\gamma = \alpha_1\alpha_2, \quad \delta = \beta_1\beta_2.$$

The case where $\gamma = \delta$ is left to the reader and we are thus at liberty to assume that

$$\gamma < \delta.$$

In the well ordered case, we define, for any d in $[\gamma,\delta]$

$$\varphi_w(d) = \frac{d(\beta_1\beta_2-\alpha_1\alpha_2)}{(\beta_1-\alpha_1)d + (\beta_2-\alpha_2)\alpha_1\beta_1}, \quad \psi_w(d) = \frac{d}{\varphi_w(d)},$$

whereas in the badly ordered case, we define,

$$\varphi_b(d) = \frac{(\beta_2-\alpha_2)d + (\beta_1-\alpha_1)\alpha_2\beta_2}{(\beta_1\beta_2-\alpha_1\alpha_2)}, \quad \psi_b(d) = \frac{d}{\varphi_b(d)}.$$

By virtue of the ordering properties of $\alpha_1,\alpha_2,\beta_1,\beta_2$ the functions (φ_w,ψ_w) (respectively (φ_b,ψ_d)) are trivially seen to satisfy (11) in the well-ordered (respectively badly ordered) case.

Theorem 3 applies to the set $K(\gamma,\delta,\varphi_w,\psi_w)$ (respectively $K(\gamma,\delta,\varphi_b,\psi_b)$) and yields the H-stability of $\mathcal{M}(K(\gamma,\delta,\varphi_w,\psi_w))$ (respectively $\mathcal{M}(K(\gamma,\delta,\varphi_b,\psi_b))$). Since

$$\begin{cases} L_w = K(\gamma,\delta,\varphi_w,\psi_w), \\ L_b = K(\gamma,\delta,\varphi_b,\psi_b), \end{cases}$$

the desired result is proved •

4. SKETCH OF THE PROOF OF THEOREM 3.

A complete proof of Theorem 3 is presented in Francfort & Murat (1987). It relies on two main ingredients : a decomposition of the set $K(\gamma,\delta,\varphi,\psi)$ as a (countable) intersection of sets with specific properties (*cf.* (14) below) and a few lemmae pertaining to the theory of H-convergence.

4.1. A decomposition of $K(\gamma,\delta,\varphi,\psi)$.

It is firstly remarked that if φ and ψ satisfy (11) there exists a sequence of real numbers d_n in $[\gamma,\delta]$ such that, upon setting

$$\varphi_n = \varphi(d_n),\ \psi_n = \psi(d_n),$$
$$\varphi'_n = \varphi'(d_n),\ \psi'_n = \psi'(d_n),$$

then

$$\begin{cases} \psi'_n\varphi_n + \varphi'_n\psi_n = 1, \\ \varphi'_n \geq 0 \text{ and/or } \psi'_n \geq 0, \\ \varphi(d) = \inf\limits_{n\geq 1} \{\varphi'_n d + \varphi_n^2\psi'_n\},\ \gamma \leq d \leq \delta, \\ \psi(d) = \sup\limits_{n\geq 1} \{\psi'_n d + \psi_n^2\varphi'_n\},\ \gamma \leq d \leq \delta. \end{cases} \tag{13}$$

A simple computation implies that for any d in $[\gamma,\delta]$

$$(\varphi'_n d + \varphi_n^2\psi'_n)(\psi'_n d + \psi_n^2\varphi'_n) = d + \varphi'_n\psi'_n(d-d_n)^2.$$

Thus, if $\varphi'_n \geq 0$ and $\psi'_n \geq 0$, the inequality $\lambda_1,\lambda_2 \geq \psi'_n\lambda_1\lambda_2 + \psi_n^2\varphi'_n$ implies that $\lambda_1,\lambda_2 \leq \varphi'_n\lambda_1\lambda_2 + \varphi_n^2\psi'_n$, whereas, if $\varphi'_n\psi'_n \leq 0$, the inequality $\lambda_1,\lambda_2 \leq \varphi'_n\lambda_1\lambda_2 + \varphi_n^2\psi'_n$ implies that $\lambda_1,\lambda_2 \geq \psi'_n\lambda_1\lambda_2 + \psi_n^2\varphi'_n$.

