*Eur. J. Mech., A/Solids*, **12**, n° 2, 149-189, 1993

# Stable damage evolution in a brittle continuous medium

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ABSTRACT. – A model of partial brutal damage is investigated for a class of brittle solids. A global stability criterion is proposed. It is shown to prohibit the existence of a undamaged-damaged type solution and to promote fine mixtures of the undamaged and damaged phases. Thus microstructures appear as a byproduct of the criterion and not as a constitutive element of the model, in striking contrast with the micromechanical approach to damage. The resulting model is studied, especially in the two dimensional problems and for the problem of cylindrical torsion.

## 0. Introduction and Notation

### 0.1. INTRODUCTION

In its infancy damage theory was most certainly brittle, and abundantly so, as witnessed by the host of brittle models used in Engineering Design. Later on however, brittle damage was sneered at and violently indicted for its inability to prevent instabilities and localization, the unavoidable companions of material softening. The evil nature of softening was asserted on the basis of various analyses of rate evolution problems together with a battery of numerical tests, *cf.* [Nguyen, 1984 and 1987; Benallal *et al.*, 1989; Stolz, 1989; Silling, 1988 and de Borst, 1989].

We propose to exhume brittle damage and exorcize part of its evilness. This is the goal of the present study. We will show that the true bandit is not material softening per se, but rather the amount of softening that may occur. In other words if, damage annihilates all stresses, catastrophic events will most certainly occur. If, on the other hand, the ultimate stiffnesses remain positive – thereby preventing the vanishing of the stress field – then stability is attainable in the large.

A better understanding of our approach necessarily involves a brief recall of the prevailing doctrines in the field. Since the end of the seventies various models that take into account the softening of materials properties through micromechanisms such as microcracking, microdebonding or microcavitation have been proposed. All of them are based on an identical postulate, namely the existence of a scalar, vectorial or tensorial

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EUROPEAN JOURNAL OF MECHANICS, A/SOLIDS, VOL. 12, N° 2, 1993
0997-7538/93/02 149 41/$ 6.10/ 

space-time dependent internal variable $\alpha(x, t)$ whose pernicious effect is to alter the original physicomechanical properties of the material under consideration, *cf.* e. g. [Lemaitre & Chaboche, 1978]. For example, damaging elastic materials will remain elastic throughout their life span but their stiffness will be irreversibly affected by the strains they are submitted to. That dependence is evidenced by the influence of the damage variable $\alpha(x, t)$ on the stiffness tensor $A(x, t)$ through the constitutive relation

$$A(x, t)=\hat{A}(\alpha(x, t)). \tag{0.1}$$

The choice of the functional dependence of A upon $\alpha$, that is of the function $\hat{A}(\alpha)$, is a matter of great debate where phenomenologists battle micromechanists.

The founding fathers of damage theory [Kachanov, 1958; Lemaitre, 1978] were undoubtedly phenomenologists and they can rightfully claim the fatherhood of the simplest models such as

$$\hat{A}(\alpha)=(1-\alpha)A^0+\alpha A^1, \tag{0.2}$$

where $\alpha$ varies in the interval [0, 1] and $A^0$, $A^1$ are two given positive well-ordered fourth order tensors, that is

$$A^0 \geqq A^1 \geqq 0, \tag{0.3}$$

in the sense of quadratic forms. If the variable $\alpha$ only takes the value 0 or 1, the damage is referred to as *brutal*; if not, it is called *progressive*. If $A^1$ is positive, the damage is called *partial* whereas if $A^1$ is only nonnegative it is labelled *total*.

The micromechanists appeared somewhat later with a new weapon: homogenization. Intent on a more microstructural understanding of damage, they proceeded to derive $\hat{A}(\alpha)$ as the macroscopic stiffness tensor associated to a relevant microstructure parametrized by $\alpha$. They investigated phenomena like the growth of microvoids ($\alpha$ being the volume fraction of voids) [Marigo, 1985], the effect of debonding in fiber composites ($\alpha$ taking the values 0 (no debonding) or 1 (total debonding)) [Lene & Leguillon, 1982; Marigo & Pideri, 1987], the growth of microcracks ($\alpha$ being the length of the microcracks) [Suquet, 1981; Andrieux *et al.*, 1986]. In doing so they generated various competing brutal/progressive partial/total models.

In any case proponents of either school were confronted with the same difficulty: once the function $\hat{A}(\alpha)$ is known, how does one choose the evolution law for $\alpha$? Numerous experiments [Lemaitre & Chaboche, 1985] suggest that the evolution of $\alpha$ is extremely material, loading and geometry dependent, and ad-hoc recipes soon followed guided by sane rather than safe general principles: rapidly varying loads do not give $\alpha$ enough time to experience its sensitivity to time; thus time independent laws based upon a yield criterion are adopted inviting a shameless plagiarism of the theory of plasticity, *cf.* e. g. [Hill, 1950; Kachanov, 1975]. Slowly varying loads on the other hand definitely call for a time dependent theory where viscosity reigns. Finally rapid cyclic loading are the realm of the theory of fatigue.

The former considerations seem to favor a three-legged procedure for the construction of a model of damage consisting of the choice of

1. the damage parameter:

2. the dependence of the mechanical properties on the damage parameter (for damaging elastic materials the choice of $\hat{A}(\alpha)$);

3. the evolution law for the damage parameter.

In this framework the adoption of a time independent law with a yield criterion – that is of a brittle damage law – at the third step was at first general practice then became more and more a target of unabashed criticism when numerical testing started producing unpleasant results (*cf.* [D, 1989], [S, 1988]) evidencing instabilities and localization.

In this paper we revisit the framework of brittle damage laws. Although our analysis holds quite generally in its principle, it remains essentially limited to the case of partial brutal damage in the present study. For the most part, the conclusions of the study would also hold true in the case of partial progressive damage. Specifically we adopt a rather natural yield criterion of the Griffith type. Recalling (0.2) we force the material to drop in stiffness from $A^0$ to $A^1$ if, at a given point and for a given strain tensor $\varepsilon$,

$$(A^0 - A^1)\varepsilon.\varepsilon > 2\kappa \tag{0.4}$$

where $\kappa$ is a positive constant representing the energy density released at the onset of damage. After formulating the problem of the evolution of damage in a body submitted to a time dependent loading, we will demonstrate that stable solutions do exist if the undamaged and the damaged materials are allowed to mix on a fine scale. Such stable solutions are out of the reach of classical finite element methods because the fineness of the mesh cannot hope to match that of the mixture. The class of admissible solutions is thus extended to the set of possible effective elasticities resulting from the fine mixing of the damaged and undamaged materials.

The paper consists of four Sections, each one being divided into Subsections. The first Section is devoted to the introduction of the mechanical model under consideration. In Subsection 1.1 we consider a brutal damaging elastic material whose stiffness tensor drops from the sound value $A^0$ to the damaged one $A^1$ when the energetic criterion (0.4) is satisfied. We formulate in Subsection 1.2 the problem of the damage evolution in a body submitted to a loading process; it consists of finding the displacement field $u(t)$ and the characteristic function $\chi(t)$ of the damaged zone (where the rigidity is $A^1$) at every time $t$ of the loading process. This problem generally admits too many solutions, *cf.* [Francfort & Marigo, 1991]. We finally introduce a stability criterion, based on the minimization, at each time, of the global energy of the body, the sum of it's potential energy and it's dissipated one, with respect to kinematically admissible displacements and admissible characteristic functions of the damaged zone. The resulting incremental problem is called $(\mathscr{M}_I)$. The second Section addresses the general two or three dimensional problem, which generically does not admit a solution. It must be "relaxed" and the aforementioned generalized stable solutions are evidenced. A relaxation process is introduced in Section 2, together with the resulting relaxed functional. To this end the proper homogenization framework is presented in Subsection 2.1. The resulting problem, called $(\mathscr{M}_I^{rel})$, is derived in Subsection 2.2. It involves the determination of the lowest possible energy density at a given strain among all possible binary mixtures of two well-ordered elastic materials mixed in predefined volume fraction. The third Section is devoted to a quasi explicit determination of the relaxed energy in the two-dimensional case. Section 4 revisits the entire process in the case of a classical torsion problem. It is in large part a compendium of previous analyses of [Murat & Tartar, 1985] or [Lurie & Cherkaev, 1986] on optimal design. After setting the problem in Subsection 4.1, the result of the relaxation process is described in Subsection 4.2. The necessary optimality conditions on the generalized minimizers are proved to forbid the existence of classical stable solutions for large enough loadings. Subsection 4.3 is devoted to the explicit determination of the unique generalized stable solution (previously sketched in [F & M, 1991] Section 5) in the case of the torsion of a circular cylinder. A time dependent zone where sound and damaged materials are finely mixed is exhibited and shown to increase with the loading.

Impervious the accusation of redundancy we recapitulate our hypotheses. Three constitutive ingredients are required:

1. A linear elastic material that undergoes brutal partial damage and whose stiffness tensor decreases from $A^0$ to $A^1$ with $A^0 > A^1 > 0$.

2. An energetic damage yield criterion, namely $(A^0 - A^1)\,\varepsilon.\varepsilon > 2\kappa$, $\kappa > 0$.

3. A stability criterion which consists in minimizing the sum of the potential energy and the dissipated energy of the body with respect to kinematically admissible displacements and admissible damage arrangements within the body.

Once acceptance, of these ingredients is granted, however reluctant it may be, the analysis leaves no room for arbitrariness.

It is also appropriate, at the close of this introduction, to point to a yet undisclosed weakness of the model: the time evolution is not continuously monitored but merely sampled at discrete times. Since inertia effects are neglected this would seem like a rather innocuous and customary simplification of the continuous mechanical model and there is nothing that prevents us from formally arguing for a continuous time variable. The mathematical implications however are not so trivial because all questions pertaining to existence become purely spatial while all measurability and regularity problems in time are done away with. In all fairness it should be noted that an incremental approach to many quasitatic problems of nonlinear mechanics is the rule rather than the exception and that the question of dependence on time is seldom broached in the literature.

Finally a remark of a general nature should be made. The less mathematically inclined reader might find several subsections somewhat too mathematical. Unfortunately this alternatives in the matter is merciless: "all or nothing". Our choice is maximalist because we believe that a careful reading of the paper and a spanning of the quoted references should permit a complete understanding of the proposed method. A more cursory reading should nevertheless deliver the essential ingredients of the method.

## 0.2. Notation

Einstein's summation convention of repeated indices is used. $\mathbb{R}$ denotes the vector space of real numbers, $A \times B$ the cartesian product of the sets A and B, $\mathbb{R}^N = \mathbb{R} \times \ldots \times \mathbb{R}$ N-times, $\mathbb{M}^N$ is the vector space of $N \times N$ real matrices, $\mathbb{M}_s^N$ the subspace of symmetric ones, $\mathscr{M}^N$ is the space of linear mappings from $\mathbb{M}_s^N$ into itself and $\mathscr{M}_s^N$ the subspace of symmetric ones. The inner products in $\mathbb{R}^N$ and $\mathbb{M}_s^N$ are represented by a dot ., that is $u.v=u_i v_i$ and $\tau.\varepsilon=\tau_{ij}\varepsilon_{ij}$. The derivative of a function $\varphi$ is noted $D\varphi$. If $\Omega$ is an open set of $\mathbb{R}^N$ and $v$ a smooth vector field defined in $\Omega$, the strain field associated with $v$ is denoted by $\varepsilon(v)$, that is $2\varepsilon(v)(x)=Dv(x)+Dv(x)^T$ where the superscript $^T$ stands for the transpose of the matrix $Dv(x)$. We have to use characteristic functions $\chi$. To this end, we recall that a characteristic function $\chi$ of a subset $\Omega^1$ of $\Omega$ is the function defined by $\chi(x)=1$ if $x\in\Omega^1$ and $\chi(x)=0$ if $x\in\Omega^0=\Omega-\Omega^1$. We also use Sobolev spaces (but it is not essential for the understanding of the paper) with their usual definition and notation, for instance $H^1(\Omega, \mathbb{R}^N)$ or $L^\infty(\Omega)$. We have to minimize functionals, that is functions J defined on a functional space $\mathscr{V}$ and taking real values. We recall that $\inf\limits_{v\in\mathscr{V}} J(v)$ denotes the greatest lower bound of the values taken by $J(v)$ when $v$ describes $\mathscr{V}$. When this bound is attained by some $u\in\mathscr{V}$, the inf becomes a min. We also distinguish the sup (the lowest upper bound) and the max (when the sup is attained).

Finally in the spirit of the remark at the end of the previous Subsection we will examine an evolution on a time interval $[0, t]$ that will have been discretized in I time intervals

$$0=t_0\leqq t_1\leqq\ldots\leqq t_I=t. \tag{0.5}$$

All quantities entering the analysis will be superscripted by the index $i$ of the time $t_i$ at which they are considered, with $i$ varying from 0 to I.

## 1. The damage evolution problem

This section is devoted to the introduction of the three ingredients of the model alluded to in the introduction. We consider a continuous medium whose reference configuration is a regular connected open subset $\Omega$ of $\mathbb{R}^N$, $1\leqq N\leqq 3$. This body is made of an homogeneous elastic damaging material whose damage law will be defined in Subsection 1.1. Starting at time $t_0=0$ when the body is assumed totally undamaged, a loading process is considered over the time interval of loading $[0, t]$.

### 1.1. The model

A space-time dependent stiffness $A^i(x)$ is considered. It can only take two values $A^0$ and $A^1$ which are ordered as follows:

$$A^0>A^1>0, \tag{1.1}$$

where the inequalities should be understood as inequalities between symmetric fourth order tensors, that is,

$$A^0\varepsilon.\varepsilon>A^1\varepsilon.\varepsilon>0, \qquad \forall\varepsilon\in\mathbb{M}_s^N,\quad \varepsilon\neq 0.$$

The assumed strict inequality $A^1>0$ is fundamental to the analysis of the damage evolution problem developed in this work. Thus

$$A^i(x)=(1-\chi^i(x))\,A^0+\chi^i(x)\,A^1, \tag{1.2}$$

with $\chi^i(x)\in\{0, 1\}$. When $\chi^i(x)=0$, the point $x$ is called sound or undamaged, while when $\chi^i(x)=1$, the point $x$ is said to be damaged. We assume that, at the initial time $t_0=0$,

$$\chi^0(x)=0. \tag{1.3}$$

In other words, at $t_0=0$ all points of the body are sound, although this last restriction is not essential to the analysis.

