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Fourth-Order Moments of Nonnegative Measures on S^2 and Applications

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0. Introduction

This paper is mainly devoted to a study of the fifteen-dimensional set of all fourth-order moments of a nonnegative Radon (Borel regular) measure on S^2. We seek a complete characterization of the set, a closed cone, with the help of the Hilbert decomposition theorem for nonnegative polynomials of degree four in two variables. Special emphasis is placed on the boundary of this set which is shown to be generated by atomic measures made of five Dirac masses or fewer, which are located on the intersection of S^2 with the zero set of a quadratic form. As a consequence, every point of the moment set is generated by atomic measures made of six Dirac masses or less.

This study may be viewed as a contribution to the moment problem; note that the analogous two-dimensional case has been analyzed in AVELLANEDA & MILTON [AM]. Potential applications for this result are however manifold, especially in the field of homogenization, which is of especial interest to us. In the setting of linearized elasticity, effective properties of fine mixtures of two phases are investigated. Specifically, the goal is to analyze the coefficients of the linear second-order elliptic system satisfied by the weak limit u of the solution fields u^ε to a sequence of elasticity problems of the form

$$-\operatorname{div}(A^\varepsilon e(u^\varepsilon)) = f \text{ in } \Omega, \quad u^\varepsilon = 0 \text{ on } \partial\Omega, \tag{0.1}$$

with

$$e_{ij}(u^\varepsilon) = \frac{1}{2}\left(\frac{\partial u_i^\varepsilon}{\partial x_j} + \frac{\partial u_j^\varepsilon}{\partial x_i}\right) \text{ in } \Omega,$$

$$A^\varepsilon = \chi^\varepsilon A_1 + (1 - \chi^\varepsilon) A_2. \tag{0.2}$$

In (0.2), A and B are elasticity tensors and χ^ε is a given arbitrary sequence of characteristic functions on Ω, while in (0.1) f is a given arbitrary element of $H^{-1}(\Omega)$.

It is a well-known result in the theory of homogenization (cf. TARTAR [T]) that u satisfies

$$-\operatorname{div}(A^0 e(u)) = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

where

$$e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{in } \Omega,$$

and where A^0 is an elasticity tensor called the effective (or homogenized) tensor.

The only knowledge of the weak $*$ limit θ (in $L^\infty(\Omega; [0, 1])$) of χ^ε does not uniquely determine A^0: a whole set G_θ of possible effective tensors is generated. As of yet, no complete characterization of this set is available in spite of the abundant literature on the topic (see, e.g., HASHIN [H]).

A special class of mixtures — laminates — which correspond to a specific kind of characteristic functions χ^ε, those that oscillate only in one direction at each scale, proves fruitful. On the one hand, explicit formulae for the effective tensor are available for them (FRANCFORT & MURAT [FM']). On the other hand, whenever the original tensors are well ordered, i.e., whenever $A_1 \leqq A_2$, the resulting effective tensors are extreme among all microstructures (AVELLANEDA [A']), i.e., for any A^0 in G_θ there exist two tensors $\underline{A}$ and $\bar{A}$ associated with laminates such that

$$\underline{A} \leqq A^0 \leqq \bar{A} \tag{0.3}$$

in the sense of quadratic forms. Hence a thorough knowledge of such tensors provides useful information on the whole set G_θ.

The connection with the study of fourth-order moments stems from the actual expression for the effective tensor of a laminate. Assume that both phases are isotropic, i.e., that

$$A_1 = \lambda_1 i \otimes i + 2\mu_1 I \quad \text{with } \mu_1 > 0, \quad NK_1 = N\lambda_1 + 2\mu_1 > 0,$$
$$A_2 = \lambda_2 i \otimes i + 2\mu_2 I \quad \text{with } \mu_2 > 0, \quad NK_2 = N\lambda_2 + 2\mu_2 > 0$$

where i is the identity matrix of R^N and I that of $R_s^{N^2}$ (the space of symmetric matrices on R^N with its Euclidean structure). Then the effective tensors associated with (finite-rank) laminates (see Subsection 3.1 for further details) are given by

$$(1-\theta)(A^0 - A_1)^{-1}h$$
$$= (A_2 - A_1)^{-1}h + \frac{\theta}{\mu_1}\int_{S^{N-1}}\left(\frac{ha \otimes a + a \otimes ha}{2} - \frac{\lambda_1 + \mu_1}{\lambda_1 + 2\mu_1}(ha.a)a \otimes a\right)dv(a), \tag{0.4}$$

$h \in R_s^{N^2}$, with

$$dv = \sum_{i=1}^{p} \eta_i \delta_{a_i}, \quad a_i \in S^{N-1}, \quad 0 \leqq \eta_i \leqq 1, \quad \sum_{i=1}^{p} \eta_i = 1,$$

or the analogous formula obtained by permuting A_1 and A_2, and θ and $(1-\theta)$. Inspection of formula (0.4) immediately makes evident the role played by the fourth-order moments of atomic probability measures on S^{N-1}. In Section 2 we establish that such measures are generic from the standpoint of the fourth-order moments among all probability measures on S^{N-1}; this, in conjunction with (0.3), provides the desired link between the moment problem and homogenization and permits us to gain further insight into the structure of G_θ. In particular, we show (see Subsection 3.2) that rank-6 laminates suffice to generate $\underline{A}$ and $\bar{A}$ in (0.3).

The paper is organized as follows: Section 1 is entirely devoted to notation and to a review of the few algebraic results needed for the subsequent study. Section 2, the main part of the paper, is devoted to the study of the set of fourth-order moments of nonnegative Radon measures on S^{N-1}. In Subsection 2.1, we derive a few preliminary results that remain true in any dimension. First the generic character of atomic measures is established (Lemma 2.1). Then a result specific to the two- and three-dimensional cases provides a complete characterization of the set under study (Theorem 2.1). Subsection 2.2 addresses the two-dimensional case; previously known results (cf. [AM]) are rederived. Subsection 2.3 addresses the three-dimensional case; the main results are Theorem 2.3 and Remark 2.7, which pertain to the intimate structure of the boundary of that set. In Subsection 2.4, we consider measures that remain invariant under the action of either one of two symmetry groups and characterize the resulting fourth-order moments. Section 3 is devoted to applications of the results of Section 2 to homogenization in linearized elasticity. After a brief description of the concept of homogenization in Subsection 3.1, Subsection 3.2 illustrates with three examples the additional information made available through an effective use of the results of Section 2.

1. Notation and algebraic preliminaries

The following notation is adopted throughout;

$\mathscr{P}_{p,N}$ is the space of polynomials of degree less than or equal to p in N variables.
$\mathscr{P}^h_{p,N}$ is the subset of $\mathscr{P}_{p,N}$ of all homogeneous polynomials of degree p.
$\mathscr{P}^+_{p,N}$ is the subset of $\mathscr{P}_{p,N}$ of all nonnegative polynomials.
$M(S^{N-1})$ is the space of all Radon (Borel regular) measures on S^{N-1}.
$M^+(S^{N-1})$ is the subset of $M(S^{N-1})$ of all nonnegative measures.
$\Pi(S^{N-1})$ is the subset of $M^+(S^{N-1})$ of all probability measures.
$M_{p,N}$ is the set of pth-order moments of all elements of $M^+(S^{N-1})$.
$O(N)$ is the set of all orthogonal matrices on R^N.
$SO(N)$ is the subset of $O(N)$ of all rotations.
$R^{N^2}_s$ is the space of all symmetric matrices on R^N.
$i(\in R^{N^2}_s)$ is the identity matrix on R^N.
I is the identity matrix on $R^{N^2}_s$.
$M(\alpha, \beta) = \{A(x) \in L^\infty(R^N; \mathscr{L}(R^{N^2}_s, R^{N^2}_s)), \alpha I \leq A(x) \leq \beta I$
in the sense of quadratic forms$\}$.
$\otimes$ denotes the tensor product of two vectors or two matrices.

Our analysis of the moment problem relies on two classical results. The first is concerned with quadrature formulae for polynomials, while the second pertains to an old question of HILBERT on whether a nonnegative real polynomial is a sum of squares of real polynomials.

The quadrature result is merely stated and the interested reader is referred to [CM, Théorème 2.2].

Lemma 1.1. *Let π be a nonnegative measure on $[\alpha, \beta]$, $-\infty < \alpha, \beta < \infty$, whose support contains strictly more than four points in (α, β) and such that $\int_{[\alpha,\beta]}(1 + |\tau|^9)d\pi < \infty$. Then there exists exactly one quintuplet of distinct points $(a_1, \ldots, a_5)$ in (α, β) and one quintuplet of positive real numbers $(\rho_1, \ldots, \rho_5)$ such that*

$$\int_\alpha^\beta p\, d\pi = \sum_{i=1}^{5} \rho_i p(a_i) \tag{1.1}$$

for all $p \in \mathscr{P}_{9,1}$.

The support condition on π easily implies that the quantity $\int_\alpha^\beta pq d\pi$, with $p, q \in \mathscr{P}_{4,1}$, defines an inner product on $\mathscr{P}_{4,1}$, and the proof of Lemma 1.1 then reduces to that of Théorème 2.2 in [CM]. The positive character of the weights ρ_i, $1 \leqq i \leqq 5$, is readily verified by choosing $\prod_{1 \leqq j \leqq 5, j \neq i}(x - a_j)^2$ as a test function in (1.1).

In 1888, HILBERT [H′] produced a complete characterization of the real polynomials for which the announced property holds. The result is strikingly simple.