We set

DEFINITION 7. *For any real numbers* a,b,ζ, $a \geq 0$ *and/or* $b \geq 0$, $\alpha^2 \leq \zeta \leq \beta^2$,

$$L_{\leq}(a,b) = \{(\lambda_1,\lambda_2) \in [\alpha,\beta]^2 \mid \lambda_1,\lambda_2 \leq a\lambda_1\lambda_2 + b\},$$
$$L_{\geq}(a,b) = \{(\lambda_1,\lambda_2) \in [\alpha,\beta]^2 \mid \lambda_1,\lambda_2 \geq a\lambda_1\lambda_2 + b\},$$
$$D_{\leq}(\zeta) = \{(\lambda_1,\lambda_2) \in [\alpha,\beta]^2 \mid \zeta \leq \lambda_1\lambda_2\},$$
$$D_{\geq}(\zeta) = \{(\lambda_1,\lambda_2) \in [\alpha,\beta]^2 \mid \zeta \geq \lambda_1\lambda_2\} \bullet$$

In view of the above computations, the set $K(\gamma,\delta,\varphi,\psi)$ decomposes as

$$K(\gamma,\delta,\varphi,\psi) = D_{\leq}(\gamma) \cap D_{\geq}(\delta) \bigcap_{\varphi'_n\psi'_n\leq 0} L_{\leq}(\varphi'_n,\varphi_n^2\psi'_n) \bigcap_{\substack{\varphi'_n\geq 0 \\ \psi'_n\geq 0}} L_{\geq}(\psi'_n,\psi_n^2\varphi'_n). \tag{14}$$

REMARK 9. The intersection of H-stable sets is H-stable. Thus, by virtue of (14), the set $\mathcal{M}(K(\gamma,\delta,\varphi,\psi))$ is H-stable if each of the sets $\mathcal{M}(D_{\leq}(\gamma))$, $\mathcal{M}(D_{\geq}(\delta))$, $\mathcal{M}(L_{\leq}(\varphi'_n,\varphi_n^2\psi'_n))$ $(\varphi'_n\psi'_n \leq 0)$, $\mathcal{M}(L_{\geq}(\psi'_n,\psi_n^2\varphi'_n))$ $(\varphi'_n \geq 0$ and $\psi'_n \geq 0)$ is H-stable, which is the object of the next subsection $\bullet$

4.2. <u>A few results of the theory of H-convergence</u>.

Our purpose here is not to give a detailed exposition of the theory of H-convergence, but rather to state and/or derive the few results needed in the proof of Theorem 3. Further details may be found in Francfort & Murat (1987).

The following theorem, *specific to the two-dimensional case*, is central to our analysis.

<u>THEOREM</u> 4. *Let* A_ε *belong to* $\mathcal{M}([\alpha,\beta]^2)$ *and* H-*converge to an element* A_o *of* $\mathcal{M}([\alpha,\beta]^3)$. *Then*

$$\frac{A_\varepsilon}{\det A_\varepsilon} \text{ H-}\textit{converges to } \frac{A_o}{\det A_o}.$$

This result traces back to Keller (1964). A proof was presented by Kohler & Papanicolaou (1982) in a probabilistic setting. The proof presented below is due to L. Tartar.

<u>Proof of Theorem 4</u>. Let w_ε, q_ε be as in Definition 2 (*cf.* (4)-(6)). If R denotes the rotation of angle $-\pi/2$, we set

$$\begin{cases} \overline{w}_\varepsilon = Rq_\varepsilon, \\ \overline{q}_\varepsilon = Rw_\varepsilon, \\ B_\varepsilon = RA_\varepsilon^{-1}\,{}^tR. \end{cases}$$

Then, as ε tends to zero,

$$\begin{cases} \overline{w}_\varepsilon \longrightarrow \overline{w}_o = Rq_o, \\ \overline{q}_\varepsilon \longrightarrow \overline{q}_o = Rw_o, \end{cases} \tag{15}$$

weakly in $[L_2(\Omega)]^2$. Furthermore, recalling (6),

$$\begin{cases} \text{curl } \overline{w}_\varepsilon = -\text{ div } q_\varepsilon, \\ \text{div } \overline{q}_\varepsilon = \text{curl } w_\varepsilon, \end{cases} \tag{16}$$

lie in a compact set of $H^{-1}_{loc}(\Omega)$.