Damage is an irreversible process. The irreversibility translates into an evolution law of $\chi^i(x)$ with respect to $i$ such that, if $\chi^i(x)=1$ for a given $i$, then $\chi^j(x)=1$ for any time $j\geqq i$. It remains to specify such an evolution law. This paper is only concerned with "rate independent" laws for which the damage process is governed by a yield criterion written in terms of the strain history $\varepsilon^i(x)$ of the point $x$. The most general such criterion consists in introducing a *fixed* open set $\mathscr{R}$ in $\mathbb{M}_s^N$ such that $\chi^i(x)=0$ as long as $\varepsilon^i(x)$ belongs to $\mathscr{R}$, while $\chi^i(x)=1$ as soon as $\varepsilon^i(x)$ leaves $\mathscr{R}$. Specifically, the law reads as

$$\chi^i(x)=\begin{cases} 0 & \text{if } \chi^{i-1}(x)=0 \text{ and } \varepsilon^i(x)\in\mathscr{R} \\ 1 & \text{if } \chi^{i-1}(x)=1 \text{ or } \varepsilon^i(x)\notin\mathscr{R} \\ 0 \text{ or } 1 & \text{otherwise.}\end{cases} \tag{1.4}$$

The specific domain $\mathscr{R}$ considered in this study is defined by

$$\mathscr{R}=\left\{\varepsilon\in\mathbb{M}_s^N \,\middle|\, \frac{1}{2}(A^0-A^1)\,\varepsilon.\varepsilon<\kappa\right\}. \tag{1.5}$$

The positive coefficient $\kappa$ is a characteristic parameter of the material. It represents the release of elastic energy due to the decrease of rigidity when the strain reaches a critical value at the boundary of $\mathscr{R}$. It can be interpreted as a density of dissipated energy of the damaged part of the body. The globally dissipated energy $D^i$ in the body from the beginning of the process to the current time $t_i$ is given by

$$D^i=\int_\Omega \kappa\,\chi^i(x)\,dx. \tag{1.6}$$

The analogy with the Griffith critical energy release rate of brittle fracture mechanics is noteworthy. In the latter setting the globally dissipated energy $D(t)$ would read as $D(t)=\kappa\, l(t)$ where $l$ is the length of the crack at time $t$ and $\kappa$ is the Griffith critical energy release rate.

Finally note that the assumed form (1.5) of the yield criterion could be deduced from a more general constitutive principle, *see* [M & P, 1987] and [Marigo, 1989]. The adopted form plays an important role in the definition of the stability criterion introduced below.

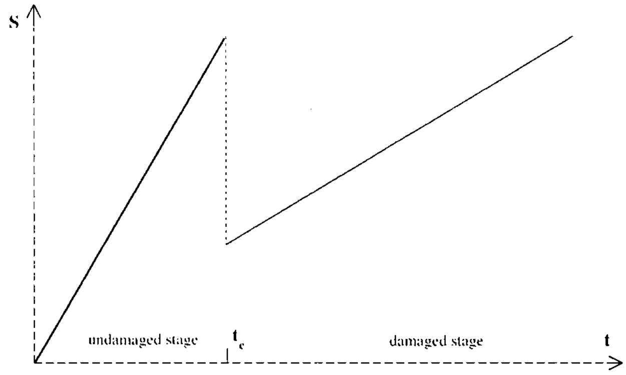


Fig. 1. — Stress versus strain response under monotone uniaxial loading.

REMARK 1.1. — At this point we have availed ourselves of two of the three ingredients needed for the construction of our model. The resulting behaviour can be illustrated on the strain-stress diagram of a material volume element subjected to the increasing strain history $\varepsilon(t)=t\,\varepsilon^0$ with $\varepsilon^0\in\mathbb{M}_s^{\mathrm{N}}$, $\varepsilon^0\neq 0$.

Setting $t_c=(2\,\kappa/(\mathrm{A}^0\,\varepsilon^0\,.\,\varepsilon^0-\mathrm{A}^1\,\varepsilon^0\,.\,\varepsilon^0))^{1/2}$, $\sigma(t)=\mathrm{A}(t)\,\varepsilon(t)$ (the stress tensor at the time $t$) and $\mathrm{S}(t)=\sigma(t)\,.\,\varepsilon^0$, we obtain the plot S versus $t$ of Figure 1.

## 1.2. FORMULATION OF THE DAMAGE EVOLUTION PROBLEM

A first formulation of the evolution of damage in $\Omega$ is now proposed. The loading process consists of time dependent body forces $f^i$, time dependent surface forces $\mathrm{T}^i$ on the part $\Gamma_s$ of the boundary $\partial\Omega$ of $\Omega$, and time dependent prescribed displacements $\mathrm{U}^i$ on the complementary part $\Gamma_c$ of $\partial\Omega$. It is assumed that the loading process is such that the deformations of the body remain small and that inertia effects can be neglected. It is fair, in our opinion, to emphasize that the quasistatic assumption is common to all analyses of damage (or brittle fracture for that matter).

Therefore, the problem is to find, at each time $t_i$, the displacement field $u^i$ of the body and the characteristic function $\chi^i$ of the damaged zone that satisfy the linearized equilibrium equations and the damage constitutive law. Formally, the problem, called $(\mathscr{P}_{\mathrm{I}})$, is formulated as follows:

PROBLEM $(\mathscr{P}_{\mathrm{I}})$. — *For* $i\in\{1,\ldots,\mathrm{I}\}$, *find* $(u^i,\chi^i)$ *such that*

$$2\,\varepsilon^i=\mathrm{D}u^i+\mathrm{D}u^{i\,\mathrm{T}}\quad in\ \Omega, \tag{1.7}$$

$$\sigma^i=((1-\chi^i)\,\mathrm{A}^0+\chi^i\,\mathrm{A}^1)\,\varepsilon^i\quad in\ \Omega, \tag{1.8}$$

$$\operatorname{div}\sigma^i+f^i=0\quad in\ \Omega,\qquad \sigma^i n=\mathrm{T}^i\quad on\ \Gamma_s, \tag{1.9}$$

$$u^i=\mathrm{U}^i\quad on\ \Gamma_c, \tag{1.10}$$

$$(1.11)\qquad \chi^i(x)=\begin{cases} 0 & \textit{if}\ \ \chi^{i-1}(x)=0 \ \ \textit{and}\ \ (\mathrm{A}^0-\mathrm{A}^1)\,\varepsilon^i(x).\varepsilon^i(x)<2\kappa \\ 1 & \textit{if}\ \ \chi^{i-1}(x)=1 \ \ \textit{or}\ \ (\mathrm{A}^0-\mathrm{A}^1)\,\varepsilon^i(x).\varepsilon^i(x)>2\kappa \\ 0\ \textit{or}\ 1 & \textit{otherwise,} \end{cases}$$

*with the initial constraint* $\chi^0=0$.

The damaged zone at the time $t_i$ is denoted by $\Omega_1^i$ and the undamaged one by $\Omega_0^i$. By definition, we have

$$\Omega_0^i=\{x\in\Omega\,|\,\chi^i(x)=0\},$$
$$\Omega_1^i=\{x\in\Omega\,|\,\chi^i(x)=1\}\qquad\text{and}\qquad \Omega=\Omega_1^i\cup\Omega_0^i,\ i\geqq 0.$$

The evolution equation (1.11) accounts for the *irreversibility of damage;* $\chi^i$ is an increasing function of $i$ and consequently $\Omega_0^i$ decreases with $i$, while $\Omega_1^i$ increases.

This definition of $\chi^i$ is implicit: to determine $\chi^i$ one must know $\varepsilon^i$, but conversely $\varepsilon^i$ depends on $\chi^i$ through the stress-strain relation (1.8) and the equilibrium equations (1.9). The problem can be rephrased as a two field partial minimization problem upon introducing the functional $\mathscr{L}^i(v,\chi)$ defined on $\mathscr{V}^i\times\mathscr{C}^i$, $\mathscr{V}^i$ and $\mathscr{C}^i$ being respectively the set of admissible displacement fields and the set of admissible characteristic functions at time $i$. Specifically we set

$$(1.12)\qquad \mathscr{V}^i=\{v\in \mathrm{H}^1(\Omega,\mathbb{R}^\mathrm{N})\,|\,v=\mathrm{U}^i \text{ on } \Gamma_c\},$$

$$(1.13)\qquad \mathscr{C}^i=\{\chi\in \mathrm{L}^\infty(\Omega,\{0,1\})\,|\,\chi(x)=1 \text{ if } \chi^{i-1}(x)=1\},$$

and

$$(1.14)\qquad \mathscr{L}^i(v,\chi)=\frac{1}{2}a(\chi;v,v)+d(\chi)-l^i(v)$$

with

$$(1.15)\qquad a(\chi;u,v)=\int_\Omega ((1-\chi)\,\mathrm{A}^0+\chi\,\mathrm{A}^1)\,\varepsilon(u).\varepsilon(v)\,dx,$$

$$(1.16)\qquad d(\chi)=\int_\Omega \kappa\chi(x)\,dx,$$

and

$$(1.17)\qquad l^i(v)=\int_\Omega f^i(x).v(x)\,dx+\int_{\Gamma_s}\mathrm{T}^i(x).v(x)\,d\Gamma(x).$$

Then $(\mathscr{P}_\mathrm{I})$ is equivalent to the following problem:

PROBLEM $(\mathscr{P}_\mathrm{I}')$. — *For* $i\in\{1,\ldots,\mathrm{I}\}$, *find* $(u^i,\chi^i)\in\mathscr{V}^i\times\mathscr{C}^i$ *such that*

$$\mathscr{L}^i(u^i,\chi^i)\leqq\mathscr{L}^i(v,\chi^i),\qquad \forall\, v\in\mathscr{V}^i,$$
$$\mathscr{L}^i(u^i,\chi^i)\leqq\mathscr{L}^i(u^i,\chi),\qquad \forall\,\chi\in\mathscr{C}^i.$$

The problem $(\mathscr{P}'_{\mathrm{I}})$ as it stands is ill-posed to the extent that it possesses in general too many solutions. The reader is referred to [F & M, 1991], Subsection 2.3, where this severe non uniqueness is illustrated by a one dimensional example. It is then reasonable to produce a physically motivated criterion that will eliminate undesirable solutions. To this end we propose to introduce our third ingredient: *the stability criterion.* A solution to the incremental problem $(\mathscr{P}'_{\mathrm{I}})$ is called a *stable solution* if it is solution to the following problem:

PROBLEM $(\mathscr{M}_{\mathrm{I}})$. — *Stable solutions for the incremental problem,*

$$(\mathscr{M}_{\mathrm{I}}) \qquad \begin{cases} \textit{For } i \in \{1, \ldots, \mathrm{I}\}, \textit{find } (u^i, \chi^i) \textit{ that minimizes } \mathscr{L}^i \textit{ over } \mathscr{V}^i \times \mathscr{C}^i \\ \qquad \textit{with the initial constraint } \chi^0 = 0. \end{cases}$$

Other equivalent formulations of $(\mathscr{M}_{\mathrm{I}})$ may be easily derived. An especially illuminating one consists in minimizing over $\mathscr{C}^i$ and rewriting the problem as a single field minimization over $\mathscr{V}^i$. The formulation so obtained may be identified with a nonlinear elastic problem at each time increment $i$ with associated potential

$$(1.18) \quad \psi^i(x, \varepsilon) = \begin{cases} \dfrac{1}{2}\mathrm{A}^1 \varepsilon . \varepsilon & \text{if } \chi^{i-1}(x) = 1 \\ \begin{aligned} &\underset{\chi \in \{0, 1\}}{\mathrm{Min}} \left\{ \dfrac{1}{2}((1-\chi)\mathrm{A}^0 + \chi \mathrm{A}^1)\varepsilon . \varepsilon + \kappa\chi \right\} \\ &= \mathrm{Min}\left\{ \dfrac{1}{2}\mathrm{A}^0 \varepsilon . \varepsilon, \dfrac{1}{2}\mathrm{A}^1 \varepsilon . \varepsilon + \kappa \right\} \end{aligned} & \text{if } \chi^{i-1}(x) = 0. \end{cases}$$

At points $x \in \Omega_0^{i-1}$ (the undamaged zone at time $t_{i-1}$) $\psi^i(x, .)$ is *not convex, cf. Fig.* 2, because $\mathrm{A}^0 > \mathrm{A}^1$. The $\kappa$ translation in the second part of the definition (1.18) of the elastic potential renders $\psi^i$ continuous. This translation represents the energy dissipated when damage occurs.

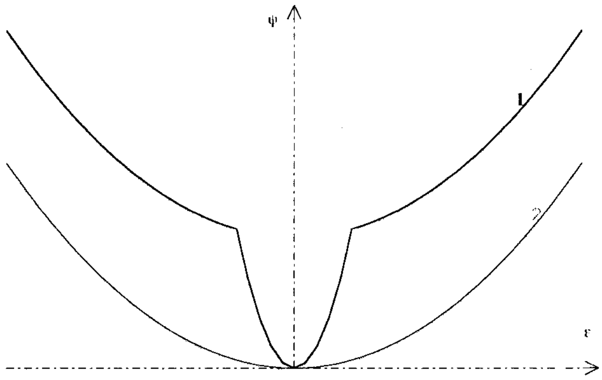


Fig. 2. — The elastic potential $\psi^i$ of the material:
1. In the undamaged zone at $t_{i-1}$. 2. In the damaged zone at $t_{i-1}$.

The associated potential energy $\Phi^i$ of the body is

$$\Phi^i(v) = \int_\Omega \psi^i(x, \varepsilon(v)(x))\,dx - l^i(v), \tag{1.19}$$

and problem $(\mathcal{M}_\mathrm{I})$ becomes

PROBLEM $(\mathcal{S}_\mathrm{I})$. — *Stable displacements evolution,*

$(\mathcal{S}_\mathrm{I})$ *For* $i \in \{1, \ldots, \mathrm{I}\}$, *find* $u^i$ *that minimizes* $\Phi^i$ *over* $\mathcal{V}^i$.

*Then* $\chi^i$ *is given by* (1.11).

Such a criterion is familiar in Nonlinear Elasticity. In Damage Mechanics, the notion of stability is often invoked but a formal statement is much more difficult to find in the literature; *see*, for instance [Ehrlacher & Fedelich, 1989] for a definition similar to ours. Other criteria could be adopted, for instance a local stability criterion. We believe however that our energy minimizing criterion is thermodynamically sound – it does not contradict any known principles –, mathematically elegant, and similar to many criteria in other fields of Continuum Mechanics.