Theorem 1.1 (HILBERT). *$\mathscr{P}^+_{n,1}$, $\mathscr{P}^+_{2,n}$, $n \in N$, and $\mathscr{P}^+_{4,2}$ are the only sets of nonnegative real polynomials in which every element can be written as a sum of squares of real polynomials.*

Remark 1.1. In the context of Theorem 1.1, the number of elements in the sum may be taken to be 2 for $\mathscr{P}^+_{4,1}$ and 3 for $\mathscr{P}^+_{4,2}$ (cf. [CL, (1.1), (1.2)]). It is worth mentioning that HILBERT's proof of Theorem 1.1 in the cases $\mathscr{P}^+_{4,3}$ and $\mathscr{P}^+_{6,2}$ — from which all other cases $\mathscr{P}^+_{n,m}$ with $n \geqq 4$, $m \geqq 2$, $(n, m) \neq (4, 2)$, are easily deduced (see, e.g., [CL, (1.4)]) — is nonconstructive. The first simple explicit counterexamples seem to have been constructed in the late 1960's (see, e.g., [M′] or [CL]).

In any case, we are only interested here in the sets $\mathscr{P}^+_{4,1}$ and $\mathscr{P}^+_{4,2}$ for which the property holds. A simple proof of the precise result used for $\mathscr{P}^+_{4,2}$ is given in the Appendix.

2. Fourth-order moments of a nonnegative Radon measure on S^1 or S^2

This section is essentially devoted to the study of the set $M_{4,N}$ of fourth-order moments of all nonnegative Radon measures $M^+(S^{N-1})$ on S^{N-1}. The first subsection investigates a general property of all points of $M_{2p,N}, P \geqq 1$, namely, that they are obtained by measures supported on a finite number of points. A first characterization of the boundary $\partial M_{2p,N}$ is also proposed, and in the case where $N = 2$ or 3, $p = 2$, a complete characterization of $M_{4,N}$ is given in Theorem 2.1.

The second subsection proposes various characterizations of $M_{4,2}$ (cf. Theorem 2.2), while the third subsection, short of providing a handy characterization of $M_{4,3}$, presents a useful analysis of $\partial M_{4,3}$ (cf. Theorem 2.3). In particular, it is shown in Theorem 2.3 that all points on $\partial M_{4,3}$ are obtained by measures supported at five or fewer points of S^2.

Finally, in the last subsection we investigate the subsets of $M_{4,2}$ or $M_{4,3}$ corresponding to various symmetry restrictions.

2.1. *The generic character of atomic measures*

In any dimension N, let $m_{p,N}$ be the dimension of the space $\mathscr{P}^h_{p,N}$ of homogeneous polynomials of degree p in N variables:

$$m_{p,N} = \binom{N+p-1}{N-1} = \frac{(N+p-1)!}{p!(N-1)!}.$$

From now on we write $m_{p,N}$ as m when used as an index or an exponent. Let $p_1, \ldots, p_m$ be a basis of $\mathscr{P}^h_{p,N}$ and define the linear mapping F from $M(S^{N-1})$ into R^m as

$$F(\mu) = \langle \mu, p_i \rangle, \quad i = 1, \ldots, m_{p,N}.$$

F maps $M^+(S^{N-1})$ into a closed positive cone $M_{p,N}$ of R^m, while it maps the set $\Pi(S^{N-1})$ of all probability measures on S^{N-1} (a convex set which is compact in the weak $*$ topology of $M(S^{N-1})$) into a compact convex subset of $M_{p,N}$ which also lies in the hyperplane

$$\langle \mu, 1 \rangle = 1$$

if p is even, since $(\sum_{i=1}^N x_i^2)^{p/2}$ is a homogeneous polynomial of degree p with value 1 on S^{N-1}.

The following lemma, based on an argument of ARTSTEIN [A], holds for any N and p.

Lemma 2.1. *Any point in $M_{2p,N}$ yields a set of 2pth-order moments of a nonnegative measure on S^{N-1} whose support is made of at most $m_{2p,N}$ points.*

Proof. Let Q be a point in $M_{2p,N}$. Then $M_Q^+ := F^{-1}(Q) \cap M^+(S^{N-1})$ is a convex subset of $M(S^{N-1})$, which is also weak $*$ compact, because $\langle \mu, 1 \rangle$ is fixed whenever $\mu \in M_Q^+$. According to the Krein-Milman theorem, M_Q^+ is the closed convex hull of its extreme points. Let v be such an extreme point. If B_W is any ball in a finite-dimensional subspace W of $M(S^{N-1})$ of dimension greater than $m_{2p,N}$, then

$$v + \mathrm{B}_W \text{ is not included in } M^+(S^{N-1}). \tag{2.1}$$

Indeed, since $\dim W > m_{2p,N}$,

$$\operatorname{Ker}(F|_W) \neq 0,$$

and there would exist a segment $[-\tau, \tau]$, $\tau \neq 0$ with

$$[-\tau, \tau] \subset B_W.$$

Since $F(\tau) = 0$,

$$[\nu - \tau, \nu + \tau] \subset M^+(S^{N-1}) \cap F^{-1}(Q) = M_Q^+,$$

which contradicts the extremality of ν.

Thus ν does not belong to the interior of a face of dimension greater than $m_{2p,N}$ of $M^+(S^{N-1})$. Assume that the support of ν is not purely atomic. Then there exists a ν-measurable subset E of S^{N-1} with $\nu(E) > 0$ and no ν-atoms in E. Since ν is a Borel measure, E can be partitioned into m' ($m' > m_{2p,N}$) Borel sets $E_1, \ldots, E_{m'}$, with

$$\nu(E_j) > 0, \quad 1 \leqq j \leqq m'.$$

If χ_j denotes the characteristic function of the set E_j, then the measures $\nu_j = \chi_j \nu$, $1 \leqq j \leqq m'$, are linearly independent elements of $M^+(S^{N-1})$ and

$$\nu = \sum_{j=1}^{m'} \nu_j + \nu|_{S^{N-1}\setminus E} = \left(\sum_{j=1}^{m'} \chi_j\right)\nu + \nu|_{S^{N-1}\setminus E}. \tag{2.2}$$

Let W be the m'-dimensional subspace generated by $\nu_1, \ldots, \nu_{m'}$. Then, in view of (2.2) there exists a small ball B_W around 0 in W such that

$$\nu + B_W \subset M^+(S^{N-1})$$

(take $\sum_{j=1}^{m'} c_j \nu_j$ with c_j close to 1). But, in view of (2.1), this cannot be so. Consequently the support of ν is purely atomic, and the same argument would demonstrate that at most $m_{2p,N}$ points lie in the support of ν, so Lemma 2.1 is proved.

We now focus our attention on the boundary of $M_{2p,N}$ and derive

Lemma 2.2. *Any point on $\partial M_{2p,N}$ yields a set of 2pth-order moments such that all nonnegative measures with those moments are supported in the set of (double) zeroes of a nonnegative homogeneous polynomial of degree $2p$ on S^{N-1}. Furthermore it may be assumed that those zeroes lie strictly inside a hemisphere of S^{N-1}.*

Remark 2.1. The last statement of Lemma 2.2 deserves a short comment. A nonnegative measure μ on S^{N-1} has moments of order $2p$ that are indistinguishable from those of the nonnegative measure obtained by symmetrizing μ about the origin. Thus it may be assumed that the support of μ is symmetric about 0. Furthermore, these moments are also undistinguishable from those of the measure

$$\hat{\mu} = \mu|_{x_N=0} + 2\mu|_{x_N>0},$$

the support of which is contained in the northern hemisphere. The last statement of Lemma 2.2 pertains to measures of the latter form.

Proof of Lemma 2.2. Let Q be an arbitrary point in $\partial M_{2p,N}$ distinct from the origin and let $\mu \in M^+(S^{N-1})$ be such that $F(\mu) = Q$. Because $M_{2p,N}$ is convex, there exists at least one tangent hyperplane to $M_{2p,N}$ at Q. It can be viewed as a homogeneous polynomial P_Q of degree $2p$, together with a constant γ, such that

$$\langle \mu, P_Q \rangle = \gamma, \tag{2.3}$$

$$\langle v, P_Q \rangle \geqq \gamma, \quad v \in M^+(S^{N-1}). \tag{2.4}$$

Define $|\mu| := \langle \mu, 1 \rangle$, and note that $|\mu| > 0$. Setting

$$P'_Q(x) = P_Q(x) - \frac{\gamma}{|\mu|} |x|^{2p}, \quad x \in S^{N-1},$$

transforms (2.3), (2.4) into

$$\langle \mu, P'_Q \rangle = 0, \tag{2.5}$$

$$\langle v, P'_Q \rangle \geqq \gamma \left(1 - \frac{|v|}{|\mu|} \right), \quad v \in M^+(S^{N-1}). \tag{2.6}$$

The choice of an arbitrary Dirac mass weighted by $|\mu|$, i.e., $|\mu| \delta_a$ ($a \in S^{N-1}$), as a test measure in (2.6) implies that

$$|\mu| P'_Q(a) \geqq 0. \tag{2.7}$$

Since a is an arbitrary point on S^{N-1}, (2.7) is equivalent to

$$P'_Q \geqq 0 \quad \text{on } S^{N-1}. \tag{2.8}$$

Denote by Z_Q the set of all zeroes of P'_Q. By virtue of (2.5), (2.8), we conclude that

$$\operatorname{supp}(\mu) \subset Z_Q.$$

The homogeneous character of P'_Q, together with (2.8), implies that all elements of Z_Q are (double) zeros of P'_Q. (Conversely any nonnegative homogeneous polynomial of degree $2p$ defines a tangent hyperplane at all moments of measure μ whose support is contained in its zero set.)