Finally,

$$\overline{q}_\varepsilon = B_\varepsilon \overline{w}_\varepsilon . \qquad (17)$$

Theorem 1 applies to B_ε and yields the existence of a subsequence $B_{\varepsilon'}$ of B_ε and of an element B_0 of $\mathcal{M}([\alpha,\beta]^2)$ such that $B_{\varepsilon'}$ H-converges to B_0 as ε' tends to zero. In view of (15)-(17) and the very definition of H-convergence (*cf.* Definition 2), we conclude that

$$\overline{q}_0 = B_0 \overline{w}_0 ,$$

or still that

$$(I - {}^tRB_0RA_0)w_0 = 0,$$

where I stands for the identity mapping on $\mathbb{R}^2$.

A classical argument in the theory of H-convergence permits to choose w_0 as an arbitrary vector of $\mathbb{R}^2$ (at least locally), from which it is easily concluded that

$$B_0(x) = R A_0^{-1}(x)\ {}^tR$$

almost everywhere.

The identity

$$R\, C^{-1}\ {}^tR = \frac{{}^tC}{\det C}$$

which holds true for any invertible two by two matrix yields the desired result •

As announced in Remark 9, the H-stability of $\mathcal{M}(K(\gamma,\delta,\varphi,\psi))$ will result from the H-stability of four kinds of sets. Specifically, the following lemmae hold in the context of Definition 7 :

<u>LEMMA 1</u>. *The sets* $\mathcal{M}(D_{\leqslant}(\zeta))$ *and* $\mathcal{M}(D_{\geqslant}(\zeta))$ *are* H-*stable* •

<u>LEMMA 2</u>. *If* $a \geqslant 0$ *and* $b \geqslant 0$, *the set* $\mathcal{M}(L_{\geqslant}(a,b))$ *is* H-*stable*. *If* $ab \leqslant 0$, *the set* $\mathcal{M}(L_{\leqslant}(a,b))$ *is* H-*stable*.•

Lemma 1 easily results from Theorem 4 together with the following comparison lemma (Tartar (1979a)) :

LEMMA 3. *Let* A_ε *and* B_ε *belong to* $\mathcal{M}([\alpha,\beta]^2)$ *and* H-*converge* A_0 *and* B_0 *respectively. If*

$$A_\varepsilon(x) \leq B_\varepsilon(x), \text{ almost everywhere on } \mathbb{R}^2,$$

then

$$A_0(x) \leq B_0(x), \text{ almost everywhere on } \mathbb{R}^2 \bullet$$

A complete proof of Lemma 2 is given in Francfort & Murat (1987). It is based on an adequate rewriting of the set $\mathcal{M}(L_{\leq}(a,b))$ (respectively $\mathcal{M}(L_{\geq}(a,b))$), namely

$$\mathcal{M}(L_{\leq(\geq)}(a,b)) = \Big\{A \in \mathcal{M}([\alpha,\beta]^2) \;\Big|\; I \leq (\geq)\, a\, A(x) + b\, \frac{A(x)}{\det A(x)} \text{ for almost any } x \text{ of } \mathbb{R}^2\Big\}.$$

The actual proof uses Theorem 4 together with Lemma 4 (*cf.* Tartar (1979a)) and Lemma 5 (Murat (1976), Boccardo & Marcellini (1978)) stated below.

LEMMA 4. *Let* A_ε *belong to* $\mathcal{M}([\alpha,\beta]^2)$ *and* H-*converge to* A_0. *If*

$$A_\varepsilon \longrightarrow \overline{A}$$

weak-$\star$ *in* $[L_\infty(\mathbb{R}^2)]^4$ *as* ε *tends to zero, then*

$$A_0(x) \leq \overline{A}(x),$$

for almost any x *of* $\mathbb{R}^2 \bullet$

LEMMA 5. *Let* A_ε *belong to* $\mathcal{M}([\alpha,\beta]^2)$ *and* H-*converge to* A_0. *Let* w_ε *be a sequence of* $[L_2(\Omega)]^2$ *such that, as* ε *tends to zero,*

$$w_\varepsilon \longrightarrow w_0 \text{ weakly in } [L_2(\Omega)]^2,$$

while curl w_ε *lies in a compact set of* $H^{-1}_{loc}(\Omega)$. *Then, for any positive* φ *in* $C^\infty_o(\Omega)$,

$$\int_\Omega \varphi\, A_o w_o \cdot w_o dx \leq \mathit{lim} \int_\Omega \varphi\, A_\varepsilon w_\varepsilon \cdot w_\varepsilon \, dx \, \bullet$$

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