Finally since $\mathcal{L}^i$ has to be minimized with respect to the pair $(v, \chi)$, it can be viewed as a Min-Min problem and the order of minimization may be changed. If we minimize $\mathcal{L}^i(v, \chi)$ with respect to $\chi$ over $\mathcal{C}^i$, at fixed $v$ in $\mathcal{V}^i$, we obtain

$$\Phi^i(v) = \min_{\chi \in \mathcal{C}^i} \mathcal{L}^i(v, \chi) \tag{1.20}$$

and we recover $(\mathcal{S}_\mathrm{I})$. Minimizing $\mathcal{L}^i$ with respect to $v$ at fixed $\chi$ leads to a classical problem of linear elasticity corresponding to an heterogeneous elastic body with stiffness tensor $(1-\chi)\,\mathrm{A}^0 + \chi\,\mathrm{A}^1$ submitted to the real loading at time $t_i$. Thus, if we denote by $v(\chi)$ the displacement field, solution to this elastic problem, it satisfies the variational problem

$$v(\chi) \in \mathcal{V}^i, \quad a(\chi;\, v(\chi),\, w - v(\chi)) = l^i(w - v(\chi)), \qquad \forall\, w \in \mathcal{V}^i. \tag{1.21}$$

In other words defining

$$\mathrm{J}^i(\chi) \overset{\mathrm{def}}{=} \mathcal{L}^i(v(\chi), \chi) = \operatorname*{Min}_{v \in \mathcal{V}^i} \mathcal{L}^i(v, \chi) \tag{1.22}$$

the problem becomes

PROBLEM $(\mathcal{D}_1)$. — *Stable damage evolution,*

$$(\mathcal{D}_i) \quad \left\{ \begin{array}{l} \textit{For } i \in \{1, \ldots, \mathrm{I}\}, \textit{ finds } \chi^i \textit{ that minimizes } \mathrm{J}^i \textit{ over } \mathcal{C}^i \\ \qquad \textit{with the initial constraint } \chi^0 = 0. \end{array} \right.$$

*Then* $u^i = v(\chi^i)$.

REMARK 1.2. — It was shown in [F & M, 1991], Subsection 3.4, that, in the previously mentioned one-dimensional example, our third ingredient, the stability criterion, eliminates all solutions but that corresponding to the well known Maxwell line in the strain-stress plane.

At this point of the study the three necessary ingredients have been introduced and the discretized evolution problem has been formulated in three equivalent forms. The remaining task should be exclusively that of a model testing. The mathematically inclined reader will however not fail to notice that the proof of existence of a minimum of the relevant energy, for any of the three problems $(\mathscr{S}_I)$, $(\mathscr{M}_I)$ and $(\mathscr{D}_I)$, is not clear since a minimizing sequence of characteristic functions does not have to converge in the appropriate topology to a characteristic function. In more intuitive words, in the absence of other constraints, minimizing sequences might not require all the strength of the sound material or all the weakness of the damaged one. The need for such considerations will be further motivated in the next section.

## 2. Generalized stable solutions

This section is devoted to the investigation of minimizers for a relaxed version of problems $(\mathscr{M}_I)$, $(\mathscr{S}_I)$ and $(\mathscr{D}_I)$. A stability criterion that reduces the great number of solutions of the evolution problem has been introduced. The existence of a stable solution is not guaranteed. The non existence of such a minimum was illustrated in [F & M, 1991], Section 5, in the case of a torsion problem. A rapid analysis of the general formulation $(\mathscr{S}_I)$ or $(\mathscr{D}_I)$ suggests that non existence is likely to be the rule rather than the exception. Indeed, consider the formulation $(\mathscr{S}_I)$ consisting of the minimization of $\Phi^i$ over $\mathscr{V}^i$. Since $\psi^i(x, \varepsilon) \geqq 1/2\, A^1 \varepsilon . \varepsilon$ and $A^1 > 0$, $\Phi^i$ is coercive over $\mathscr{V}^i$, minimizing sequences are bounded in $\mathscr{V}^i$ and we can extract a weakly converging subsequence in $\mathscr{V}^i$. Existence of a minimum will be guaranteed if weak lower semicontinuity of the functional is established. But in the present context the lack of convexity of the elastic potential $\psi^i$ with respect to $\varepsilon$ forbids the use of elementary convexity arguments.

Let us consider the formulation $(\mathscr{D}_I)$ which consists in the minimization of $J^i$ over $\mathscr{C}^i$. Considering a minimizing sequence $(\chi_n^i)_{n \in \mathbb{N}}$, we denote by $(v_n^i)_{n \in \mathbb{N}}$ the associated sequence of displacements $v_n^i = v(\chi_n^i)$, that is,

$$(2.1) \qquad v_n^i \in \mathscr{V}^i, \quad a(\chi_n^i; v_n^i, w - v_n^i) = l^i(w - v_n^i), \qquad \forall w \in \mathscr{V}^i.$$

Because $(\chi_n^i)$ is a sequence of characteristic functions, it is a bounded sequence in $L^\infty(\Omega)$. We can extract a subsequence (always denoted by $\chi_n^i$) that converges to $\alpha^i$ for the weak-* topology of $L^\infty(\Omega)$. But in general, $\alpha^i$ is not a characteristic function. In fact $\alpha^i(x)$ could *a priori* take any value in the whole interval [0, 1]. This situation corresponds to *a fine mixture* of the damaged and the undamaged material at the point $x$ of $\Omega$ with local volume fraction $\alpha(x)$ of damaged material. Let us now examine the behaviour of the associated sequence $v_n^i$ of displacement fields. Since $(1 - \chi_n^i) A^0 + \chi_n^i A^1 \geqq A^1 > 0$, the sequence $v_n^i$ is bounded in $\mathscr{V}^i$. After possible extraction of a subsequence, $v_n^i$ converges to $u^i \in \mathscr{V}^i$ for the weak topology of $H^1(\Omega, \mathbb{R}^N)$. It remains to find the limit problem whose solution is $u^i$. In general $\lim a(\chi_n^i; v_n^i, w) \neq a(\alpha^i, u^i, w)$. The correct result is furnished by the theory of homogenization, a few results of which are recalled in Subsection 2.1.

In essence minimizing sequences force us to seek generalized solutions which correspond to a relaxed formulation of our original problem through the introduction of fine mixtures. The main steps of this relaxation process are described in Subsection 2.2.

## 2.1. Homogenization

This subsection is a brief incursion into the theory of homogenization and recalls the various results needed for a better understanding of the subsequent analysis. Our approach to homogenization is that of H-convergence developed by Murat and Tartar — *see* e.g. [Tartar, 1977; Murat, 1978] and also [Francfort & Murat, 1986] — which is very close to that of G-convergence developed by Spagnolo in the symmetric case (*see* [Spagnolo, 1968]). We thus consider the set

$$\mathcal{M}(m, \mathrm{M}) = \{ \mathrm{A} \in \mathrm{L}^\infty(\Omega, \mathcal{M}_s^{\mathrm{N}}) \,|\, m\,\mathbb{I} \leqq \mathrm{A}(x) \leqq \mathrm{M}\,\mathbb{I} \text{ a.e. in } \Omega \}, \tag{2.2}$$

where $m$ and M are two given positive real numbers, $\mathbb{I}$ is the identity of $\mathcal{M}_s^{\mathrm{N}}$ and the inequalities have to be understood in the sense of quadratic forms. The adopted definition of H-convergence is

Definition 2.1. — *Let $\mathcal{V}$ be a rigid-motion-free closed affine subspace of* $\mathrm{H}^1(\Omega, \mathbb{R}^{\mathrm{N}})$. *A sequence $\mathrm{A}_n$ of elements of $\mathcal{M}(m, \mathrm{M})$ is said to* H-*convergence to an element* $\mathrm{A}^*$ *of $\mathcal{M}(m, \mathrm{M})$ if and only if, for every continuous linear form* 1 *on the linear subspace associated to $\mathcal{V}$, the unique pair $(u_n, \sigma_n)$ of $\mathcal{V} \times \mathrm{L}^2(\Omega, \mathbb{M}_s^{\mathrm{N}})$ satisfying*

$$\begin{cases} \displaystyle\int_\Omega \mathrm{A}_n \varepsilon(u_n) \,.\, \varepsilon(v - u_n)\, dx = l(v - u_n), & \forall v \in \mathcal{V} \\ \sigma_n = \mathrm{A}_n \varepsilon(u_n) \ \text{ a.e. } in\ \Omega, & \end{cases} \tag{2.3}$$

*is such that, as $n$ tends to infinity, $(u_n, \sigma_n)$ converges weakly in $\mathcal{V} \times \mathrm{L}^2(\Omega, \mathbb{M}_s^{\mathrm{N}})$ to $(u, \sigma)$ the unique solution of*

$$\begin{cases} \displaystyle\int_\Omega \mathrm{A}^* \varepsilon(u) \,.\, \varepsilon(v - u)\, dx = l(v - u), & \forall v \in \mathcal{V} \\ \sigma = \mathrm{A}^* \varepsilon(u) \ \text{ a.e. } in\ \Omega. & \end{cases} \tag{2.4}$$

The motivation for such a definition resides in the following compactness theorem due to Spagnolo [S, 1968] or Tartar [T, 1977]:

Theorem 2.1. — *If $\mathrm{A}_n$ is a sequence of $\mathcal{M}(m, \mathrm{M})$, there exists $\mathrm{A}^*$ in $\mathcal{M}(m, \mathrm{M})$ and a subsequence of $\mathrm{A}_n$ which* H-*converges to* $\mathrm{A}^*$.

A few properties of H-convergence are loosely listed here. For precise statements and proofs *see* e.g. [M, 1978], [Tartar, 1985] or [F & M, 1986] in the case of linearized elasticity.

PROPERTY 1. — Any kind of boundary condition that ensures existence and uniqueness of the solution $u_n$ will do. In fact H-convergence is local and convergence in local spaces (such as $L^2_{loc}$...) suffices.

PROPERTY 2. — H-convergence is metrizable. Almost pointwise convergence of $A_n(x)$ to $A^*(x)$ implies H-convergence of $A_n$ to $A^*$.

PROPERTY 3. — The elastic energy of the converging subsequence converges to the homogenized one, that is,

$$\lim_{n \to \infty} \int_\Omega A_n \varepsilon(u_n) . \varepsilon(u_n)\, dx = \int_\Omega A^* \varepsilon(u) . \varepsilon(u)\, dx. \tag{2.5}$$

PROPERTY 4. — If $A(y)$ is defined on the unit torus $\mathscr{C}$ then the whole sequence $A_n(x) = A(nx)$ is found to H-converges to a constant elasticity $A^*$ given by

$$A^* \varepsilon . \varepsilon = \underset{\varphi \in H^1(\mathscr{C}, \mathbb{R}^N)}{\operatorname{Min}} \int_{\mathscr{C}} A(y)(\varepsilon + \varepsilon(\varphi)(y)) . (\varepsilon + \varepsilon(\varphi)(y))\, dy, \qquad \varepsilon \in \mathbb{M}_s^N.$$

Let us now specialize the sequence $A_n$ to be of the form

$$A_n = (1 - \chi_n) A^0 + \chi_n A^1, \tag{2.6}$$

where $\chi_n$ is a sequence of characteristic functions. Since $\chi_n$ is a bounded sequence in $L^\infty(\Omega, [0, 1])$, there exists $\alpha \in L^\infty(\Omega, [0, 1])$ and a subsequence $\chi_{\sigma(n)}$ of $\chi_n$ such that $\chi_{\sigma(n)}$ converges to $\alpha$ for the weak-* topology of $L^\infty(\Omega)$. We set

$$m = \operatorname{Min}\{A^1 \varepsilon . \varepsilon : \varepsilon \in \mathbb{M}_s^N, \varepsilon . \varepsilon = 1\}, \qquad M = \operatorname{Max}\{A^0 \varepsilon . \varepsilon : \varepsilon \in \mathbb{M}_s^N, \varepsilon . \varepsilon = 1\}, \tag{2.7}$$

and observe that the sequence $A_n$ belongs to $\mathscr{M}(m, M)$. We can apply Theorem 2.1 and conclude that there exists $\alpha$ in $L^\infty(\Omega, [0, 1])$, $A^*$ in $\mathscr{M}(m, M)$ and subsequences $\chi_{\sigma(n)}$, $A_{\sigma(n)}$ such that

$$\begin{cases} A_{\sigma(n)} \text{ H-converges to } A^* \\ \chi_{\sigma(n)} \text{ weak-* converges to } \alpha. \end{cases} \tag{2.8}$$

The tensor field $A^*$ is the effective elastic tensor field for the binary mixture of $A^0$ and $A^1$ in volume fraction $\alpha$ of $A^1$. We denote by $\mathscr{G}(\alpha)$ the set of all tensor fields $A^*$ such that a subsequence $(\chi_{\sigma(n)}, A_{\sigma(n)})$ satisfies (2.8). Several structural properties of $\mathscr{G}(\alpha)$ are proved in [Dal Maso & Kohn, to appear] and listed in Remark 2.1 below. They will be of use in the sequel.

REMARK 2.1. — The set $\mathscr{G}(\alpha)$ satisfies the following properties:

1. If $\alpha$ is a constant field, then there exists a compact subset $\mathbb{G}(\alpha)$ of $\mathscr{M}_s^N$ such that $\mathscr{G}(\alpha) = L^\infty(\Omega, \mathbb{G}(\alpha))$;

2. If $\alpha$ is an $L^\infty(\Omega, [0, 1])$-function then

$$\mathscr{G}(\alpha) = \{A^* \text{ measurable} \mid A^*(x) \in \mathbb{G}(\alpha(x)) \text{ a. e. in } \Omega\}. \tag{2.9}$$

3. $\mathbb{G}^p(\alpha) \stackrel{\text{def}}{=} \{ A^* \in \mathcal{M}_s^N \mid m\,\mathbb{I} \leqq A^* \leqq M\,\mathbb{I}$ and $A^*$ is a H-limit of periodic sequences$\}$ is dense in $\mathbb{G}(\alpha)$.

This last remark is in essence a statement of the irrelevance of scale separation (or of characteristic length of the heterogeneities) as far as bounds for elastic moduli at fixed volume fraction are concerned.

In more intuitive words these results can be interpreted as follows. If the body is filled with the damaged and the sound materials in a manner such that there is a fine mixture of the two materials at a point $x$ with a proportion $\alpha(x)$ of the damaged material, then the local behaviour of the mixture at $x$ is like that of an homogeneous elastic material with a rigidity tensor $A^*(x)$. This tensor is called the effective rigidity tensor of the mixture at $x$. $A^*(x)$ does not only depend on $\alpha(x)$, but also on the microgeometry (the $\chi_{\sigma(n)}$'s) with which the mixture in proportion $\alpha(x)$ is realized. In other words, there may exist several effective tensors associated to the same local volume fraction $\alpha(x)$. The set of effective elastic tensors that we can obtain by such local mixtures of the materials $A^0$ and $A^1$ is $\mathbb{G}(\alpha(x))$. For the body $\Omega$ in which we assume that there exists a field $\alpha(x)$ of mixture of the sound and damaged materials the set of effective elastic tensor fields $A^*(x)$ that can be obtained is denoted by $\mathcal{G}(\alpha)$.