Furthermore P'_Q is even; thus the set Z_Q of all its zeroes may be assumed to lie in the northern hemisphere $x_N \geqq 0$. If $a_N = 0$ for a point a in Z_Q, then we are at liberty to assume that $a_{N-1} \geqq 0$. Repeating this argument until a positive component of a is found, we have identified a set Z'_Q of zeroes of P'_Q which is such that

$$
\begin{aligned}
& Z'_Q \subset \{a \in S^{N-1}\colon a_N \geqq 0\}, \\
& \text{if } a \in Z'_Q \text{ with } a_N = \cdots = a_k = 0, \text{ then } a_{k-1} \geqq 0, \quad 2 \leqq k \leqq N, \\
& \text{if } a \in Z'_Q, \text{ with } a_N = \cdots = a_2 = 0, \text{ then } a_1 > 0.
\end{aligned} \tag{2.9}
$$

In view of (2.9), we immediately see that the origin does not belong to the closed convex hull of Z'_Q, which permits us to conclude the existence of an equatorial hyperplane $L(x) = 0$ such that $L > 0$ on Z'_Q and complete the proof of Lemma 2.2.

Remark 2.2. In the context of Lemma 2.2 and upon relabeling the Nth direction so that it is normal to the hyperplane $L(x) = 0$, we may take Z'_Q to be such that

$$a_N > 0, \quad a \in Z'_Q. \tag{2.10}$$

Under the transformation Γ_N from R^N into $R^N \cup \{+\infty\}$ defined as

$$\Gamma_N(x_1, \ldots, x_N) := \left(\frac{x_1}{x_N}, \ldots, \frac{x_{N-1}}{x_N}\right), \tag{2.11}$$

P'_Q is transformed into a nonnegative polynomial P''_Q of degree less than or equal to $2p$ in $(N-1)$ variables (an element of $\mathscr{P}^+_{2p,N-1}$), and, in view of (2.10), the points a/a_N, with $a \in Z'_Q$, are (double) zeroes of P''_Q.

Remark 2.3. In the two-dimensional case ($N = 2$) any element of $\mathscr{P}^+_{2p,1}$ has at most p double zeroes. Thus in the context of Remark 2.2,

$$\operatorname{card}(Z'_Q) \leqq p.$$

By virtue of Remark 2.2, the degree of the nonnegative homogeneous polynomial whose zero set contains the support of the inverse image under F of any point of $\partial M_{2p,N}$ may be drastically reduced whenever Hilbert's theorem is applicable. In the case of interest to us in the remainder of this study, p is taken to be equal to 2 and we obtain

Theorem 2.1. *If $N = 2$ or 3, the set $M_{4,N}$ is the set of all matrices F_{ijkm}, $1 \leqq i, j, k, m \leqq N$, invariant under any permutation of the indices and such that*

$$\sum_{i,j,k,m=1}^{N} F_{ijkm} A_{ij} A_{km} \geqq 0 \tag{2.12}$$

for all symmetric matrices A on R^N. Furthermore, the nonnegative measures on S^{N-1} whose moments of order 4 lie on $\partial M_{4,N}$ are those that are supported on the zero set of a homogeneous polynomial, say $R(x) = \sum_{i,j=1}^{N} B_{ij} x_i x_j$ of degree 2. Inequality (2.12) becomes an equality at such points of $\partial M_{4,N}$ for the symmetric matrix B.

Proof. We recall Remark 2.2 and conclude the existence, for any point Q of $\partial M_{4,N}$, of a nonnegative polynomial P''_Q of degree less than or equal to 4 in $N-1$ variables. If $N = 2$ or 3, then Hilbert's theorem (cf. Theorem 1.1) implies that P''_Q is a sum of squares of real polynomials. The same holds for P'_Q, which thus reads

$$P'_Q = \sum_{i=1}^{k_Q} R_i^2,$$

where R_i is a homogeneous polynomial of degree 2 and k_Q is an integer (which, in view of Remark 1.1, may be taken to be 2 if $N = 2$ or 3 if $N = 3$). Recalling (2.5) we obtain

$$\langle \mu, R_i^2 \rangle = 0, \quad i = 1, \ldots, k_Q. \tag{2.13}$$

Furthermore, note that if ν is an arbitrary element of $M^+(S^{N-1})$ and R is an arbitrary homogeneous polynomial of degree 2 (an element of $\mathscr{P}^h_{2,N}$), then

$$\langle \nu, R^2 \rangle \geqq 0. \tag{2.14}$$

Any element R of $\mathscr{P}^h_{2,N}$ has the form

$$R(x) = \sum_{i,j=1}^{N} A_{ij} x_i x_j, \tag{2.15}$$

with A a symmetric matrix on R^N. Thus, if we set

$$F(\nu)_{ijkm} = \int_{S^{N-1}} x_i x_j x_k x_m \, d\nu, \quad \nu \in M^+(S^{N-1}), \tag{2.16}$$

then relation (2.14) states that, for any ν in $M^+(S^{N-1})$ and any symmetric matrix A,

$$\sum_{i,j,k,m=1}^{N} F(\nu)_{ijkm} A_{ij} A_{km} \geqq 0, \tag{2.17}$$

while (2.13) is equivalent to the existence for each R_i of an associated symmetric matrix B such that

$$\sum_{i,j,k,m=1}^{N} F(\mu)_{ijkm} B_{ij} B_{km} = 0 \tag{2.18}$$

(Q is the point with components $F(\mu)_{ijkm}$ for the choice of $x_i x_j x_k x_m$ as the generating set for $\mathscr{P}^h_{4,N}$). Conversely, let A be any symmetric matrix; any nonnegative measure μ with its support on the zero set of R defined through (2.15) satisfies

$$\sum_{i,j,k,m}^{N} F(\mu)_{ijkm,} A_{ij} A_{km} = 0,$$

while (2.17) is obviously satisfied. Thus R^2 defines a tangent hyperplane to $M_{4,N}$ ($N = 2$ or 3).

We have thus established, when $N = 2$ or 3, a one-to-one correspondence between the hyperplanes tangent to $M_{4,N}$ and the symmetric matrices on R^N. Since a convex closed set ($M_{4,N}$) is the intersection of the closed half spaces located above its tangent hyperplanes, (2.17), (2.18) provide the desired characterization (2.12) of $M_{4,N}$ ($N = 2$ or 3), while consideration of (2.15) and (2.18) allows us to complete the proof of Theorem 2.1.

2.2. The case of $M_{4,2}$

In the two-dimensional case ($N = 2$) a characterization of $M_{4,2}$ can be proposed with the help of Theorem 2.1 and Remark 2.3 specialized to $p = 2$. This is the object of Theorem 2.2, which also describes the support of the measures whose moments lie on the boundary of $M_{4,2}$.

Theorem 2.2. *If $N = 2$, then the set*

$$M_{4,2} = \left\{ F_j = \int_{S^1} x_1^{4-j} x_2^j \, d\nu, \; j = 0, \ldots, 4; \; \nu \in M^+(S^1) \right\} \tag{2.19}$$

is the subset of all $F_0, \ldots, F_4$ in R^5 such that

$$F_0, F_2, F_4 \geqq 0, \quad F_1^2 \leqq F_0 F_2, \quad F_2^2 \leqq F_0 F_4, \quad F_3^2 \leqq F_2 F_4,$$
$$F_0 F_2 F_4 + 2F_1 F_2 F_3 - F_0 F_3^2 - F_2^3 - F_4 F_1^2 \geqq 0. \tag{2.20}$$

The elements of $\partial M_{4,2}$ are the fourth-order moments of the measures supported at at most two points of S^1.

Proof. We first establish (2.20). To this end we recall Theorem 2.1 and remark that a 2×2-symmetric matrix A is characterized by three coefficients:

$$A_{11} = \alpha, \quad A_{12} = A_{21} = \beta, \quad A_{22} = \gamma.$$

Upon recalling (2.12) and the definition of F_j in (2.19) (together with (2.16)), we obtain

$$\alpha^2 F_0 + 4\beta^2 F_2 + \gamma^2 F_4 + 4\alpha\beta F_1 + 2\alpha\gamma F_2 + 4\beta\gamma F_3 \geqq 0$$

for every triplet (α, β, γ) in R^3. Thus the matrix

$$\begin{pmatrix} F_0 & 2F_1 & F_2 \\ 2F_1 & 4F_2 & 2F_3 \\ F_2 & 2F_3 & F_4 \end{pmatrix}$$

must be nonnegative. But a 3×3 matrix is nonnegative if and only if its diagonal elements, its diagonal 2×2 minors, and its determinant are nonnegative; thus (2.20) must hold.

The remainder of Theorem 2.2 immediately follows upon recalling Remark 2.3 (specialized to $p = 2$) and Lemma 2.2.

Remark 2.4. Another characterization of $M_{4,2}$ may be found in [AM].

Remark 2.5. In the context of Theorem 2.2, assume that the inertia matrix of a given nonnegative measure μ is known. At the possible expense of a change of basis, we are at liberty to consider a diagonal matrix. Its diagonal elements are

$$I_1 = \int_{S^1} x_1^2 \, d\mu, \quad I_2 = \int_{S^1} x_2^2 \, d\mu, \tag{2.21}$$

and we suppose, with no loss of generality, that $I_1 \leqq I_2$. Then the set of the fourth-order moments of all nonnegative measures on S^1 with (2.21) as the inertia matrix is given by

$$F_2 = I_1 - F_0, \quad F_3 = -F_1, \quad F_4 = I_2 - I_1 + F_0,$$
$$I_1^2 \leqq F_0(I_1 + I_2), \quad F_0 \leqq I_1,$$
$$F_1^2(I_1 + I_2) \leqq (I_1 - F_0)(F_0(I_1 + I_2) - I_1^2),$$

as can be immediately checked by specializing (2.20) to the case under consideration and taking (2.21) into account.