In one dimension the set $\mathbb{G}(\alpha)$ is reduced to one element, namely the harmonic mean $A_*(\alpha)$ of the stiffness coefficient $A^0$ and $A^1$ in respective proportion $1-\alpha$ and $\alpha$, *i.e.*,

$$\frac{1}{A_*(\alpha)} = \frac{1-\alpha}{A^0} + \frac{\alpha}{A^1}. \tag{2.10}$$

In dimension $N \geqq 2$ however, there are no existing characterizations of $\mathbb{G}(\alpha)$ (even in the case of two well-ordered isotropic elastic materials). Such a characterization is available in the case of two isotropic conducting materials, see [M & T, 1985]. Fortunately a complete characterization of $\mathbb{G}(\alpha)$ will not be necessary to our purpose and we will merely need an optimal lower bound on the energy associated to the possible binary mixtures of $A^0$ and $A^1$ in volume fraction $\alpha$, $1-\alpha$ for a given strain $\varepsilon$. A convenient way of deriving a lower bound is to use the Kohn-Milton version [Kohn & Milton, 1986] of the variational method of Hashin and Shtrikman [Hashin & Shtrikman, 1962] and then to prove optimality by adopting a specific microgeometry. Specifically, when $A^0$ and $A^1$ are two *isotropic positive well-ordered* stiffness tensors, the following theorem holds true:

THEOREM 2.2. — *When* $A^0$ *and* $A^1$ *are two strictly well ordered isotropic positive fourth order tensors*, i.e. *when they are such that*

$$A^0 = \lambda^0 I \otimes I + 2\mu^0\,\mathbb{I}, \qquad A^1 = \lambda^1 I \otimes I + 2\mu^1\,\mathbb{I}, \tag{2.11}$$

*with*

$$K^0 = \lambda^0 + \frac{2}{N}\mu^0 > K^1 = \lambda^1 + \frac{2}{N}\mu^1 > 0, \qquad \mu^0 > \mu^1 > 0, \tag{2.12}$$

*where* I *and* $\mathbb{I}$ *are respectively the identity mapping on* $\mathbb{M}_s^N$ *and* $\mathcal{M}_s^N$ *and* $\otimes$ *denotes the tensor product, then*

$$(2.13)\qquad \min_{A^* \in \mathbb{G}^p(\alpha)} \frac{1}{2} A^* \varepsilon . \varepsilon = \frac{1}{2} A^1 \varepsilon . \varepsilon + \sup_{\sigma \in \mathbb{M}_s^N} \inf_{n \in \mathbb{R}^N,\, n.n=1} (1-\alpha) f(\varepsilon, \alpha; \sigma, n),$$

*with*

$$(2.14)\quad f(\varepsilon, \alpha; \sigma, n) = \sigma.\varepsilon - \frac{1}{2}(A^0 - A^1)^{-1} \sigma.\sigma - \frac{\alpha}{2\mu^1}\left(\sigma n.\sigma n - \frac{\lambda^1+\mu^1}{\lambda^1+2\mu^1}(\sigma n.n)^2\right).$$

A derivation of the term on the right hand side of (2.13) as a lower bound of the left hand side can be found in [Kohn & Lipton, 1988] Sections 2 A-B in the incompressible case, in [Kohn, 1988] Section 3, or in [Allaire & Kohn, 1991] in the case under consideration. Optimality (*i.e.* the equality) is found to hold true in [K, 1988] or [A & K, 1991]. The argument uses subdifferential calculus so as to compute the optimality conditions for the maximum in the right hand side of (2.13), together with the layering formula obtained in [F & M, 1986]. The reader should consult [K, 1988], Section 4, for details, although we present at the end of Section 3 an alternative proof in the two dimensional case which shows that the lower bound is attained by rank-one or rank-two laminates.

REMARK 2.2. – The right hand side of (2.13) is a convex function of $\varepsilon$, which can be further checked to be isotropic and positively homogeneous of degree 2 in $\varepsilon$. See the proof of Theorem 3.1 in Section 3 for a justification of these assertions in the two-dimensional case.

### 2.2. THE RELAXED FORMULATION

Let us return to the damage evolution problem $(\mathcal{M}_1)$ and focus for now on the initial step $t_1$. We construct a minimizing sequence $(v_n, \chi_n) \in \mathscr{C}^1 \times \mathscr{V}^1$ of the Lagrangian $\mathscr{L}^1$, with $v_n$ related to $\chi_n$ by (2.1), such that $\chi_n$ converges to $\alpha$ in $L^\infty(\Omega, [0, 1])$. According to the result of the previous Subsection, $v_n$ converges to $u$ which is the solution of an elastic problem written in terms of an effective elastic tensor field $A^*$, that is, $v_n \rightharpoonup u \in \mathscr{V}^1$ such that

$$(2.15)\qquad a^*(u, v-u) = l^1(v-u), \qquad \forall v \in \mathscr{V}^1$$

where

$$(2.16)\qquad a^*(u, v) = \int_\Omega A^*(x)\, \varepsilon(v)(x) . \varepsilon(v)(x)\, dx.$$

Moreover, recalling Property 3 in Subsection 2.1,

$$(2.17)\qquad \lim a(\chi_n; v_n, v_n) = a^*(u, u).$$

Thus the search for stable solutions to the damage evolution problem forces us to extend the type of possible distributions of damage in the body, so as to allow for locally fine mixtures. In turn, because such mixtures behave "like" a homogenized elastic material, we must change the expression for the energy. The resulting formulation is called the relaxed problem and it is obtained through a procedure which is similar to that used in optimal compliance design, *see* [M & T, 1985] and [Kohn & Strang, 1986] for instance. Note however that optimal compliance design results in a problem of stiffness maximization whereas our setting should be viewed as associated to stiffness minimization.

We introduce the new set $\mathscr{C}^1_{\mathrm{rel}}$ of admissible distributions of damage, namely

$$\mathscr{C}^1_{\mathrm{rel}} = \mathrm{L}^\infty(\Omega, [0, 1]). \tag{2.18}$$

We then define the relaxed energy $\mathrm{J}^1_{\mathrm{rel}} : \mathscr{C}^1_{\mathrm{rel}} \to \mathbb{R}$ by

$$\mathrm{J}^1_{\mathrm{rel}}(\alpha) \overset{\mathrm{def}}{=} \inf\{\liminf_{n \to \infty} \mathrm{J}^1(\chi_n) \mid \chi_n \in \mathrm{C}^1 \text{ with } \chi_n \overset{*}{\rightharpoonup} \alpha\}. \tag{2.19}$$

The definition (1.22) of $\mathrm{J}^1(\chi)$ permits a simplified expression for $\mathrm{J}^1_{\mathrm{rel}}$. For any sequence $\chi_n$ converging to $\alpha$ and such that the associated sequence $\mathrm{A}_n$ H-converges to $\mathrm{A}^*$ the following equality holds true as a result of Property 3:

$$\lim \mathrm{J}^1(\chi_n) = \min_{v \in \mathscr{V}^i} \left\{ \frac{1}{2} \int_\Omega \mathrm{A}^* \varepsilon(v) . \varepsilon(v)\, dx + \int_\Omega \kappa\alpha\, dx - l^1(v) \right\}. \tag{2.20}$$

By the very definition of $\mathscr{G}(\alpha)$ we thus obtain that

$$\mathrm{J}^1_{\mathrm{rel}}(\alpha) = \operatorname*{Min}_{\mathrm{A}^* \in \mathscr{G}(\alpha)} \operatorname*{Min}_{v \in \mathscr{V}^1} \left\{ \frac{1}{2} \int_\Omega \mathrm{A}^* \varepsilon(v) . \varepsilon(v)\, dx + \int_\Omega \kappa\alpha\, dx - l^1(v) \right\}, \tag{2.21}$$

and commuting the order of minimizations yields

$$\mathrm{J}^1_{\mathrm{rel}}(\alpha) = \operatorname*{Min}_{v \in \mathscr{V}^i} \left\{ \operatorname*{Min}_{\mathrm{A}^* \in \mathscr{G}(\alpha)} \left\{ \frac{1}{2} \int_\Omega \mathrm{A}^* \varepsilon(v) . \varepsilon(v)\, dx \right\} + \int_\Omega \kappa\alpha\, dx - l^1(v) \right\}. \tag{2.22}$$

Upon recalling Remark 2.1 we are at liberty to commute the first integral with the minimization with respect to $\mathrm{A}^*$. We obtain

$$\mathrm{J}^1_{\mathrm{rel}}(\alpha) = \operatorname*{Min}_{v \in \mathscr{V}^i} \left\{ \frac{1}{2} \int_\Omega \operatorname*{Min}_{\mathrm{A}^*(x) \in \mathbb{G}^p(\alpha(x))} \mathrm{A}^*(x) \varepsilon(v)(x) . \varepsilon(v)(x)\, dx + \int_\Omega \kappa\alpha\, dx - l^1(v) \right\}.$$

Finally

$$\mathrm{J}^1_{\mathrm{rel}}(\alpha) = \operatorname*{Min}_{v \in \mathscr{V}^1} \mathscr{L}^1_{\mathrm{rel}}(v, \alpha) \tag{2.23}$$

with

$$\mathscr{L}^1_{\mathrm{rel}}(v, \alpha) = \int_\Omega \varphi^*(\varepsilon(v)(x), \alpha(x))\, dx + \int_\Omega \kappa\alpha(x)\, dx - l^1(v) \tag{2.24}$$

where the function $\varphi^*: \mathbb{M}_s^N \times [0, 1] \to \mathbb{R}$ is defined by

$$\varphi^*(\varepsilon, \alpha) = \operatorname*{Min}_{A^* \in \mathbb{G}^p(\alpha)} \frac{1}{2} A^* \varepsilon . \varepsilon. \tag{2.25}$$

At the first time step $t_1$ the relaxed problem can thus be formulated as

(2.26) *Find* $(u^1, \alpha^1)$ *that minimizes* $\mathscr{L}^1_{\text{rel}}$ *over* $\mathscr{V}^1 \times \mathscr{C}^1_{\text{rel}}$.

Note that the relaxed problem is by construction guaranteed to possess at least one solution. Once $(u^1, \alpha^1)$ is determined, $\varepsilon(u^1)$ can be computed for almost every $x$ in $\Omega$ and, recalling (2.25), one can determine $A^*(x)$ such that

$$\varphi^*(\varepsilon(u^1(x)), \alpha^1(x)) = \frac{1}{2} A^*(x)\, \varepsilon(u^1(x)) . \varepsilon(u^1(x)) \tag{2.27}$$

so that, at almost every point x of $\Omega$, a local volume fraction $\alpha^1(x)$ and an associated effective tensor $A^*(x)$ are available.

It is now time to pass to the subsequent time steps $t_i (i \geqq 2)$. We thus assume that $\alpha^1(x)$ has been determined. The reader is reminded that the admissible characteristic functions $\chi$ at subsequent time steps have to satisfy the irreversibility constraint $\chi \in C^i$, *i.e.* $\chi(x) \geqq \chi^{i-1}(x)$ a.e. in $\Omega$. Bearing this constraint in mind we propose in the spirit of the relaxation performed at the first time step, the following *two-field relaxed formulation*:

Problem $(\mathscr{M}_I^{\text{rel}})$:

$$(\mathscr{M}_I^{\text{rel}}) \qquad \begin{cases} \textit{For } i \in \{1, \ldots, I\}, \textit{find } (u^i, \alpha^i) \textit{ that minimizes } \mathscr{L}^i_{\text{rel}} \textit{ over } \mathscr{V}^i \times \mathscr{C}^i_{\text{rel}} \\ \qquad\qquad \textit{with the initial constraint } \alpha^0 = 0, \end{cases}$$

where

$$\mathscr{L}^i_{\text{rel}}(v, \alpha) = \int_\Omega \varphi^*(\varepsilon(v)(x), \alpha(x))\, dx + \int_\Omega \kappa \alpha(x)\, dx - l^i(v) \tag{2.28}$$

and

$$\mathscr{C}^i_{\text{rel}} = \{\alpha \in L^\infty(\Omega, [0, 1]) \mid \alpha(x) \geqq \alpha^{i-1}(x) \text{ a.e. in } \Omega\}. \tag{2.29}$$

The problem $(\mathscr{M}_I^{\text{rel}})$ is easily seen to admit a solution. The existence of a solution at time $t_1$ was mentioned above. For subsequent times the existence of a solution is guaranteed upon considering at each fixed time $t_i$ a minimizing sequence and upon recalling Property 2 of H-convergence which essentially states that mixtures of binary mixtures are also binary mixtures of those same materials.

Remark 2.3. – It would however be improper to view $(\mathscr{M}_I^{\text{rel}})$ as a relaxation of $(\mathscr{M}_I)$ because it is only a *bona fide* relaxation at the first time step while the susequent time steps should be seen as natural extensions of the relaxation performed at time $t_1$ once the irreversibility constraint has been weakened so as to hold true on "average". Thus it is implicitly assumed here that the material only remembers the volume fraction of each phase, when going from one time step to the next. The complete space-time relaxation of the problem, which may or may not, upon time discretization, yield ($\mathscr{M}_{\mathrm{I}}^{\mathrm{rel}}$) has yet to be investigated.

From now onwards the solutions of ($\mathscr{M}_{\mathrm{I}}^{\mathrm{rel}}$) will be called *generalized stable solutions*.

REMARK 2.4. — Classical stable solutions are also generalized ones, *i.e.*, if ($u^i$, $\chi^i$) is a solution of ($\mathscr{M}_{\mathrm{I}}$), it is also a solution of ($\mathscr{M}_{\mathrm{I}}^{\mathrm{rel}}$).

This "relaxed" problem consists in seeking stable evolutions of damage in the body $\Omega$ made of an elastic damaging material whose damage parameter $\alpha$ can evolve *progressively* from 0 to 1 and whose elastic potential is $\varphi^*$. We started with a brutal damage model where the material drops in stiffness from $A^0$ to $A^1$ and looked for stable solutions. We were then naturally led to expect fine evolving mixtures that can be seen as a progressive damage of the material. A generalized stable solution determines a local volume fraction $\alpha(x)$ and a local effective tensor $A^*(x)$. It does not however determine which microstructure is preferred for a given $A^*$ or, in other words, which minimizing sequence $\chi_n$ is the preferred one. We would like to stress once again the fundamental difference between our approach and the usual "homogenization" approach to progressive damage. The latter presupposes a microstructure that depends on a certain number of parameters and uses homogenization to produce the dependence of the stiffness upon the damage parameters, while our approach forces mixtures to take place but only determines their effective behaviour; there is no *a priori* assumption on the kind of relevant microstructure that should be used. Actually one need not even think of a microstructure (or a mesostructure) although certainly the true material will experience a constraint on the fineness of the possible mixtures because of surface tension and possibly other mechanisms. The reader is referred to the abundant literature on phase transitions, *cf.* e.g. [Ball & James, 1987], [James & Kinderlehrer, 1987], [Kohn, 1990] and references therein, in which such kinds of fine mixtures of different phases of a material have firstly been studied.

The meticulous reader will not fail to notice that uniqueness is not ensured. For instance it was found in [F & M, 1991] that the one-dimensional problem of the traction of a bar admits an infinite number of stable solutions. This lack of uniqueness means that, at each time step, one will only follow one of the possible bifurcation branches, but that other stable paths are equally possible.

Equivalent formulations of the relaxed problem ($\mathscr{M}_{\mathrm{I}}^{\mathrm{rel}}$) may be obtained in a manner similar to that of Section 1. For instance, minimizing first with respect to $\alpha$ at fixed $v$ yields the following relaxed version ($\mathscr{S}_{\mathrm{I}}^{\mathrm{rel}}$) of ($\mathscr{S}_{\mathrm{I}}$).

PROBLEM ($\mathscr{S}_{\mathrm{I}}^{\mathrm{rel}}$). — *For* $i \in \{1, \ldots, \mathrm{I}\}$, *find* ($u^i$, $\alpha^i$) *such that*

1. $u^i$ *is a minimizer of* $\Phi^{*i}$ *over* $\mathscr{V}^i$,
2. $\alpha^i(x)$ *is a minimizer of* $\varphi^*(\varepsilon(u^i(x)), \alpha) + \kappa\alpha$ *over* $[\alpha^{i-1}(x), 1]$, *with*

$$\psi^{*i}(x, \varepsilon) = \operatorname*{Min}_{\alpha \in [\alpha^{i-1}(x),\, 1]} \{\varphi^*(\varepsilon, \alpha) + \kappa\alpha\} \tag{2.30}$$

and

$$\Phi^{*i}(v)=\int_{\Omega}\psi^{*i}(x,\varepsilon(v)(x))\,dx-l^i(v). \tag{2.31}$$

The remaining difficulty lies in a more explicit determination of $\varphi^*(\varepsilon,\alpha)$ and $\psi^{*i}(\varepsilon)$ than the somewhat mysterious definitions (2.13), (2.25) and (2.30). This is the aim of the next Section.

## 3. The relaxed energy

In this section we propose to determine an explicit expression for $\varphi^*(\varepsilon,\alpha)$ in the one and two dimensional cases and in an isotropic setting. The definition (2.25) for $\varphi^*(\varepsilon,\alpha)$ involves the minimization of the energy density $1/2\ A^*\varepsilon.\varepsilon$ over all $A^*$'s that belong to $\mathbb{G}^p(\alpha)$.

In the one dimensional case $\mathbb{G}^p(\alpha)$ is reduced to the harmonic mean of the elastic moduli and we obtain

$$\varphi^*(\varepsilon,\alpha)=\frac{1}{2}A_*(\alpha)\,\varepsilon^2, \tag{3.1}$$

with $A_*(\alpha)$ given by (2.10).

The problem of stiffness minimization for binary mixtures of isotropic elastic materials has been addressed in Theorem 2.2 of Section 2. We specialize the result to the two dimensional case and propose to derive the following

THEOREM 3.1. — *The hypotheses are those of Theorem* 2.2 *in the two dimensional case. For* $0\leqq\alpha\leqq 1$, $\mathbb{M}_s^2$ is partitioned into three cones $\mathbb{M}_i(\alpha)$, $i=\mathrm{I}$, II *or* III, *defined as follows:*

$$\mathbb{M}_{\mathrm{I}}(\alpha)=\{\varepsilon\,|\,(K^0-K^1)(\alpha\mu^0+(1-\alpha)\mu^1)\,|\mathrm{Tr}\,\varepsilon| < (\mu^0-\mu^1)(\alpha K^0+(1-\alpha)K^1)\sqrt{2}\,\|\varepsilon^D\|\}, \tag{3.2}$$

$$\mathbb{M}_{\mathrm{III}}(\alpha)=\{\varepsilon\,|\,\alpha(K^0-K^1)\,|\mathrm{Tr}\,\varepsilon|\geqq(\mu^1+\alpha K^0+(1-\alpha)K^1)\sqrt{2}\,\|\varepsilon^D\|\}, \tag{3.3}$$

$$\mathbb{M}_{\mathrm{II}}(\alpha)=\mathbb{M}_s^2-\mathbb{M}_{\mathrm{I}}(\alpha)\cup\mathbb{M}_{\mathrm{III}}(\alpha), \tag{3.4}$$

*where* $\mathrm{Tr}\,\varepsilon$ *is the Trace of* $\varepsilon$, $\varepsilon^D=\varepsilon-(\mathrm{Tr}\,\varepsilon/2)\,\mathrm{I}$ *its deviatoric part and* $\|\varepsilon^D\|=\sqrt{\varepsilon^D.\varepsilon^D}$ *the norm of* $\varepsilon^D$. *Then the explicit formula for the relaxed energy* $\varphi^*(\varepsilon,\alpha)$ *is*

$$\varphi^*(\varepsilon,\alpha)=\frac{1}{2}K_*(\alpha)(\mathrm{Tr}\,\varepsilon)^2+\mu_*(\alpha)\,\varepsilon^D.\varepsilon^D, \qquad \varepsilon\in\mathbb{M}_{\mathrm{I}}(\alpha); \tag{3.5}$$

$$\varphi^*(\varepsilon,\alpha)=\frac{1}{2}K^*(\alpha)(\mathrm{Tr}\,\varepsilon)^2+\mu^*(\alpha)\,\varepsilon^D.\varepsilon^D -\frac{1}{2}C(\alpha)((K^0-K^1)\,|\mathrm{Tr}\,\varepsilon|+(\mu^0-\mu^1)\sqrt{2}\,\|\varepsilon^D\|)^2, \qquad \varepsilon\in\mathbb{M}_{\mathrm{II}}(\alpha); \tag{3.6}$$

$$\varphi^*(\varepsilon, \alpha) = \frac{1}{2} K_{HS}(\alpha)(\operatorname{Tr}\varepsilon)^2 + \mu^1 \varepsilon^D . \varepsilon^D, \qquad \varepsilon \in \mathbb{M}_{III}(\alpha). \tag{3.7}$$

*In* (3.5), $K_*(\alpha)$ *and* $\mu_*(\alpha)$ *represent the harmonic means of* $K^0$, $K^1$ *and* $\mu^0$, $\mu^1$ *in proportion* $(1-\alpha)$, $\alpha$, *while in* (3.6) $K^*(\alpha)$ *and* $\mu^*(\alpha)$ *represent their arithmetic means. In* (3.6), *the coefficient* $C(\alpha)$ *is defined by*

$$C(\alpha) = \frac{\alpha(1-\alpha)}{\alpha(K^0+\mu^0)+(1-\alpha)(K^1+\mu^1)}, \tag{3.8}$$

*and, in* (3.7), $K_{HS}(\alpha)$ *represents the lower Hashin-Shtrikman bulk modulus bound,*

$$K_{HS}(\alpha) = K^1 + (1-\alpha)\left(\frac{\alpha}{K^1+\mu^1} + \frac{1}{K^0-K^1}\right)^{-1}. \tag{3.9}$$

REMARK 3.1. — This result was also obtained by Allaire & Kohn [1991] who derive the minimum of the effective energies for binary mixtures of two (not necessarily well ordered) elastic materials. Note that disregarding the assumption that $A^0$ is greater than $A^1$ leads to non trivial difficulties in proving a result of the type of Theorems 2.2 and 3.1. Actually the Hashin-Shtrikman variational principle must be abandoned and replaced by what is usually referred to as the translation method (*cf.* [A & K, 1991] for further details). In any case we are only concerned with well ordered materials in the present study.

*Sketch of the proof.* — The major part of the proof is purely computational. Firstly it is remarked that $\mathbb{M}_I(\alpha)$, $\mathbb{M}_{II}(\alpha)$ and $\mathbb{M}_{III}(\alpha)$ are disjoint sets covering $\mathbb{M}_s^2$. It is clearly true because the inequality

$$\frac{\mu^1+K^1+\alpha(K^0-K^1)}{K^1+\alpha(K^0-K^1)} > \frac{\alpha(\mu^0-\mu^1)}{\mu^1+\alpha(\mu^0-\mu^1)} \tag{3.10}$$

holds true under the hypotheses of the theorem. Set, for any given $\varepsilon$ in $\mathbb{M}_s^2$ and $\alpha \in (0, 1)$ (the cases $\alpha = 0$ or 1 are trivial),

$$f^*(\varepsilon, \alpha) = \sup_{\sigma \in \mathbb{M}_s^2} \inf_{n \in \mathbb{R}^2,\, n.n=1} f(\varepsilon, \alpha; \sigma, n), \tag{3.11}$$

with $f(\varepsilon, \alpha; \sigma, n)$ given by (2.14). Introduce the following notations

$$\delta\lambda = \lambda^0 - \lambda^1, \qquad \delta\mu = \mu^0 - \mu^1, \qquad \delta K = \delta\lambda + \delta\mu \tag{3.12}$$

and denote by $\sigma_1$, $\sigma_2$ (respectively $\varepsilon_1$, $\varepsilon_2$) the eigenvalues of $\sigma$ (respectively $\varepsilon$). Further, for a fixed $\sigma$ in $\mathbb{M}_s^2$, denote by $n_1$, $n_2$ the components of the unit vector $n$ in an orthonormal eigenbasis of $\sigma$ and set

$$n_1^2 = \nu, \qquad \text{which} \quad 0 \leqq \nu \leqq 1. \tag{3.13}$$

It is well known, that for given eigenvalues $\sigma_1$, $\sigma_2$, $\varepsilon_1$, $\varepsilon_2$, the quantity $\sigma.\varepsilon$ reaches its maximum when $\sigma$ and $\varepsilon$ are diagonal in the same basis. It is then easily checked that

$$f^*(\varepsilon, \alpha) = \sup_{(\sigma_1, \sigma_2) \in \mathbb{R}^2} \inf_{\nu \in [0, 1]} \{\sigma_1 \varepsilon_1 + \sigma_2 \varepsilon_2 - g(\alpha, \sigma_1, \sigma_2, \nu)\}, \tag{3.14}$$

with

$$g(\alpha, \sigma_1, \sigma_2, \nu) = \frac{1}{8\delta\mu}(\sigma_1 - \sigma_2)^2 + \frac{1}{8\delta K}(\sigma_1 + \sigma_2)^2 + \frac{\alpha}{2\mu^1}\left(\nu\sigma_1^2 + (1-\nu)\sigma_2^2 - \frac{K^1}{K^1+\mu^1}(\nu\sigma_1 + (1-\nu)\sigma_2)^2\right). \tag{3.15}$$

Let us remark that $g$ is a convex function of $(\sigma_1, \sigma_2)$ and a concave function of $\nu$. According to a classical min-max theorem (*cf.* [Ekeland & Temam, 1974] Chapter 6), this sup-inf problem possesses a saddle point (the inf and the sup are min and max) and we can exchange the max and the min in the right hand side of (3.14). We denote by $(\sigma_1^*, \sigma_2^*, \nu^*)$ an optimal triplet. Since $(\sigma_1^*, \sigma_2^*)$ maximizes the bracket in (3.14) over $\mathbb{R}^2$ with $\nu = \nu^*$, it satisfies the optimality condition

$$\left\{\begin{aligned} \varepsilon_1 &= \frac{1}{4\delta\mu}(\sigma_1^* - \sigma_2^*) + \frac{1}{4\delta K}(\sigma_1^* + \sigma_2^*) + \frac{\alpha\nu^*}{\mu^1}\left(\sigma_1^* - \frac{K^1}{K^1+\mu^1}(\nu^*\sigma_1^* + (1-\nu^*)\sigma_2^*)\right) \\ \varepsilon_2 &= \frac{1}{4\delta\mu}(\sigma_2^* - \sigma_1^*) + \frac{1}{4\delta K}(\sigma_1^* + \sigma_2^*) + \frac{\alpha(1-\nu^*)}{\mu^1}\left(\sigma_2^* - \frac{K^1}{K^1+\mu^1}(\nu^*\sigma_1^* + (1-\nu^*)\sigma_2^*)\right). \end{aligned}\right. \tag{3.16}$$

Moreover, once $(\sigma_1^*, \sigma_2^*)$ is known, the relaxed energy reads as

$$\varphi^*(\varepsilon, \alpha) = \frac{1}{2}K^1(\varepsilon_1 + \varepsilon_2)^2 + \frac{1}{2}\mu^1(\varepsilon_1 - \varepsilon_2)^2 + \frac{1-\alpha}{2}(\sigma_1^*\varepsilon_1 + \sigma_2^*\varepsilon_2). \tag{3.17}$$

In order to determine $(\sigma_1^*, \sigma_2^*, \nu^*)$ it remains to write the optimality condition in $\nu$. We have to maximize $g(\alpha, \sigma_1^*, \sigma_2^*, \nu)$ with respect to $\nu$ in the interval [0, 1]. Two cases must be considered according to whether $\sigma_1^* = \sigma_2^*$ or not.

If $\sigma_1^* = \sigma_2^* = \sigma^*$ then $g$ does not depend on $\nu$ and its maximization does not yield any constraint on $\nu^*$. Thus the optimal pair $(\sigma^*, \nu^*)$ is given by (3.16). Easy computations lead to

$$\sigma^* = \frac{(K^0 - K^1)(\mu^1 + K^1)}{\mu^1 + \alpha K^0 + (1-\alpha)K^1}\operatorname{Tr}\varepsilon \tag{3.18}$$

and

$$\alpha(K^0 - K^1)\operatorname{Tr}\varepsilon(2\nu^* - 1) = (\mu^1 + \alpha K^0 + (1-\alpha)K^1)(\varepsilon_1 - \varepsilon_2). \tag{3.19}$$

Insertion of (3.18) into (3.17) results in expression (3.7). Note however that $\nu^*$ is constrained to remain in the interval [0, 1] (or equivalently $|2\nu^* - 1| \leq 1$) which requires in turn that $\varepsilon$ belongs to $\mathbb{M}_{\text{III}}(\alpha)$. In such a case, (3.19) determines the optimal value $v^*$ except when $\varepsilon=0$.