Remark 2.6. Every point in the interior of $M_{4,2}$ can be attained by an element μ of $M^+(S^1)$ which is supported at three points, one of the points being chosen arbitrarily. Indeed, let ν be an arbitrary element of $M^+(S^1)$ such that its moments of order 4, $F(\nu)_j$, lie in the interior of $M_{4,2}$. If a_0 is an arbitrary point of S^1, the measure $\nu - t\delta_{a_0}$ $(t \geqq 0)$ — perhaps not an element of $M^+(S^1)$ — has moments of order 4 that form a half line $F(\nu) - tF(\delta_{a_0})$ which must intersect $\partial M_{4,2}$ for a value t_0 of t, because $M_{4,2}$ is a closed positive cone and thus contains no straight line, while $M_{4,2}$ certainly contains the half line $F(\nu) + tF(\delta_{a_0})$.

The application of Theorem 2.2 yields the existence of a measure

$$\bar{\nu} = \sum_{i=1}^{2} t_j \delta_{a_j}, \quad t_1, t_2 \geqq 0$$

such that

$$F(\nu) - t_0 F(\delta_{a_0}) = F(\bar{\nu}).$$

Hence

$$F(\nu) = \sum_{i=0}^{2} t_j F(\delta_{a_j}).$$

2.3. *The case of* $M_{4,3}$

The three-dimensional case is more intricate than its two-dimensional counterpart, and an analytical characterization of $M_{4,3}$ seems to be algebraically intractable. We shall however prove the following theorem, which may be viewed as the three-dimensional analogue of the part of Theorem 2.2 concerned with $\partial M_{4,2}$.

Theorem 2.3. *Assume that $N = 3$. All elements of $\partial M_{4,3}$ are the fourth-order moments of a measure supported at at most five points of S^2. If $M_{4,3}$ admits a single tangent hyperplane at the point of $\partial M_{4,3}$ under consideration, then the intersection of $\partial M_{4,3}$ with that hyperplane is a nine-dimensional closed positive cone of $M_{4,3}$.*

Remark 2.7. Theorem 2.3 can be enriched by adjoining the result of Theorem 2.1. The support of any element of $M^+(S^2)$ whose fourth-order moments lie at a given point Q of $\partial M_{4,3}$ must be contained in the intersection C_Q of the zero set of the homogeneous polynomial R_Q of degree 2 described in Theorem 2.1 (and denoted there by R) with S^2. Furthermore, since, according to Lemma 2.1, those zeroes may be taken to lie strictly inside a hemisphere of S^2, Remark 2.2 permits us to view the image of C_Q under the transformation Γ_3, defined in (2.11) by

$$\Gamma_3(x_1, x_2, x_3) := \left(\frac{x_1}{x_3}, \frac{x_2}{x_3}\right),$$

as lying on the image of R_Q under that transformation, i.e., on a conic section.

Further, if the support of an element μ of $M^+(S^2)$ has its image under Γ_3 lying on a conic section, then μ belongs to $\partial M_{4,3}$. In particular, all measures supported at five or fewer points of S^2 belong to $\partial M_{4,3}$.

Remark 2.8. If, in the context of Remark 2.7, the symmetric form B associated with R_Q has three nonzero eigenvalues, then Theorem 2.3 can be strengthened. Specifically, Q corresponds to the fourth-order moments of a one-parameter family of measures supported at five points of S^2.

Proof of Theorem 2.3. According to Theorem 2.1, specialized to the case $N = 3$, any point Q of $\partial M_{4,3}$ is the image under F of a measure μ of $M^+(S^2)$, whose support lies in (the intersection of) the zero sets of nonzero homogeneous quadratic polynomials. Let R_Q be such a polynomial. It is diagonalizable (or in the notation of Theorem 2.1, the associated matrix B_{ij} is diagonalizable) in an orthonormal basis of R^3. In such a basis, R_Q reads

$$R_Q(x_1, x_2, x_3) = \alpha x_1^2 + \beta x_2^2 + \gamma x_3^2.$$

Since the support of μ is not empty, one of the coefficients α, β, or γ is nonpositive. There is no loss of generality in assuming that

$$\alpha \geqq 0, \quad \beta \geqq 0, \quad \gamma \leqq 0.$$

Denote by C_Q the intersection of the zero set of R_Q with the sphere S^2. Three cases are distinguished.

Case 1. $\alpha > 0, \beta > 0, \gamma < 0$. Then C_Q is the intersection of the surface

$$x_3^2 = \alpha^* x_1^2 + \beta^* x_2^2, \quad \left(\alpha^* = -\frac{\alpha}{\gamma} > 0,\ \beta^* = -\frac{\beta}{\gamma} > 0\right) \tag{2.22}$$

with the sphere S^2. Since $C_Q \subset S^2$, we may replace x_3^2 by $1 - x_1^2 - x_2^2$ in the moments of order 4 of μ. Consequently nine moments are to be considered, namely, those with the integrands

$$x_1^4,\ x_1^3x_2,\ x_1^2x_2^2,\ x_1x_2^3,\ x_2^4,\ x_1^3x_3,\ x_1^2x_2x_3,\ x_1x_2^2x_3,\ x_2^3x_3.$$

Because C_Q is invariant under a sign change of any of the $x_i's$ $(1 \leqq i \leqq 3)$, the four sets of homogeneous polynomials

$$S_1 = \{x_1^4, x_1^2x_2^2, x_2^4\},$$
$$S_2 = \{x_1^3x_2, x_1x_2^3\},$$
$$S_3 = \{x_1^3x_3, x_1x_2^2x_3\},$$
$$S_4 = \{x_2^3x_3, x_1^2x_2x_3\}$$

are linearly independent from one another on C_Q.

Further assume that either S_2, S_3, or S_4 is not made of linearly independent polynomials on C_Q. Then C_Q is imbedded in a union of equatorial planes of the type

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad \delta x_1 + \eta x_2 = 0.$$

The same holds for S_1, except if $x_1^2x_2^2$ is a linear combination of x_1^4 and x_2^4, i.e., if

$$x_1^2x_2^2 = \zeta x_1^4 + \lambda x_2^4, \quad \zeta\lambda \neq 0.$$

In such a case, solving this second-degree equation for $(x_1/x_2)^2$ (or $(x_2/x_1)^2$), we end up with an equation of the type $\delta x_1 + \eta x_2 = 0$ (except when $x_1 = 0$ and $x_2 = 0$).

Thus S_i $(1 \leqq i \leqq 4)$ consists of linearly independent polynomials on C_Q unless C_Q is included in a finite union of equatorial planes. Such is not the case, as is easily checked by recalling (2.22), together with the constraint $x_3^2 = 1 - x_1^2 - x_2^2$.

We conclude that if $\alpha\beta\gamma < 0$, then

$$x_1^4, x_1^3x_2, x_1^2x_2^2, x_1x_2^3, x_2^4, x_1^3x_3, x_1^2x_2x_3, x_1x_2^2x_3, x_2^3x_3$$

are linearly independent on C_Q. (2.23)

Case 2. $\alpha = 0$ and $\beta \neq 0 \neq \gamma$, or $\alpha \neq 0 \neq \gamma$ and $\beta = 0$. The zeroes of R_Q lie on the union of two equatorial planes. If, for example, $\alpha = 0$, then the two planes are

$$x_2 = \pm\sqrt{-\frac{\gamma}{\beta}}\,x_3. \tag{2.24}$$

The measure μ introduced at the beginning of the proof is decomposed into two elements of $M^+(S^2)$, μ_1 and μ_2, each supported on one and only one equatorial plane.

Since μ_i, $i = 1, 2$, is supported on the intersection of a plane with S^2, it is supported on a circle. An appropriate change of coordinates (specific to i) reduces the analysis of μ_i, $i = 1, 2$, to the two-dimensional case. According to Remark 2.6, the moments $F(\mu_i)$ can be realized by an element μ_i^* of $M^+(S^1)$ supported at three points of S^1, one of the points being chosen arbitrarily. But the two circles corresponding to μ_1 and to μ_2 intersect at two (antipodal) points. Choosing one of these points as the third support point for the measures μ_1^* and μ_2^* permits us to construct a measure $\mu^* = \mu_1^* + \mu_2^*$ such that

$$F(\mu^*) = F(\mu) = Q,$$

while its support contains at most five points, which proves, in Case 2, the first assertion of Theorem 2.3.

Note that in the case where relation (2.24) holds, only nine moments are to be considered, namely,

$$1, x_1^2, x_1^4, x_1x_2, x_1^3x_2, x_1x_3, x_1^3x_3, x_2x_3, x_1^2x_2x_3,$$

and that an argument similar to that used in Case 1 would show that

$$1, x_1^2, x_1^4, x_1x_2, x_1^3x_2, x_1x_3, x_1^3x_3, x_2x_3, x_1^2x_2x_3, \text{ are linearly independent on } C_Q. \tag{2.25}$$

A conclusion similar to (2.25) would be reached if $\beta = 0$ instead of $\alpha = 0$.

Case 3. $\alpha = \beta = 0$ or $\gamma = 0$. Then the zeroes of R_Q lie on a single equatorial plane as well as on S^2. An appropriate change of coordinate reduces this case to the two-dimensional one; the zeroes of R_Q may thus be assumed to lie on

$$x_1^2 + x_2^2 = 1, \quad x_3 = 0.$$

Remark 2.6 applies and yields a measure μ^* supported at three points or fewer of S^1 such that

$$F(\mu^*) = F(\mu) = Q,$$

which proves, in Case 3, the first assertion of Theorem 2.3.