Assume now that $\sigma_1^* \neq \sigma_2^*$. Then $g(\alpha, \sigma_1^*, \sigma_2^*, v)$ is a strictly concave function of $v$ which attains its maximum at an unique value $v^*$ of $v$. Three subcases must be considered according to whether $v^*=0$, $v^*=1$ or $0<v^*<1$. Specifically, we have

$$(3.20)\quad 0<v^*<1 \quad \text{when } \sigma_1^* \neq \sigma_2^* \text{ and } \frac{\partial g}{\partial v}(\alpha, \sigma_1^*, \sigma_2^*, 0)>0>\frac{\partial g}{\partial v}(\alpha, \sigma_1^*, \sigma_2^*, 1),$$

$$(3.21)\qquad v^*=0 \quad \text{when } \sigma_1^* \neq \sigma_2^* \text{ and } \frac{\partial g}{\partial v}(\alpha, \sigma_1^*, \sigma_2^*, 0)\leqq 0,$$

$$(3.22)\qquad v^*=1 \quad \text{when } \sigma_1^* \neq \sigma_2^* \text{ and } \frac{\partial g}{\partial v}(\alpha, \sigma_1^*, \sigma_2^*, 1)\geqq 0.$$

In the case (3.20), $g(\alpha, \sigma_1^*, \sigma_2^*, v)$ attains its maximum at the point $v^*$ of the open interval (0, 1) such that $\partial g/\partial v(\alpha, \sigma_1^*, \sigma_2^*, v^*)=0$ which gives

$$(3.23)\qquad v^*=\frac{(K^1+\mu^1)(\sigma_1^*+\sigma_2^*)-2K^1\sigma_2^*}{2K^1(\sigma_1^*-\sigma_2^*)}.$$

Inserting (3.23) into (3.16) yields

$$(3.24)\qquad \begin{cases} \sigma_1^*=\dfrac{K^1\,\delta K}{K^1+\alpha\delta K}(\varepsilon_1+\varepsilon_2)+\dfrac{\mu^1\,\delta\mu}{\mu^1+\alpha\delta\mu}(\varepsilon_1-\varepsilon_2) \\ \sigma_2^*=\dfrac{K^1\,\delta K}{K^1+\alpha\delta K}(\varepsilon_1+\varepsilon_2)+\dfrac{\mu^1\,\delta\mu}{\mu^1+\alpha\delta\mu}(\varepsilon_2-\varepsilon_1). \end{cases}$$

Inserting in turn (3.24) into (3.23) we obtain

$$(3.25)\qquad 2v^*-1=\frac{(\mu^1+\alpha\delta\mu)\,\delta K\,(\varepsilon_1+\varepsilon_2)}{(K^1+\alpha\delta K)\,\delta\mu\,(\varepsilon_1-\varepsilon_2)}.$$

Finally inserting (3.24) into (3.17) leads to (3.5). This latter relation is only valid when the constraints in (3.20) are satisfied. These constraints are equivalent to $v^*\in(0, 1)$ with $v^*$ given by (3.25). Consequently (3.5) is valid if and only if $\varepsilon$ belongs to $\mathbb{M}_{\text{I}}(\alpha)$.

In the case (3.21), $(\sigma_1^*, \sigma_2^*)$ is obtained through solving (3.16) with $v^*=0$ which yields

$$(3.26)\qquad \begin{cases} \sigma_1^*=\dfrac{\delta K\,(K^1+\mu^1+2\,\alpha\delta\mu)\,(\varepsilon_1+\varepsilon_2)+\delta\mu\,(K^1+\mu^1+2\,\alpha\delta K)\,(\varepsilon_1-\varepsilon_2)}{K^1+\mu^1+\alpha\,(\delta K+\delta\mu)} \\ \sigma_2^*=\dfrac{(K^1+\mu^1)\,(\delta K\,(\varepsilon_1+\varepsilon_2)-\delta\mu\,(\varepsilon_1-\varepsilon_2))}{K^1+\mu^1+\alpha\,(\delta K+\delta\mu)}. \end{cases}$$

The pair $(\sigma_1^*, \sigma_2^*)$ defined by (3.26) must satisfy the constraints in (3.21). After tedious computations we obtain that these constraints are satisfied if and only if $\varepsilon$ belongs to the subset of $\mathbb{M}_{\text{II}}(\alpha)$ where $|\varepsilon_2|\geqq|\varepsilon_1|$. Then inserting (3.26) into (3.17) yields (3.6).

The last case (3.22) is symmetric to the previous one. It suffices to exchange $\varepsilon_1$ with $\varepsilon_2$ and $\sigma_1^*$ with $\sigma_2^*$ in (3.26) as well as in the constraint. Thus (3.6) remains unchanged and is valid in the full set $\mathbb{M}_{\text{II}}(\alpha)$.

In the present setting the optimality of the bound (2.13) can be readily checked. To this effect we propose to show that, for every given pair $(\varepsilon, \alpha)$, $\alpha \in (0, 1)$, there exists an effective tensor $A^* \in \mathbb{G}(\alpha)$ such that

$$\frac{1}{2} A^* \varepsilon . \varepsilon = \frac{1}{2} A^1 \varepsilon . \varepsilon + (1-\alpha) f^*(\varepsilon, \alpha). \tag{3.27}$$

Let us first recall the following Lemma ([F & M, 1986], Proposition 4.2):

LEMMA 3.2. — *Let us consider two isotropic well-ordered stiffness tensors* $A^0$ *and* $A^1$ *and* $\alpha \in (0, 1)$, $\theta^1 \in [0, 1]$, $\theta^2 = 1 - \theta^1$, *and* $n^1$, $n^2$ *two unit vectors of* $\mathbb{R}^2$. *Then the tensor* $A^*$ *given, for any* $\sigma \in \mathbb{M}_s^2$, *through*

$$(1-\alpha)(A^* - A^1)^{-1} \sigma = (A^0 - A^1)^{-1} \sigma + \frac{\alpha}{\mu^1} \left( \sum_{i=1}^{2} \theta^i \left( \sigma n^i \otimes_s n^i - \frac{K^1}{K^1 + \mu^1} (\sigma n^i . n^i) n^i \otimes n^i \right) \right) \tag{3.28}$$

*is an element of* $\mathbb{G}(\alpha)$.

*In* (3.28) $\otimes$ *and* $\otimes_s$ *respectively denote the tensor product and the symmetrized tensor product. The tensor* $A^*$ *is the effective stiffness tensor associated to a process of "rank-two layering" which consists in the layering of* $A^0$ *with* $A^1$, *in the direction* $n^1$ *and in respective volume fraction* $1 - \theta^1 \alpha$ *and* $\theta^1 \alpha$, *followed by the layering of the resulting composite with* $A^1$, *in the direction* $n^2$ *and in respective volume fraction* $(1-\alpha)/(1-\theta^1 \alpha)$ *and* $\theta^2 \alpha/(1-\theta^1 \alpha)$. *Rank-one laminates are obtained upon setting* $\theta^1 = 1$.

Let us assume that there exists $\theta^1 \in [0, 1]$ and two unit vectors of $\mathbb{R}^2$, say $n^1$ and $n^2$, such that the optimality condition (3.16) can be written in the following tensorial form:

$$\varepsilon = (A^0 - A^1)^{-1} \sigma^* + \frac{\alpha}{\mu^1} \left( \sum_{i=1}^{2} \theta^i \left( \sigma^* n^i \otimes_s n^i - \frac{K^1}{K^1 + \mu^1} (\sigma^* n^i . n^i) n^i \otimes n^i \right) \right). \tag{3.29}$$

Then, according to Lemma 3.2 and upon denoting by $A^*$ the effective stiffness tensor obtained by a rank-two (or a rank-one if $\theta^1 = 1$) layering of $A^0$ and $A^1$ in volume fraction $\alpha$ and $1-\alpha$, relation (3.29) reads as

$$\varepsilon = (1-\alpha)(A^* - A^1)^{-1} \sigma^*. \tag{3.30}$$

Moreover, since $f^*(\varepsilon, \alpha) = \frac{1}{2} \sigma^* . \varepsilon$, we obtain

$$f^*(\varepsilon, \alpha) = \frac{1}{2(1-\alpha)} (A^* - A^1) \varepsilon . \varepsilon, \tag{3.31}$$

which is nothing but (3.27). Hence, it remains to transform (3.16) into (3.29). Consider $\varepsilon \in \mathbb{M}_s^2$ and examine the different cases, $e^1$ and $e^2$ being two orthonormal eigenvectors of $\varepsilon$. If $\varepsilon$ belongs to $\mathbb{M}_{\mathrm{I}}(\alpha)$, set $\theta^1=1$, $\theta^2=0$ and $n^1=n^*$ (one of) the direction(s) corresponding to the optimal $\nu^*$ given by (3.25), that is

$$n^*=\sqrt{\nu^*}\,e^1+\sqrt{1-\nu^*}\,e^2. \tag{3.32}$$

Easy computations that use (3.16), (3.23) and the fact that $\sigma^*$ is diagonal in the basis $(e^1, e^2)$ lead to (3.29). Assume now that $\varepsilon$ belongs to $\mathbb{M}_{\mathrm{II}}(\alpha)$. Set, as in the previous case, $\theta^1=1$, $\theta^2=0$ and $n^1=n^*$ given by (3.32), with $\nu^*=1$ when $|\varepsilon_1|>|\varepsilon_2|$ and $\nu^*=0$ when $|\varepsilon_1|<|\varepsilon_2|$. Then (3.16) is exactly (3.29). Finally assume that $\varepsilon$ belongs to $\mathbb{M}_{\mathrm{III}}(\alpha)$. Set $\theta^1=\nu^*$, $\theta^2=1-\nu^*$ with $\nu^*$ given by (3.19) and $n^1=e^1$, $n^2=e^2$. Because $\sigma^*$ is a spherical tensor [$\sigma_1^*=\sigma_2^*$ given by (3.18)] it immediately follows that (3.16) is exactly (3.29). The proof of Theorem 3.1 is complete.

Remark 3.2. – In the course of proving the optimality of (3.13) we have further established that the relaxed energy $\varphi^*(\varepsilon, \alpha)$ corresponds to the elastic energy of

1. a rank-one layering of $A^0$ and $A^1$ in the direction $n^*$, given by (3.32) and (3.25), and in volume fraction $1-\alpha$ and $\alpha$, when $\varepsilon \in \mathbb{M}_{\mathrm{I}}(\alpha)$;

2. a rank-one layering of $A^0$ and $A^1$ in the direction of the eigenvectors of $\varepsilon$ corresponding to the greatest absolute eigenvalue and in volume fraction $1-\alpha$ and $\alpha$, when $\varepsilon \in \mathbb{M}_{\mathrm{II}}(\alpha)$;

3. a rank-two layering of $A^0$ and $A^1$ in the directions of two orthogonal eigenvectors of $\varepsilon$, in volume fraction $1-\nu^*\alpha$ and $\nu^*\alpha$ at the first layering, $\nu^*$ given by (3.19), and final volume fraction $1-\alpha$ and $\alpha$, when $\varepsilon \in \mathbb{M}_{\mathrm{III}}(\alpha)$.

The optimal direction of layering depends *a priori* on $\varepsilon$ and $\alpha$, and thus the optimal effective tensor changes with $\varepsilon$ and $\alpha$. On the other hand, for fixed $\varepsilon$ and $\alpha$, several layering directions may give rise to the optimal result. For instance, when $\varepsilon_1=\varepsilon_2$ any orthonormal basis is an eigenbasis and the directions of layering can be chosen arbitrarily. Revisiting the proof permits us to assert that the optimal $\nu^*$ which gives the optimal direction of layering $n^*$ for rank-one laminates or the optimal first layering volume fraction for rank-two laminates is unique except when $\alpha=0$, $\alpha=1$ or $\varepsilon=0$. Moreover, by virtue of (3.19) and (3.25), $\nu^*$ is immediately seen to be a continuous function of $\varepsilon$ (except at $\varepsilon=0$ where it is not defined).

Remark 3.3. – In the context of Remark 2.2, $\varphi^*$ could be proved to be continuously differentiable in $\varepsilon$. We do not know whether this last property holds true in higher dimensions.

Remark 3.4. – Since $\varphi^*$ is not a quadratic function of $\varepsilon$, the relaxed damage model does not correspond to the damaging process for a *linear* elastic material. With relaxation both the brutal character of the damage and the linear character of the elasticity disappear.

In view of Theorem 3.1 the potential $\psi^{*i}(\varepsilon)$ associated with the relaxed problem $(\mathscr{S}_{\mathrm{I}}^{\mathrm{rel}})$ defined in Section 2, namely

$$(3.33)\qquad \psi^{*i}(x,\varepsilon) \stackrel{\text{def}}{=} \underset{\alpha\in[\alpha^{i-1}(x),\,1]}{\mathrm{Min}} \{\varphi^*(\varepsilon,\alpha)+\kappa\alpha\},$$

is in principle explicitly computable at each time step $t_i$, $1 \leqq i \leqq \mathrm{I}$. At the first time step, $\psi^{*1}$ is the bona fine relaxed from the elastic potential $\psi^1(\varepsilon)$ introduced in (1.18), namely

$$(3.34)\qquad \psi^1(\varepsilon) = \underset{\alpha\in\{0,1\}}{\mathrm{Min}} \left\{\frac{1}{2}((1-\alpha)\mathrm{A}^0+\alpha\mathrm{A}^1)\varepsilon.\varepsilon+\kappa\alpha\right\}.$$

In the one dimensional case, $\psi^{*1}$ is easily computed from (3.1). It is the convex envelop of the nonconvex potential $\psi^1$. Thus, in this case, the relaxation is precisely the convexification of the potential energy, which conforms to the analysis of the problem of the traction of a bar presented in [F & M, 1991] as well as to classical results of convex analysis (*cf.* e.g. [E & T, 1974], Ch. X, Corollary 3.8).

In two dimensions, an explicit determination of $\psi^{*1}$ is not so easy because of the intricate dependence of $\varphi^*$ on $\alpha$. In any case, $\psi^{*1}$ is not convex any more as demonstrated by the following Proposition:

PROPOSITION 3.3. – *Let us consider the set $\mathbb{S}^2$ of spherical tensors,*

$$(3.35)\qquad \mathbb{S}^2=\{\varepsilon\in\mathbb{M}_s^2 \mid \varepsilon=e\,\mathrm{I},\ e\in\mathbb{R}\}.$$

*Denote by $e\to\psi^{*1}(e\mathrm{I})$ the restriction of $\psi^{*1}$ to $\mathbb{S}^2$. Then $\psi^{*1}(e\mathrm{I})$ is the following continuously differentiable but non-convex function of e:*

$$(3.36)\qquad \psi^{*1}(e\mathrm{I})=\begin{cases} 2\mathrm{K}^0 e^2 & \text{when } |e|\leqq \dfrac{e_c}{q} \\ \dfrac{2\mathrm{K}^0}{q^2-1}(-(1-(1-p)q^2)e^2+2pqe_c|e|-pe_c^2) & \text{when } \dfrac{e_c}{q}\leqq|e|\leqq qe_c \\ 2\mathrm{K}^1e^2+\kappa & \text{when } |e|\geqq qe_c \end{cases}$$

*with $p=1-\mathrm{K}^1/\mathrm{K}^0$, $q=\sqrt{(\mathrm{K}^0+\mu^1)/(\mathrm{K}^1+\mu^1)}$ and $e_c=\sqrt{\kappa/(2p\mathrm{K}^0)}$.*

PROOF. – Since $e\mathrm{I}$ belongs to $\mathbb{M}_{\mathrm{III}}(\alpha)$ for any $e$ and $\alpha$, $\varphi^*(e\mathrm{I},\alpha)=2\mathrm{K}_{\mathrm{HS}}(\alpha)e^2$. The case $e=0$ is trivial. Since $\mathrm{K}_{\mathrm{HS}}$ is a strictly convex function of $\alpha$, when $e\neq 0$, the minimum in (3.33) is attained for an unique value $\alpha^*(e)$ of $\alpha$. Further $\alpha^*(e)=0$ when $2\mathrm{DK}_{\mathrm{HS}}(0)\,e^2+\kappa\geqq 0$, that is when $|e|\leqq e_c/q$, $\alpha^*(e)=1$ when $2\mathrm{DK}_{\mathrm{HS}}(1)\,e^2+\kappa\leqq 0$, that is when $|e|\leqq qe_c$, while, when $e_c/q<|e|<qe_c$, $\alpha^*(e)$ is the unique solution of $2\mathrm{DK}_{\mathrm{HS}}(\alpha)\,e^2+\kappa=0$, which yields

$$(3.37)\qquad \alpha^*(e)=\frac{q|e|-e_c}{(q^2-1)\,e_c}.$$

Replacing $\alpha^*(e)$ by its expression in $\psi^{*1}(e\,\mathrm{I})$ yields (3.36). To show that $\psi^{*1}(e\,\mathrm{I})$ is a continuously differentiable function of $e$, it suffices to remark that $\alpha^*(e)$ is Lipchitz continuous in $e$ and to use the associated optimality properties [the reader can also simply differentiate (3.36)]. Finally the nonconvexity of $\psi^*(e\,\mathrm{I})$ is a consequence of the inequality

$$(3.38) \qquad (1-p)\,q^2<1$$

which holds true because $K^0>K^1>0$ and $\mu^1>0$. Thus the restriction to $\mathbb{S}^2$ of the relaxed elastic potential $\psi^{*1}$ is not the convex hull of $\psi^1(e\,\mathrm{I})$, *cf. Fig.* 3.