Note that there are many quadratic polynomials which are zero on the support of μ, for example, all quadratic expressions with a common x_3 term. In other words, the set $M_{4,3}$ admits several tangent hyperplanes at the point Q. By virtue of (2.23), (2.25) we infer that, if $M_{4,3}$ admits only one tangent hyperplane at the point Q, then the setting is that of Case 1 or Case 2, yielding nine linearly independent homogeneous integrands on C_Q. Therefore, when the point a of S^2 spans the curve C_Q, the moments $F(\lambda\delta_a)$, $\lambda \geqq 0$, span a nine-dimensional closed positive cone of R^{15}. Since $R_Q(a) = 0$, the point $F(\lambda\delta_a)$ belongs to $\partial M_{4,3}$ and the point Q belongs to a nine-dimensional closed positive cone. This establishes the second assertion of Theorem 2.3.

It remains to prove the first assertion of Theorem 2.3 in Case 1. To this end we recall (2.22) and propose a natural parametrization of C_Q, immediately derived from that of the conic section $\Gamma_3(C_Q)$ (cf. Remark 2.7). Specifically, we set

$$x_1(t) = \frac{S(t)\cos t}{\sqrt{\alpha^*}}, \quad x_2(t) = \frac{S(t)\sin t}{\sqrt{\beta^*}}, \quad x_3(t) = S(t), \quad -\pi \leqq t \leqq \pi \tag{2.26}$$

with

$$S(t) = \frac{1}{\sqrt{\dfrac{\cos^2 t}{\alpha^*} + \dfrac{\sin^2 t}{\beta^*} + 1}}.$$

Note that (2.22) is then identically satisfied and that

$$x_1(t)^2 + x_2(t)^2 + x_3(t)^2 = 1,$$

as it should be.

If we set $\tau = \tan(t/2)$, then (2.26) becomes

$$x_1(\tau) = \frac{(1-\tau^2)f(\tau)}{\sqrt{\alpha^*}}, \quad x_2(\tau) = \frac{2\tau f(\tau)}{\sqrt{\beta^*}}, \quad x_3(\tau) = (1+\tau^2)f(\tau), \qquad -\infty \leqq \tau \leqq +\infty, \tag{2.27}$$

with

$$f(\tau) = \frac{S(2\arctan\tau)}{(1+\tau^2)} = \frac{1}{\sqrt{\dfrac{(1-\tau^2)^2}{\alpha^*} + \dfrac{4\tau^2}{\beta^*} + (1+\tau^2)^2}}. \tag{2.28}$$

Under the parametrization (2.27), a homogeneous polynomial $P(x_1, x_2, x_3)$ of degree 4 in x_1, x_2, x_3 becomes a polynomial $q(\tau)$ of degree between 1 and 8, multiplied by $f^4(\tau)$ with $f(\tau)$ defined in (2.28), i.e.,

$$P(x_1(\tau), x_2(\tau), x_3(\tau)) = q(\tau)f^4(\tau), \quad 1 \leqq d^0(q) \leqq 8. \tag{2.29}$$

We introduce the mapping T from S^2 into $\bar{R}_+$ defined as

$$T(x_1, x_2, x_3) := \frac{\sqrt{|x_3 - \sqrt{\alpha^*}x_1|}}{\sqrt{|x_3 + \sqrt{\alpha^*}x_1|}}. \tag{2.30}$$

Its restriction to C_Q is the inverse of the mapping $\tau \mapsto (x_1(\tau), x_2(\tau), x_3(\tau))$.

If μ_T denotes the image measure of a measure μ supported on C_Q under T defined through (2.30), then by virtue of (2.29), we have

$$\int_{S^2} P\, d\mu = \int_{-\infty}^{+\infty} q(\tau)f^4(\tau)d\mu_T. \tag{2.31}$$

We define the measure $\pi_T = f^4(\tau)\mu_T$, so that (2.31) becomes

$$\int_{S^2} P\, d\mu = \int_{-\infty}^{+\infty} q(\tau)d\pi_T. \tag{2.32}$$

We now appeal to a variant of Lemma 1.1 and conclude that, if the support of π_T contains strictly more than four points, then there exists a one-parameter family of measures π_T^λ with

$$\pi_T^\lambda = \sum_{j=1}^{5} \alpha_j^\lambda \delta_{\tau_j}^\lambda, \qquad -\infty < \tau_j^\lambda < +\infty, \ \alpha_j^\lambda \geqq 0, \ j = 1, \ldots, 5,$$

such that the moments of order less than or equal to 8 of π_T^λ are, for every value of λ, those of π_T. Notice that although we only know that $\int(1 + |\tau|^8)d\pi_T < \infty$, we however only need a quadrature formula which is exact on $\mathscr{P}_{8,1}$. Let L_4 and L_5 be two polynomials of degrees exactly 4 and 5 which are orthogonal to $\mathscr{P}_{3,1}$ (for the scalar product defined by $\langle f, g\rangle = \int f(\tau)g(\tau)d\pi$, for which one cannot always compute the scalar product of L_4 and L_5 since L_4L_5 is a polynomial of degree 9 and not of degree 8). Then for $\lambda \in R$, the points $\tau_j^\lambda, j = 1, \ldots, 5$ are the zeroes of $L_5 + \lambda L_4$, with the corresponding weights which give a quadrature formula exact on $\mathscr{P}_{4,1}$, automatically exact on $\mathscr{P}_{8,1}$ by the orthogonality property. To each τ_j^λ there corresponds a unique point α_j^λ on C_Q through (2.27). Consider the measure

$$\mu^\lambda = \sum_{i=1}^{5} \frac{\alpha_j^\lambda}{f^4(\tau_j^\lambda)} \delta_{\alpha_j^\lambda}.$$

According to (2.32), for any element of $\mathscr{P}_{4,3}^h$ one has

$$\int_{S^2} P\, d\mu = \int_{-\infty}^{+\infty} q(\tau) d\pi_T = \int_{-\infty}^{+\infty} q(\tau) d\pi_T^\lambda = \sum_{j=1}^{5} \alpha_j^\lambda q(\tau_j^\lambda) = \sum_{j=1}^{5} \frac{\alpha_j^\lambda}{f^4(\tau_j^\lambda)} f^4(\tau_j^\lambda) q(\tau_j^\lambda)$$

$$= \sum_{j=1}^{5} \frac{\alpha_j^\lambda}{f^4(\tau_j^\lambda)} P(x_1(\tau_j^\lambda), x_2(\tau_j^\lambda), x_3(\tau_j^\lambda)) = \sum_{j=1}^{5} \frac{\alpha_j^\lambda}{f^4(\tau_j^\lambda)} P(\alpha_j^\lambda) = \int_{S^2} P\, d\mu^\lambda,$$

where the third equality holds because $d^0(q) \leqq 8$ and the fourth because all the τ_j^λ's $(1 \leqq j \leqq 5)$ are finite.

We have thus proved that in Case 1,

$$Q = F(\mu) = F(\mu^\lambda),$$

where μ^λ is a one-parameter family of elements of $M^+(S^2)$ supported at five points of S^2, which establishes the first assertion in Case 1 and completes the proof of Theorem 2.3 and of Remark 2.8.

Remark 2.9. If in the context of Theorem 2.3, the point Q of $\partial M_{4,3}$ admits two distinct tangent hyperplanes, then it corresponds to the fourth-order moments of a measure in $M^+(S^2)$ supported at four points at most. Indeed there exist two nonzero homogeneous quadratic polynomials R_{1Q} and R_{2Q} with common zeroes. The number of zeroes is thus limited to four unless both polynomials are degenerate and admit a common affine factor L. Then all the zeroes lie on the equatorial plane $L = 0$. Otherwise, setting

$$R_{1Q} = LL_1, \quad R_{2Q} = LL_2,$$

with L_1 and L_2 affine, we would conclude that L_1 and L_2 have a common zero; thus either $L_1 = L_2$ and $R_{1Q} = R_{2Q}$, which contradicts the premise, or the support of μ belongs to $L = 0$ or to the intersection of $L_1 = 0$ with $L_2 = 0$, i.e., a line passing through the origin. But then Q admits an infinity of tangent hyperplanes. Consequently the support of μ is contained in the equatorial plane $L = 0$. But this case corresponds to Case 3 in the proof of Theorem 2.3, and in such a case, $M_{4,3}$ admits an infinity of tangent hyperplanes, which once again is a contradiction.

Remark 2.10. The three-dimensional counterpart of Remark 2.6 is also true. Every point in the interior of $M_{4,3}$ can be attained by an element μ of $M^+(S^2)$ supported at six points of S^2, one of the points being chosen arbitrarily. The proof of this assertion is identical to its two-dimensional analogue, with Theorem 2.3 replacing Theorem 2.2.

2.4. Symmetry restrictions

This short subsection illustrates the previous results. Specifically, Theorem 2.3 and Remarks 2.7 and 2.10 are used to characterize subsets of $M_{4,3}$ that correspond to fourth-order moments of nonnegative measures that remain invariant under certain subgroups of $SO(3)$.

As a first example, consider the isotropic measure da, which is invariant under the action of $SO(3)$ itself. On S^2, consider the north pole, together with five equidistributed points on the intersection of S^2 with a plane located at $1/\sqrt{5}$ above the equatorial plane. Note that the six directions defined in such a manner are those of the northern vertices of a regular icosahedron. Their coordinates in the canonical basis of R^3 are

$$a_1 = (0, 0, 1), \quad a_{i+1} = (\sin 2\beta \cos 2i\alpha, \sin 2\beta \sin 2i\alpha, \cos 2\beta), \quad i = 1, \ldots, 5$$

with

$$\alpha = \frac{\pi}{5}, \quad \beta = \frac{1}{2} \arccos\left(\frac{1}{\sqrt{5}}\right).$$

Remark 2.11. The six directions $a_1, \ldots, a_6$ are precisely those used in FRANCFORT & MURAT [FM′, Subsection 4.2] to demonstrate that HASHIN & SHTRIKMAN's bounds on the bulk and shear moduli of an isotropic two-phase composite are optimal. See Section 3, Corollary 3.1, for further details.