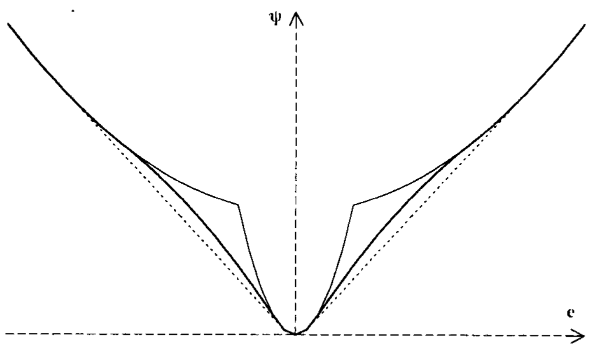


Fig. 3. — The elastic potential $\psi^1$ and the relaxed elastic potential $\psi^{*1}$ for spherical tensors $\varepsilon=e\,\mathrm{I}$
— Elastic potential — Relaxed potential --- convexified potential.

REMARK 3.5. — The absence of convexity of $\psi^{*i}$ renders the numerical search for stable solutions non trivial although their existence is ensured as a byproduct of the relaxation process.

Since the explicit determination of $\psi^{*i}$ seems to be somewhat intricate, a different approach can be followed, in the spirit of the work of Bendsoe, Kikushi and Suzuki on structural optimization (*cf.* e.g. [Bendsoe & Kikushi, 1988] or [Kikushi & Suzuki, to appear]). It consists (in its most recent version) in taking advantage of the optimal character of certain microstructures with respect to energy minimization so as to specialize the search for $\varphi^*$ within the class of those microstructures. Specifically, the proof of the optimality of the bound (2.13) demonstrates that, in two dimensions, rank-two layering of $A^0$ and $A^1$ about $A^1$ is sufficient to generate an effective tensor whose associated elastic energy is precisely the bound (2.13). Actually a simple counting argument would show that, in dimension N, N pivotal layering (that is rank-N layering) about the damaged material are needed. The effective stiffness tensors associated to this kind of multiple layering process is explicitly computable (*cf.* Lemma 3.2 above and [F & M, 1986] Proposition 4.2). Thus a computation of the minimum of the relaxed energy can be organized as follows:

1. Discretize $\Omega$ with your favorite finite element mesh.

2. Place in each element a material with elasticity $A^*$ that depends on the layering parameters (directions and individual volume fractions).

3. Perform a finite element computation.
4. Optimize the energy with respect to the layering parameters.

Of course the delicate step is the fourth one. The reader is invited to refer to [K & S, to appear] for more details. In a two dimensional setting the optimization would involve 4 parameters (2 directions and 2 volume fractions).

## 4. The torsion problem

In this Section a torsion problem is investigated in order to illustrate the notions introduced in the previous Sections. Specifically, after stating the problem in Subsection 4.1, the relaxation procedure is implemented in Subsection 4.2 and is shown not to admit any smooth classical stable solution (for sufficiently large loadings) but only generalized ones. Finally, in Subsection 4.3, in the case of a circular cross section the unique generalized stable solution is explicitly computed.

### 4.1. Statement of the problem

We consider a body whose reference configuration $\Omega$ is a cylinder $S \times (0, 1)$. The section S is a bounded smooth simply connected open set of $\mathbb{R}^2$. No lateral or body loadings are applied and the boundary conditions on the lower and upper cross sections are the following:

$$\left\{ \begin{array}{lll} u_1 = u_2 = 0 & \text{and} & \sigma_{33} = 0 \quad \text{on } S \times \{0\} \\ u_1 = -t x_2,\ u_2 = t x_1 & \text{and} & \sigma_{33} = 0 \quad \text{on } S \times \{1\}, \end{array} \right. \tag{4.1}$$

where $t$ denotes the increasing angle of torsion applied to the upper cross section. The cylinder is made of an elastic damaging isotropic material characterized by its undamaged Lamé moduli $\lambda^0$, $\mu^0$, its damaged ones $\lambda^1$, $\mu^1$, with $\mu^0 > \mu^1 > 0$, and its critical energy release rate $\kappa$. We assume that the cylinder is not damaged before the onset of torsion.

An increasing sequence of torsion angles $(t_i)_{0 \leq i \leq I}$ is considered and the $t_i$'s are identified with discrete time steps,

$$0 = t_0 < t_1 < \ldots < t_i < \ldots < t_I = t.$$

The incremental torsion problem reads as follows

PROBLEM ($\mathscr{P}_I$). – *For* $i \in \{1, \ldots, I\}$ *find* $(u^i, \chi^i)$ *such that*

$$\operatorname{div} \sigma^i = 0 \quad in\ \Omega, \qquad \sigma^i n = 0 \quad on\ \partial S \times (0, 1), \tag{4.2}$$

$$\sigma^i = ((1 - \chi^i) A^0 + \chi^i A^1) \varepsilon^i \quad in\ \Omega, \tag{4.3}$$

$$2 \varepsilon^i = D u^i + (D u^i)^T \quad in\ \Omega, \tag{4.4}$$

*where* $u^i$ *and* $\sigma^i$ *satisfy* (4.1) *and* $\chi^i$ *satisfies the damage law* (1.11).

A solution of ($\mathscr{P}_I$) is stable if it minimizes, at each $t_i$, the Lagrangian $\mathscr{L}$ over the sets $\mathscr{V}^i$ of admissible displacement fields and $\mathscr{C}^i$ of admissible characteristic functions of the damaged zone. Here $\mathscr{V}^i$, $\mathscr{C}^i$ and $\mathscr{L}$ read as:

$$(4.5)\quad \mathscr{V}^i=\{v\in \mathrm{H}^1(\Omega, \mathbb{R}^3)\,|\, v_1=v_2=0 \text{ on } \mathrm{S}\times\{0\} \text{ and } v_1=-t_i x_2,\ v_2=t_i x_1 \text{ on } \mathrm{S}\times\{1\}\},$$

$$(4.6)\qquad \mathscr{C}^i=\{\chi\in \mathrm{L}^\infty(\Omega)\,|\,\chi(x)\in\{0,1\},\ \chi\geqq\chi^{i-1}\}$$

and

$$(4.7)\qquad \mathscr{L}(v,\chi)=\frac{1}{2}\int_\Omega ((1-\chi)\mathrm{A}^0+\chi\mathrm{A}^1)\varepsilon(v).\varepsilon(v)\,dx+\int_\Omega \kappa\chi\,dx.$$

It is assumed, as always, that $\chi^0=0$.

We will not attempt an analysis of the full three-dimensional minimization problem but will only consider damage fields that are independent of $x_3$, so as to be left with a more tractable formulation. A full three dimensional analysis is beyond reach at this stage, for it would involve a three dimensional relaxed energy whose expression has yet to be computed (*see* Section 3). We thus restrict the geometry of the damage by minimizing $\mathscr{L}$, not over $\mathscr{V}^i\times\mathscr{C}^i$, but over the subset $\mathscr{V}^i\times\mathscr{C}^i_a$ with $\mathscr{C}^i_a$ defined as

$$(4.8)\qquad \mathscr{C}^i_a=\{\chi\in\mathscr{C}^i\,|\,\chi=\chi(x_1,x_2)\}.$$

A first step consists in minimizing $\mathscr{L}$ with respect to $v$ at fixed $\chi$. For a fixed $\chi$ in $\mathscr{C}^i_a$, the minimization of $\mathscr{L}(.,\chi)$ over $\mathscr{V}^i$ is a classical linear elastic torsion problem which admits a solution, noted $v^i(\chi)$, unique to within an axial translation; it is given by

$$(4.9)\qquad \begin{cases} v^i_1(\chi)(x)=-t_i x_3 x_2, \quad v^i_2(\chi)(x)=t_i x_3 x_1,\\ v^i_3(\chi)(x)=t_i w(\chi)(x_1,x_2),\end{cases}$$

where $w(\chi)$ is the normalized axial displacement, a minimizer over $\mathrm{H}^1(\mathrm{S})$ of the functional $\mathrm{J}(.,\chi)$ defined by

$$(4.10)\qquad \mathrm{J}(w,\chi)=\int_\mathrm{S} ((1-\chi)\mu^0+\chi\mu^1)\left(\left(\frac{\partial w}{\partial x_1}-x_2\right)^2+\left(\frac{\partial w}{\partial x_2}+x_1\right)^2\right)d\mathrm{S}.$$

Insertion of (4.9) into (4.7) eventually leads to

$$(4.11)\qquad \mathscr{L}(v^i(\chi),\chi)=\frac{1}{2}\mathrm{T}(\chi)\,t_i^2+\int_\mathrm{S}\kappa\chi\,d\mathrm{S}$$

where

$$(4.12)\qquad \mathrm{T}(\chi)\overset{\mathrm{def}}{=}\mathrm{J}(w(\chi),\chi)$$

represents the torsional modulus of rigidity of the section when its damage field is $\chi$.

It now remains to minimize $\mathscr{L}(v^i(\chi),\chi)$ over $\chi$ in $\mathscr{C}^i_a$, which is, as we have learned in the previous Sections, a hopeless task.

REMARK 4.1. – Let us consider the first step $t_1>0$ for which $\mathscr{C}_a^1=L^\infty(S, \{0, 1\})$. The search for a stable solution $\chi^1$ proceeds as follows. Set, for $D\in[0, 1]$,

$$(4.13)\qquad \mathscr{C}(D)=\left\{\chi\in L^\infty(S, \{0, 1\})\,\middle|\, \int_S \chi\, dS = D\,\text{meas}(S)\right\}.$$

Then,

$$(4.14)\qquad \underset{\chi\in\mathscr{C}_a^1}{\text{Min}}\ \mathscr{L}(v^1(\chi), \chi)=\underset{D\in[0, 1]}{\text{Min}}\left\{\frac{1}{2}T_*(D)\,t_1^2+\kappa\, D\,\text{meas}(S)\right\}$$

with

$$(4.15)\qquad T_*(D)=\underset{\chi\in\mathscr{C}(D)}{\text{Min}}\ T(\chi).$$

The minimization problem (4.15) is nothing but a classical problem of optimal design which consists in the *minimization of the torsional rigidity of the section* S. This "classical" problem is completely solved in [M & T, 1985] in the case of a circular section. It does not admit any classical solutions (except when $D=0$ or 1). Generalized solutions require to finely mix both materials in some region of S. The problem must be relaxed in the manner detailed in the previous sections.

It is however traditional to address a torsion problem in a dual form, that is to express it in terms of stresses rather than displacements. For the mechanically inclined reader's convenience we yield to the custom and proceed to rewrite (4.12) with the help of the Prandtl stress functions. We should stress however that the attained results could be also derived from the above described "primal" formulation. Although the dualization procedure is well-known (*see* [Germain, 1973]) we briefly recall it for the reader's sake.

For any $\chi$, the only non zero components of the stress field $\sigma^i(\chi)$ associated to $v^i(\chi)$ are the shear stresses $\sigma^i(\chi)_{13}$ and $\sigma^i(\chi)_{23}$, which only depend on $(x_1, x_2)$. The equilibrium equations (4.2) are easily seen to ensure the existence of a scalar function, called a (Prandtl) stress function, $\eta(\chi)\in H_0^1(S)$, such that

$$(4.16)\qquad \sigma^i(\chi)_{13}=t_i\frac{\partial\eta(\chi)}{\partial x_2}\quad\text{and}\quad \sigma^i(\chi)_{23}=-t_i\frac{\partial\eta(\chi)}{\partial x_1}.$$

The stress-strain relation (4.3) becomes

$$(4.17)\qquad \begin{cases} ((1-\chi)\mu^0+\chi\mu^1)\left(\dfrac{\partial w(\chi)}{\partial x_1}-x_2\right)=+\dfrac{\partial\eta(\chi)}{\partial x_2} \\ ((1-\chi)\mu^0+\chi\mu^1)\left(\dfrac{\partial w(\chi)}{\partial x_2}+x_1\right)=-\dfrac{\partial\eta(\chi)}{\partial x_1}, \end{cases}$$

or equivalently,

$$(4.18)\qquad -\text{div}\left(\left(\frac{1-\chi}{\mu^0}+\frac{\chi}{\mu^1}\right)D\eta(\chi)\right)=2.$$

Thus $\eta(\chi)$ is the unique element of $H_0^1(S)$ that maximizes the functional

$$(4.19)\qquad J^*(\eta, \chi)=4\int_S \eta\, dS-\int_S\left(\frac{1-\chi}{\mu^0}+\frac{\chi}{\mu^1}\right)D\eta . D\eta\, dS$$

over $H_0^1(S)$ at fixed $\chi$. Furthermore, by virtue of (4.10), (4.11), (4.17), (4.18) and (4.19),

$$(4.20)\qquad T(\chi)=J^*(\eta(\chi), \chi)=\underset{\eta\in H_0^1(S)}{\operatorname{Max}}\ J^*(\eta, \chi),$$

which completes the dualization process.