A mere algebraic computation would demonstrate that the measure

$$\mu = \tfrac{1}{6} \sum_{i=1}^{6} \delta_{a_i} \tag{2.33}$$

has the same fourth-order moments as the normalized Lebesgue measure $da/4\pi$, i.e., that

$$F\left(\frac{da}{4\pi}\right) = F(\mu).$$

If $F(da)$ were to belong to $\partial M_{4,3}$, then, according to Remark 2.7, the image of the set $A := \{a_i, i = 1, \ldots, 6\}$ under Γ_3 would lie on a conic section. The coordinates of the points in $\Gamma_3(A)$ are

$$\Gamma_3(a_1) = (0, 0), \quad \Gamma_3(a_{i+1}) = \tan 2\beta(\cos 2i\alpha, \sin 2i\alpha), \quad i = 1, \ldots, 5$$

and these points cannot lie on a conic section since the last five points are distinct and belong to half a circle whose center is the first point.

Thus, by virtue of Remark 2.7,

$$F(da) \notin \partial M_{4,3}$$

and the number of Dirac masses in (2.33) is minimal. We have then proved

Corollary 2.1. *The fourth-order moments of the Lebesgue measure on S^2 belong to the interior of $M_{4,3}$ and cannot be obtained as fourth-order moments of any element of $M^+(S^2)$ supported at fewer than six points of S^2. The construction* (2.33) *is then optimal.*

As a second example we seek a complete characterization of all fourth-order moments of nonnegative *transversely isotropic measures*, i.e., of all measures which remain invariant under any rotation about a given axis (which we take to be the x_3-axis).

The assumed symmetry dramatically reduces the number of independent moments of order 4. In fact, three independent nonnegative numbers characterize the set of fourth-order moments, namely,

$$\begin{aligned} a &= \int_{S^2} x_1^4 d\mu = \int_{S^2} x_2^4 d\mu = 3 \int_{S^2} x_1^2 x_2^2 d\mu, \\ b &= \int_{S^2} x_1^2 x_3^2 d\mu = \int_{S^2} x_2^2 x_3^2 d\mu, \\ c &= \int_{S^2} x_3^4 d\mu, \end{aligned} \tag{2.34}$$

all other moments being equal to zero. This last statement is immediately established by rewriting x_1 as $(x.e)$ and x_2 as $(x.e^\perp)$ with $|e| = |e^\perp| = 1$, e orthogonal to $e^\perp$, e and $e^\perp$ orthogonal to e_3 (the unit vector in the x_3 direction). Thus, for example,

$$\begin{aligned} \int_{S^2} x_1^2 x_2^2 d\mu &= \int_{S^2} (x.e)^2 (x.e^\perp)^2 d\mu = \frac{1}{2\pi} \int_{e \perp e_3, |e|=1} \left(\int_{S^2} (x.e)^2 (x.e^\perp)^2 d\mu \right) de \\ &= \frac{1}{2\pi} \int_{S^2} \left(\int_{e \perp e_3, |e|=1} (x.e)^2 (x.e^\perp)^2 de \right) d\mu \\ &= \frac{1}{2\pi} \int_{S^2} (x_1^2 + x_2^2)^2 \left(\int_0^{2\pi} \cos^2\theta \sin^2\theta \, d\theta \right) d\mu \\ &= \frac{1}{8} \int_{S^2} (x_1^2 + x_2^2)^2 d\mu \end{aligned}$$

where the second equality holds because of the transversely isotropic character of μ and the third results from Fubini's theorem. Thus

$$\int_{S^2} x_1^2 x_2^2 d\mu = \tfrac{1}{3} \int_{S^2} x_1^4 d\mu = \tfrac{1}{3} \int_{S^2} x_2^4 d\mu$$

which establishes (2.34). Since $x_1^2 + x_2^2 = 1 - x_3^2$ on S^2, we obtain

$$a = \tfrac{3}{8}(|\mu| - 2J_2 + J_4), \quad b = \tfrac{1}{2}(J_2 - J_4), \quad c = J_4,$$

where

$$J_2 = \int_{S^2} x_3^2 d\mu, \quad J_4 = \int_{S^2} x_3^4 d\mu,$$

i.e.,

$$|\mu| = \tfrac{8}{3} a + 4b + c, \quad J_2 = 2b + c, \quad J_4 = c. \tag{2.35}$$

For a given total mass $|\mu|$, the set of all possible pairs (J_2, J_4) as μ varies over all nonnegative measures on S^2 with fixed total mass is given by

$$0 \leqq J_2 \leqq |\mu|, \quad J_2^2 \leqq |\mu| J_4 \leqq |\mu| J_2. \tag{2.36}$$

In view of (2.34)–(2.36), the set of values for a, b, c is

$$a, b, c \geqq 0, \quad ac \geqq \tfrac{3}{2} b^2. \tag{2.37}$$

We have thus proved

Corollary 2.2. *The set of all fourth-order moments of the nonnegative transversely isotropic measures with fixed transverse axis, namely, the x_3 axis, is characterized by three nonnegative real numbers a, b, c satisfying*

$$\tfrac{3}{2} b^2 \leqq ac$$

with

$$a = \int_{S^2} x_1^4 d\mu = \int_{S^2} x_2^4 d\mu = 3 \int_{S^2} x_1^2 x_2^2 d\mu,$$

$$b = \int_{S^2} x_1^2 x_3^2 d\mu = \int_{S^2} x_2^2 x_3^2 d\mu,$$

$$c = \int_{S^2} x_3^4 d\mu,$$

and with all other moments equal to zero.

Remark 2.12. Note that this last result should not, strictly speaking, be labeled a corollary since its derivation, which is elementary, did not appeal to the previously obtained results.

3. Fourth-order moments and homogenization in linearized elasticity

The knowledge, acquired in Section 2, of the intimate structure of $M_{4,2}$ and $M_{4,3}$ has immediate consequences in the field of homogenization when applied to linearized elasticity. It is not our purpose to discuss that theory in great details. We merely recall the few needed definitions and theorems in Subsection 3.1. One of the outstanding problems in "elastic homogenization" is that of finding bounds — and whenever possible, optimal bounds — on various quantities involving the effective tensor (the limit in the sense of homogenization) associated with (sequences of) two-phase mixtures of isotropic elastic materials (cf., e.g., MILTON [M] for a compendium of available results). We demonstrate in Subsection 3.2 how the results of Section 2 permit us to better circumscribe the task at hand.

3.1. H-convergence: a brief review in the framework of linearized elasticity

Homogenization theory aims at describing the weak limits of solutions to partial differential equations with oscillating coefficients. Within the framework of linearized elasticity, the following definition and theorem are applied to an arbitrary sequence A^ε of elasticity tensors in $M(\alpha, \beta)$ (defined in Section 1).

Definition 3.1. A sequence A^ε in $M(\alpha, \beta)$ H-*converges* to A^0, an element of $M(\alpha, \beta)$, if and only if for every bounded domain Ω of R^N and any element f in $(H^{-1}(\Omega))^N$ the solution u^ε — unique in $(H_0^1(\Omega))^N$ — of

$$-\operatorname{div}(A^\varepsilon e(u^\varepsilon)) = f \text{ in } \Omega, \quad u^\varepsilon = 0 \text{ on } \partial\Omega,$$

with

$$e_{ij}(u^\varepsilon) = \frac{1}{2}\left(\frac{\partial u_i^\varepsilon}{\partial x_j} + \frac{\partial u_j^\varepsilon}{\partial x_i}\right) \quad \text{in } \Omega,$$

is such that

$$u^\varepsilon \rightharpoonup u^0 \quad \text{weakly in } (H_0^1(\Omega))^N,$$

$$A^\varepsilon e(u^\varepsilon) \rightharpoonup A^0 e(u^0) \quad \text{weakly in } (L^2(\Omega))^N,$$

where u^0 is the solution — unique in $(H_0^1(\Omega))^N$ — of

$$-\operatorname{div}(A^0 e(u^0)) = f \text{ in } \Omega, \quad u^0 = 0 \text{ on } \partial\Omega,$$

with

$$e_{ij}(u^0) = \frac{1}{2}\left(\frac{\partial u_i^0}{\partial x_j} + \frac{\partial u_j^0}{\partial x_i}\right) \quad \text{in } \Omega.$$

A^0 is called the H-*limit* of the sequence A^ε.

Theorem 3.1. (TARTAR [T]). *Consider a a family A^ε of elements of $M(\alpha, \beta)$. There exists a subsequence of A^ε which H-converges to an element A^0 of $M(\alpha, \beta)$.*

In physical terms, Definition 3.1 proposes a mathematical conceptualization of the intuitive notion of effective behavior while Theorem 3.1 asserts the existence of the notion of effective properties for any kind of microscopically heterogeneous material.

A specific type of sequence is of particular interest in applications, that corresponding to mixtures of two phases. If A_1 and A_2 denote the elasticity tensors associated with each phase, the sequence A^ε has the form

$$A^\varepsilon(x) = \chi^\varepsilon(x) A_1 + (1 - \chi^\varepsilon(x)) A_2, \tag{3.1}$$

where χ^ε is the characteristic function of the A-phase for fixed ε. Assume that it is a priori known that

$$\chi^\varepsilon \rightharpoonup \theta \quad \text{weak} * \text{ in } L^\infty(R^N) \tag{3.2}$$

as ε tends to zero. Then the problem of bounds is: What is the set G_θ of all possible H-limits of sequences of the form (3.1) — the existence of such H-limits is guaranteed by Theorem 3.1 — for a given weak $*$ limit θ (a given local volume fraction of the A_1 phase)? Note that H-limits are actually local, so that it may be assumed, without loss of generality, that θ is a constant element of [0, 1] (cf. Dal Maso & Kohn [DK]).

This problem, sometimes referred to as the G_θ-closure problem, is the subject of a vast literature; the reader is referred to Hashin [H] for an application-oriented overview and to Milton [M] for a more theoretical standpoint. It remains an unsolved problem, even in the simplest case where A_1 and A_2 are both isotropic, i.e., when

$$\begin{aligned} A_1 &= K_1 i \otimes i + 2\mu_1\left(I - \frac{i \otimes i}{N}\right), \quad K_1 \equiv \lambda_1 + \frac{2\mu_1}{N}, \\ A_2 &= K_2 i \otimes i + 2\mu_2\left(I - \frac{i \otimes i}{N}\right), \quad K_2 \equiv \lambda_2 + \frac{2\mu_2}{N}, \end{aligned} \tag{3.3}$$

and when only isotropic H-limits are considered. If A^0 is such a H-limit, it has the form

$$A^0 = K i \otimes i + 2\mu\left(I + \frac{i \otimes i}{N}\right),$$

and optimal bounds are known on K and μ separately whenever A_1 and A_2 are well ordered, i.e., whenever

$$K_1 \leqq K_2, \quad \mu_1 \leqq \mu_2.$$

These are the celebrated Hashin-Shtrikman bounds (cf., e.g., Hashin & Shtrikman [HS], Francfort & Murat [FM′], Milton & Kohn [MK]).

The optimal character of the bounds has been demonstrated through the use of a special kind of composite (a special kind of characteristic functions): multiple rank laminates [FM′]. The specific characteristic function is

$$\chi^\varepsilon(x) = \tilde{\chi}^\varepsilon((x.a))$$

where $a \in S^{N-1}$ and $\tilde{\chi}^\varepsilon$ is a sequence of characteristic functions with θ as weak $*$ limit. Then the associated sequence A^ε (cf. (3.1)) is shown (cf. [FM′], Corollary 4.1) to H-converge to A^0 given by

$$(1-\theta)(A^0-A_1)^{-1}h$$

$$= (A_2-A_1)^{-1}h + \frac{\theta}{\mu_1}\left(\frac{ha\otimes a + a\otimes ha}{2} - \frac{\lambda_1+\mu_1}{\lambda_1+2\mu_1}(ha.a)a\otimes a\right), \quad h\in R_s^{N^2}. \tag{3.4}$$

Note that it is implicitly assumed in (3.4) that A_1 is isotropic and given by (3.3) and that $A_2 - A_1$ is invertible. A more general formula holds without such restrictions. The resulting A^0 is a rank-1 laminate. Similarly a rank-p laminate is given by

$$(1-\theta)(A^0-A_1)^{-1}h = (A_2-A_1)^{-1}h$$

$$+\frac{\theta}{\mu_1}\sum_{i=1}^{p}\eta_i\left(\frac{ha_i\otimes a_i + a_i\otimes ha_i}{2} - \frac{\lambda_1+\mu_1}{\lambda_1+2\mu_1}(ha_i.a_i)a_i\otimes a_i\right), \quad h\in R_s^{N^2}, \tag{3.5}$$

with

$$\eta_i \geqq 0,\ i=1,\ldots,p;\ \sum_{i=1}^{p}\eta_i = 1.$$

The vectors a_i, $i = 1, \ldots, p$, are the directions of lamination and θ is the total volume fraction of the A_1-phase in the resulting mixture. The phase with elasticity tensor A_1 in formulae (3.4), (3.5) is the matrix phase, that with A_2 the inclusion phase. Rank-p layers with the phase with elasticity tensor A_2 as matrix phase are defined in a similar manner.

It was shown in [FM′, Proposition 4.3] that, in a three-dimensional setting, six suitably chosen directions of lamination give rise to an isotropic H-limit that saturates both bounds on K and μ simultaneously. Corollary 3.1 below establishes that this number is optimal.

As already mentioned the full G_θ-closure, or even the full G-closure (the union of the G_θ-closures as θ varies between 0 and 1) is not known. A simpler problem with many applications, most notably in the fields of relaxation and structural optimization, is the investigation of quantities such as

$$\sup_{A\in G_\theta}\ (\text{or} \inf_{A\in G_\theta})\tfrac{1}{2}\sum_{i=1}^{p}(Ae_i.e_i), \quad e_i\in R_s^{N^2}.$$

These are called energy bounds (cf., e.g., ALLAIRE & KOHN [AK]). Whenever $p = 1$, the problem is well understood (cf. [AK] or [FM]). Bounds are known and their optimality is obtained — at least in the well-ordered case, i.e., when $A_1 \leqq A_2$ as quadratic forms — by virtue of the following result:

Theorem 3.2 (AVELLANEDA [A′]). *Assume that A_1 and A_2 are isotropic (cf. (3.3)) and that $A_1 \leqq A_2$. Let Γ be a subgroup of $O(N)$, the group of orthogonal matrices, and* *consider the subset $G_\theta(\Gamma)$ of G_θ of all possible H-limits of A^0 of the form* (3.1), (3.2) *that remain invariant under the action of Γ. For every element A^0 of $G_\theta(\Gamma)$, there exist two finite-rank laminates with elasticity tensors $\underline{A}$ and $\bar{A}$ such that $\underline{A}$ and $\bar{A}$ remain invariant under the action of Γ and satisfy*

$$\underline{A} \leqq A^0 \leqq \bar{A}.$$

Moreover, $\underline{A}$ corresponds to a finite-rank laminate with the A_1-phase as matrix phase, and $\bar{A}$ to a finite-rank laminate with the A_2-phase as matrix phase.

Corollary 3.2 delivers the rank of such a laminate in the case $N = 3$ without any symmetry restrictions ($\Gamma = i$); the case $N = 2$ was first derived by AVELLANEDA & MILTON [AM]. Corollary 3.3 treats the case where $N = 3$ and where Γ corresponds to rotations about a given axis, the x_3 axis.

3.2. *A few results about bounds*

This last subsection refers extensively to the terminology and to the results mentioned in Subsection 3.1.

First, it should be noted that the finite-rank lamination formula (3.5) reads

$$(1-\theta)(A^0 - A_1)^{-1}h$$
$$= (A_2 - A_1)^{-1}h + \frac{\theta}{\mu_1}\int_{S^{N-1}} \left(\frac{(ha \otimes a + a \otimes ha)}{2} - \frac{\lambda_1 + \mu_1}{\lambda_1 + 2\mu_1}(ha.a)a \otimes a\right) d\mu(a), \tag{3.6}$$

where $\mu = \sum_{i=1}^{p} \eta_i \delta_{a_i}$ is a probability measure on S^{N-1}. One of the main (although not explicitly stated) ingredients in the proof of Theorem 3.2 in AVELLANEDA [A′] is the following remark, which becomes obvious in view of Subsection 2.1.

Remark 3.1. According to Lemma 2.1, all fourth-order moments of a probability measure on S^{N-1} can be achieved by an atomic measure with a finite number of atoms. Thus formulae (3.5) and (3.6) yield the same set of tensors A^0 and can be used interchangeably.

If both phases, as well as the resulting effective tensor A^0, are isotropic, then the tensor X, defined by

$$Xh = \frac{\theta}{\mu_1}\int_{S^{N-1}} \left(\frac{(ha \otimes a + a \otimes ha)}{2} - \frac{\lambda_1 + \mu_1}{\lambda_1 + 2\mu_1}(ha.a)a \otimes a\right) d\mu(a), \quad h \in R_s^{N^2},$$

must be isotropic. Hence, upon setting

$$X = \Lambda i \otimes i + 2MI, \quad N\Lambda + 2M \equiv NK \tag{3.7}$$

and choosing h to be of the form

$$h = \tfrac{1}{2}(e_i \otimes e_j + e_j \otimes e_i), \quad 1 \leqq i, j \leqq N,$$

we see that the fourth-order moment tensor J defined as

$$J_{ijkm} = \int_{S^{N-1}} a_i a_j a_k a_m \, d\mu(a), \quad 1 \leqq i, j, k, m \leqq N,$$

is easily identified as

$$J_{ijkm} = \frac{\lambda_1 + 2\mu_1}{2(\lambda_1 + \mu_1)}((NK_1(\lambda_1 + 2\mu_1) - 2M\mu_1)(\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk}) - 2\mu_1 A \delta_{ij}\delta_{km}). \tag{3.8}$$

We remark that, because of the symmetry properties of J, together with the relation

$$\sum_{i,k=1}^{N} J_{iikk} = 1,$$

(3.8) holds if and only if

$$K = \frac{1}{N^2(\lambda_1 + 2\mu_1)}, \quad M = \frac{K_1 + 2\mu_1}{2(N+2)\mu_1(\lambda_1 + 2\mu_1)}, \tag{3.9}$$

in which case,

$$J_{ijkm} = \frac{1}{N(N+2)}(\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk} + \delta_{ij}\delta_{km}). \tag{3.10}$$

In the notation of Section 2, (3.10) reads

$$F(d\mu) = F\left(\frac{da}{|S^{N-1}|}\right), \tag{3.11}$$

where $|S^{N-1}|$ denotes the surface area of S^{N-1} and da the Lebesgue measure on S^{N-1}.

Further, according to [FM′, Subsection 4.2], there exists a unique tensor X of the form (3.7) compatible with the finite-rank lamination formula (3.5), in the sense that the resulting tensor A^0 satisfies

$$(1-\theta)(A^0 - A_1)^{-1} = (A_2 - A_1)^{-1} + \theta X. \tag{3.12}$$

Equation (3.12), in which A_1 and A_2 are isotropic, yields the value of X through formulae (4.30) and (4.31) of [FM′], and the resulting Lamé coefficients K and M are immediately checked to be those determined in (3.9).

Recalling (3.11) we have thus proved that finite-rank layering of two isotropic materials with the A_1 phase as matrix phase may at most produce — at fixed volume fraction — one isotropic tensor A^0; the tensor A^0 is given through (3.6) with

$$d\mu = \frac{da}{|S^{N-1}|}. \tag{3.13}$$

Remark 3.2. Application of the finite-rank layering formula with the A_2-phase as the matrix phase would yield an analogous result.

Remark 3.3. The only two possible isotropic tensors A_1 that could be produced through finite-rank layering (cf. Remark 3.2) are known to achieve Hashin-Shtrikman bounds in both bulk and shear (K and μ); see [FM′, Subsection 4.2]. Note however that only a specific kind of multiple-rank laminate has been considered, namely, that where the same phase is used as a pivot at each layering stage (cf. (3.5)).

We now restrict our attention to the three-dimensional setting and recall Corollary 2.1. The following corollary is then an immediate consequence of (3.6), (3.13).

Corollary 3.1. *The number of directions (six) used in* [FM′, *Proposition 4.3*] *to generate through finite-rank lamination an isotropic effective material (whose bulk and shear moduli saturate Hashin-Shtrikman bounds) in three dimensions is optimal. Those directions are given by the north pole together with five equidistributed points on the intersection of* S^2 *with a plane located at* $1/\sqrt{5}$ *above the equatorial plane.*

Next, Remark 2.10 immediately yields a corollary to Theorem 3.2 in [A′]. Specifically we obtain

Corollary 3.2. *In the context of Theorem 3.2, the two finite-rank laminates* $\underline{A}$ *and* $\bar{A}$ *may be chosen to be at most of rank* 6 *when* $N = 3$.

Remark 3.4. By virtue of Corollary 3.2, the infimum or supremum over $A \in G_\theta$ of $\frac{1}{2}\sum_{i=1}^{p}(Ae_i . e_i)$, with $e_i \in R_s^{N^2}$, is always achieved by rank-6 laminates in the three-dimensional setting (rank-3 in the two-dimensional analogue). Note that, in the case $p = 1$, it is implicit in [AK] that rank-N laminates are optimal, for any N. The best possible lamination rank for $N \neq 1, 2, 3$ and $p > 1$ is not known. Note also that we still do not have a simple characterization of the set of all tensors A^0 satisfying $\underline{A} \leqq A^0 \leqq \bar{A}$.

We conclude this study with a brief incursion into the set of laminates with transverse isotropic symmetry. This problem was examined at length in Lipton [L]. The second-order moments of a nonnegative transversely isotropic measure μ are immediately derived in terms of a, b, c given in Corollary 2.2. In the notation of that corollary, we obtain

$$\int_{S^2} x_1^2 d\mu = \int_{S^2} x_2^2 d\mu = \tfrac{4}{3}a + b,$$

$$\int_{S^2} x_3^2 d\mu = 2b + c,$$

$$\int_{S^2} x_1 x_2 d\mu = \int_{S^2} x_1 x_3 d\mu = \int_{S^2} x_2 x_3 d\mu = 0.$$

Thus, for any such measure μ, (3.6) reads

$$(1-\theta)(A^0 - A_1)^{-1}_{ijkm} = (A_2 - A_1)^{-1}_{ijkm} + \frac{\theta}{\mu_1} Y_{ijkm}$$

with

$$Y_{1111} = Y_{2222} = \frac{4a}{3} + b - \frac{\lambda_1 + \mu_1}{\lambda_1 + 2\mu_1} a,$$

$$Y_{3333} = 2b + c - \frac{\lambda_1 + \mu_1}{\lambda_1 + 2\mu_1} c, \qquad (3.14)_1$$

$$Y_{1122} = -\frac{\lambda_1 + \mu_1}{3(\lambda_1 + 2\mu_1)} a,$$

$$Y_{1133} = Y_{2233} = -\frac{\lambda_1 + \mu_1}{\lambda_1 + 2\mu_1} b, \qquad (3.14)_2$$

$$Y_{1212} = \frac{2a}{3} + \frac{b}{2} - \frac{\lambda_1 + \mu_1}{3(\lambda_1 + 2\mu_1)} a,$$

$$Y_{1313} = Y_{2323} = \frac{a}{3} + \frac{3b}{4} + \frac{c}{4} - \frac{\lambda_1 + \mu_1}{\lambda_1 + 2\mu_1} b, \qquad (3.14)_3$$

and with all other components equal to zero.

The constraint that μ be a probability measure translates into

$$\tfrac{8}{3} a + 4b + c = 1.$$

We appeal to Remark 3.1 and Corollary 2.2 to conclude that

Corollary 3.3. *The set of all transversely isotropic finite-rank laminates made from two isotropic phases with elasticity tensors A_1 and A_2 given by* (3.3) *with the A_1 phase at volume fraction θ is the set of all tensors A^0 satisfying*

$$(1-\theta)(A^0 - A_1)^{-1} = (A_2 - A_1)^{-1} + \frac{\theta}{\mu_1} Y$$

with Y defined by (3.14), *or*

$$\theta(A^0 - A_2)^{-1} = (A_1 - A_2)^{-1} + \frac{1-\theta}{\mu_2} Z$$

with Z obtained from Y by replacing λ_1 and λ_2 by μ_1 and μ_2 in (3.14). *In* (3.14), *a, b, c are three nonnegative real numbers satisfying*

$$\tfrac{3}{2} b^2 \leqq ac, \quad \tfrac{8}{3} a + 4b + c = 1.$$

Remark 3.5. Combining Corollary 3.3 with Theorem 3.2 would permit us to obtain energy bounds on the set of transversely isotropic effective elastic tensors. See [L, Section 4] for details.

Appendix

After treating the two-dimensional case using the fact that a nonnegative polynomial in one variable is the sum of squares of polynomials, we wondered if this property was also true for nonnegative polynomials of degree at most four in two variables in order to treat the three-dimensional case. We first learned of the classical result that a nonnegative polynomial is always the sum of squares of rational fractions but not always the sum of squares of polynomials, but the counterexample that we saw dealt with a polynomial of degree six in two variables. Thus the sought result could still be true, and we indeed derived a proof for it. A few months later we learned that the result had been proved by HILBERT; we did not try to obtain the smallest number of squares, which HILBERT had proved to be three. Since HILBERT's proof is not so transparent, we deem it useful to sketch the only part of the proof which is of use to us, namely, that if P is a nonnegative polynomial of degree 4 which has five distinct zeroes $a_j, j = 1, \ldots, 5$, then P is a sum of (three) squares.

Let us consider the case where three of the points a_j are on a line of equation $L(x, y) = 0$. In that case the intersection with $P(x, y) = 0$ with the line has three double zeroes, and therefore the degree of P being at most four, P is divisible by L; so $P = LS$. Since P is nonnegative, we must have $S = 0$ on the line $L = 0$; so $P = L^2 T$ and T has degree $\leqq 2$ and is nonnegative and so is the sum of (three) squares.

Assuming that we are not in the above-mentioned degenerate case, we want to construct a conic section going through all the points $a_j, j = 1, \ldots, 5$. This is certainly possible since a quadratic polynomial is defined by six homogeneous coefficients and since one writes one linear relation to express the fact that the conic section goes through a point, and five linear relations can be imposed. Let $Q(x, y) = 0$ be the equation of that conic section. In the intersection of the zero set of P and the zero set of Q, each of the points a_j counts for two, and this gives a counting of ten intersection points instead of eight for intersecting two algebraic curves of degree two and four. Thus there is a degeneracy, and Q and P should have a common factor, i.e., P is a multiple of Q. Thus there exists a polynomial R of degree at most two such that $P = QR$. Because P is nonnegative, R must change sign when Q does, and so $R = 0$ when $Q = 0$ and R is divisible by Q. Thus $P = cQ^2$ with $c > 0$.

The preceding argument can be made more analytical by parametrizing the conic section $Q(x, y) = 0$ by $x = a(t)/c(t)$, $y = c(t)/c(t)$ with a, b, c polynomials of degree $\leqq 2$, since Q is a nondegenerate quadratic polynomial. (If we take the origin on the conic section, then any line of equation $y = tx$ intersects the conic section at two points, one value of x being 0 and the other being expressed as a rational fraction in t.) Writing $P = 0$ gives an equation of degree eight in t with five double zeroes and we deduce that $Q = 0$ implies $P = 0$. In order to deduce that P is divisible by Q, we first change basis so that Q can be written as $Q(x, y) = ax^2 + q(y)$ with $a \neq 0$, and we write P as $P(x, y) = P_1(x, y) + xP_2(x, y)$ where P_1 and P_2 only contain even powers of x. Since $Q(x, y) = 0$ also implies that $Q(-x, y) = 0$, it implies not only that $P(x, y) = 0$ but also that $P_1(x, y) = P_2(x, y) = 0$. By replacing then each occurrence of x^2 in P_1 or P_2 by $\frac{1}{a}(Q(x, y) - q(y))$, we obtain a multiple of Q plus a polynomial in y which must be 0 since $Q = 0$ implies that $P_1 = P_2 = 0$.

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