### 4.2. The relaxed problem

Equipped with (4.20) we implement the relaxation of $\mathscr{L}(v^i(\chi), \chi)$. The relaxed set of admissible damage fields is

$$(4.21)\qquad \mathscr{C}^i_{\text{rel}}=\{\alpha\in L^\infty(S, [0, 1])\,|\,\alpha^{i-1}(x)\leqq\alpha(x)\leqq 1 \text{ a.e. in } S\}.$$

The relaxed functional $J^i_{\text{rel}}$, defined on $\mathscr{C}^i_{\text{rel}}$, is easily seen, in the spirit of Subsection 2.2, to be

$$(4.22)\qquad J^i_{\text{rel}}(\alpha)=\inf\{\liminf_{n\to\infty}\mathscr{L}(v^i(\chi_n), \chi_n)\,|\,\chi_n\in L^\infty(S, \{0, 1\}), \chi_n\rightharpoonup\alpha\}.$$

Using (4.11) it is immediately seen that

$$(4.23)\qquad J^i_{\text{rel}}(\alpha)=\frac{1}{2}T_{\text{rel}}(\alpha)\,t_i^2+\int_S \kappa\alpha\, dS$$

with

$$(4.24)\qquad T_{\text{rel}}(\alpha)=\inf\{\liminf_{n\to\infty} T(\chi_n)\,|\,\chi_n\in L^\infty(S, \{0,1\}), \chi_n\rightharpoonup\alpha\}.$$

It only remains to determine $T_{\text{rel}}(\alpha)$. In view of (4.20), the relaxation of $T(\chi)$ coincides with that of the conductivity problem (4.18). The reader is referred to [M & T, 1985], where this problem is fully analyzed. Note however that the obtained result could be rederived here at the expense of adapting the definitions of H-convergence to a scalar setting and of using the analogue of the layering formula (3.28) (*cf.* [T, 1985]). The result is

$$(4.25)\qquad T_{\text{rel}}(\alpha)=\underset{\eta\in H_0^1(S)}{\operatorname{Max}}\ J^*_{\text{rel}}(\eta, \alpha)$$

with

$$(4.26)\qquad J^*_{\text{rel}}(\eta, \alpha)=4\int_S \eta\, dS-\int_S\left(\frac{1-\alpha}{\mu^0}+\frac{\alpha}{\mu^1}\right)D\eta . D\eta\, dS.$$

The relaxed torsion problem can now be formulated as one of the following three equivalent problems:

THE RELAXED DAMAGE PROBLEM. — *For* $i \in \{1, \ldots, I\}$ *find* $\alpha^i$ *which minimizes the functional* $\frac{1}{2} t_i^2 \mathrm{T}_{\mathrm{rel}}(\alpha) + \int_{\mathrm{S}} \kappa\alpha \, d\mathrm{S}$ *over* $\mathscr{C}_a^i$, *under the constraint* $\alpha^0 = 0$;

THE RELAXED STRESS-DAMAGE PROBLEM. — *For* $i \in \{1, \ldots, I\}$ *find* $(\alpha^i, \eta^i)$ *as a saddle point in* $\mathscr{C}_a^i \times \mathrm{H}_0^1(\mathrm{S})$ (*with the constraint* $\alpha^0 = 0$) *of the functional* $\mathscr{L}_{\mathrm{rel}}^i$ *defined by*

$$\mathscr{L}_{\mathrm{rel}}^i(\alpha, \eta) = 2 t_i^2 \int_{\mathrm{S}} \eta \, d\mathrm{S} - \frac{t_i^2}{2} \int_{\mathrm{S}} \left( \frac{1-\alpha}{\mu^0} + \frac{\alpha}{\mu^1} \right) \mathrm{D}\eta \, . \, \mathrm{D}\eta \, d\mathrm{S} + \int_{\mathrm{S}} \kappa\alpha \, d\mathrm{S}; \tag{4.27}$$

THE RELAXED STRESS PROBLEM. — *For* $i \in \{1, \ldots, I\}$ *find* $\eta^i$ *as a maximizer of the functional* $\Phi_{\mathrm{rel}}^i$ *over* $\mathrm{H}_0^1(\mathrm{S})$, *with*

$$\Phi_{\mathrm{rel}}^i(\eta) = 2 t_i^2 \int_{\mathrm{S}} \eta \, d\mathrm{S} - \int_{\mathrm{S}} \psi_{\mathrm{rel}}^i(x, \mathrm{D}\eta(x)) \, d\mathrm{S} \tag{4.28}$$

*where* $\psi_{\mathrm{rel}}^i(x, .)$ *is the function defined on* $\mathbb{R}^2$ *by*

$$\psi_{\mathrm{rel}}^i(x, s) = \underset{\alpha^{i-1}(x) \leq \alpha \leq 1}{\mathrm{Max}} \left\{ \frac{1}{2} t_i^2 \left( \frac{1-\alpha}{\mu^0} + \frac{\alpha}{\mu^1} \right) \| s \|^2 - \kappa\alpha \right\}. \tag{4.29}$$

The first formulation is obtained by maximizing $\mathscr{L}_{\mathrm{rel}}^i$ with respect to $\eta$ and then minimizing with respect to $\alpha$, whereas the third one is obtained by exchanging the order of min(max)imization, which is a licit operation because $\mathscr{L}_{\mathrm{rel}}^i$ is quadratic and concave in $\eta$ while it is linear in $\alpha$.

By construction these problems admit at least one solution. From a practical standpoint it is convenient to give another characterization of the solutions. This is the aim of the following proposition whose proof is an immediate consequence of the existence of a saddle point:

PROPOSITION 4.1. — *A sequence of pairs* $(\alpha^i, \eta^i) \in \mathscr{C}_a^i \times \mathrm{H}_0^1(\mathrm{S})$ *is a solution of the Relaxed Stress-Damage Problem of torsion if and only if it satisfies the following set of optimality conditions*

$$-\mathrm{div}\left( \left( \frac{1-\alpha^i}{\mu^0} + \frac{\alpha^i}{\mu^1} \right) \mathrm{D}\eta^i \right) = 2 \tag{4.30}$$

*and*

$$\alpha^i(x) = \begin{cases} \alpha^{i-1}(x) & \textit{if} \quad \| \mathrm{D}\eta^i(x) \| < \dfrac{k}{t_i} \\ 1 & \textit{if} \quad \| \mathrm{D}\eta^i(x) \| > \dfrac{k}{t_i}, \end{cases} \tag{4.31}$$

*where* $\| . \|$ *denotes the euclidian norm of* $\mathbb{R}^2$ *and* $k = \sqrt{2 \kappa \mu^0 \mu^1 / (\mu^0 - \mu^1)}$.

REMARK 4.2. — For generalized solutions, if partial damage appears or increases at a point $x$ of S and at a time step $t_i$, then $\alpha^{i-1}(x)<\alpha^i(x)<1$ and (4.31) requires that $\|D\eta^i(x)\|=k/t_i$. In other words, the norm of the stresses remains constant in the domain where fine mixtures evolve.

While it was already mentioned that generalized stable solutions are not necessarily unique, the following uniqueness result for the stresses holds true (*cf.* also [M & T, 1985]):

PROPOSITION 4.2. — *To a given damage state at time $t_{i-1}$ there corresponds a unique stable stress function $\eta^i$ at the next time $t_i$.*

PROOF. — Let $\alpha^{i-1}$ be a given damage state at $t_{i-1}$. Then, according to the Relaxed Stress Problem, a stable stress function $\eta^i$ at $t_i$ maximizes $\Phi^i_{\mathrm{rel}}$ over $H^1_0(S)$. By virtue of (4.29) and (4.31), $\psi^i_{\mathrm{rel}}$ reads as

$$(4.32)\quad \psi^i_{\mathrm{rel}}(x, s)=\begin{cases}\dfrac{1}{2}t_i^2\left(\dfrac{1-\alpha^{i-1}(x)}{\mu^0}+\dfrac{\alpha^{i-1}(x)}{\mu^1}\right)\|s\|^2-\kappa\alpha^{i-1}(x) & \text{when } \|s\|\leqq\dfrac{k}{t_i}\\[2ex] \dfrac{1}{2\mu^1}t_i^2\|s\|^2-\kappa & \text{when } \|s\|\geqq\dfrac{k}{t_i}.\end{cases}$$

Since, when $\|s\|>k/t_i$, the coefficient of $\|s\|^2$ is larger than when $\|s\|<k/t_i$, $\psi^i_{\mathrm{rel}}$ is a strictly convex function of $s$ which ensures the uniqueness of $\eta^i$.

Unfortunately the above proposition does not ensure uniqueness of the stress response for the entire loading history. Although $\eta^i$ is uniquely determined once $\alpha^{i-1}$ is given, we do not know whether $\alpha^i$ is unique. If it is not, we will obtain at $t_{i+1}$ as many stable stress functions as $\alpha^i$, and so on. However the optimality conditions (4.31) permit us to know the three regimes of damage at $t_i$ from the knowledge of $\alpha^{i-1}$. Specifically the domains of S where $\|D\eta^i\|$ is respectively less than, equal to and greater than $k/t_i$ are known once $\alpha^{i-1}$ is known. Therefore $\alpha^i$ is equal respectively to $\alpha^{i-1}$ and 1 in respectively the first and the third domain whereas in the second domain $\alpha^i$ must satisfy (4.30). Thus the issue of uniqueness is reduced to that of the uniqueness of the solution (in $\alpha^i$) to (4.30) in a domain where $\eta^i$ is given with $\|D\eta^i\|=k/t_i$.

REMARK 4.3. — The optimal design problem of maximization of the torsional rigidity — the opposite of the problem described in Remark 4.1 — has been shown to possess unique solutions for certain specialized classes of cross sections (*cf.* [Kawohl *et al.*, to appear].

To conclude this subsection we investigate the existence of classical stable solutions to the torsion problem, *i.e.*, of solutions without fine mixtures. If such a solution exists, it is also a solution to the Relaxed Stress-Damage Problem defined in the previous subsection. Consequently, it must satisfy (4.30) and (4.31). We label a sequence of pairs $(\chi^i, \eta^i)_{1\leqq i\leqq I}$ *a smooth classical stable solution* of the torsion problem any sequence that belongs to $(L^\infty(S, \{0, 1\})\times H^1_0(S))^I$, satisfies (4.30) and (4.31) for every $i$ with $\chi^0=0$, and is such that the undamaged part $S^i_0$ (where $\chi^i=0$) and the damaged part $S^i_1$ (where $\chi^i=1$) are either empty or sufficiently smooth subsets of S so that their common boundary $\partial S^i_0\cap\partial S^i_1$ is a finite family of non intersecting smooth curves.

PROPOSITION 4.3. — *Let* $\bar{\eta}$ *be the unique element of* $H_0^1(S)$ *which satisfies* $\Delta\bar{\eta}+2=0$ *in* S *and set* $\sup_{x \in S} \| \mu^0 D\bar{\eta}(x) \| = k/t_m$. *Then*

1. *As long as* $t_i \leqq t_m$, *the torsion problem admits as unique stable solution the so-called elastic solution* $(\chi^i, \eta^i) = (0, \mu^0 \bar{\eta})$.

2. *As soon as* $t_i > t_m$, *it does not admit any smooth classical stable solution in the sense of the above definition.*

PROOF. — The reference stress function $\bar{\eta}$ is in $C^\infty(S)$. If its gradient is bounded, then $t_m > 0$, whereas if it is not, $t_m = 0$. It is bounded if the boundary $\partial S$ of S is smooth enough, but it could be unbounded otherwise. We will assume that $t_m > 0$ (otherwise the first part of the Proposition is of no interest) and consider a discretization of the increasing loading process such that $t_j = t_m$ for some $j \in \{2, \ldots, I-1\}$. At time $t_1$, $(0, \mu^0 \bar{\eta})$ satisfies (4.30)-(4.31) and hence is a stable solution. According to Proposition 4.2, $\mu^0 \bar{\eta}$ is the unique stable stress function at $t_1$. Therefore we conclude from (4.31) that $\alpha^1 = 0$ is the unique stable damage field at $t_1$. The same result holds true at any time $t_i (i<j)$ and the elastic solution is the unique stable solution for $i<j$. At time $t_j = t_m$ it is also solution, hence $\eta^j = \mu^0 \bar{\eta}$ is the unique stable stress function and since the set where $\| D\bar{\eta}(x) \|$ reaches its maximum is of measure zero, $t_m \mu^0 \| D\bar{\eta}(x) \| < k$ almost everywhere and hence $\alpha^j = 0$ almost everywhere. Thus the elastic response is the unique stable solution in the interval $[0, t_m]$, which establishes the first part of the Proposition.

At the next time $t_{j+1} > t_m$, the previous elastic solution is no longer valid because $t_{j+1} \mu^0 \| D\bar{\eta}(x) \| > k$ in a set of nonzero measure which contradicts (4.31). Further there does not exist a stable solution for which the section would be totally damaged. Indeed assume that $\alpha^{j+1} = 1$ is a stable damage field. Then, according to (4.30), the corresponding stable stress function should be $\eta^{j+1} = \mu^1 \bar{\eta}$ and according to (4.31) it should satisfy $t_{j+1} \mu^1 \| D\bar{\eta}(x) \| \geqq k$ almost everywhere in S. But $\bar{\eta}$ is continuous on $\bar{S}$ and nonnegative, according to the maximum principle. Since $\bar{\eta} \in C^\infty(S)$, $\bar{\eta}$ attains its maximum at a point $x_M$ of S at which $D\bar{\eta}(x_M) = 0$. Thus $\| D\bar{\eta}(x) \|$ is very small in a neighbourhood of $x_M$, which leads to a contradiction. By induction, that remains true for every $i>j$.

We return to the time step $t_{j+1}$ and drop the index $j+1$ for notational convenience. According to the analysis of the previous paragraph, if a stable classical solution $(\chi, \eta)$ exists, then the section is divided into two non empty domains $S^0$ and $S^1$ where the damage field $\chi$ respectively takes the values 0 and 1. Let us assume that they are smooth in the sense of the above definition and denote by $\gamma$ the connected component of a given point of their common boundary $\partial S^0 \cap \partial S^1$. By hypothesis $\gamma$ is a smooth curve and the stress function $\eta$ is smooth on each side of $\gamma$. Let us denote by $f^0$ and $f^1$ the restrictions of a field $f$ on S to $S^0$ and $S^1$. We denote by $\nu$ the unit normal to $S^1$ on $\gamma$ and by $\tau$ the associated unit tangent vector. Since $\eta$ belongs to $H_0^1(S)$, $\eta^1 = \eta^0$ on $\gamma$ and the tangential derivatives are also continuous across $\gamma$, that is

$$\frac{\partial \eta^0}{\partial \tau} = \frac{\partial \eta^1}{\partial \tau} \quad \text{on } \gamma. \tag{4.33}$$

On the other hand, the normal derivatives satisfy the following relation [by virtue of (4.30)]:

$$\mu^1 \frac{\partial \eta^0}{\partial \nu} = \mu^0 \frac{\partial \eta^1}{\partial \nu} \quad \text{on } \gamma. \tag{4.34}$$

According to (4.31), $D\eta$ must also satisfy

$$\| D\eta \| \leqq k/t_{j+1} \quad \text{in } S^0 \qquad \text{and} \qquad \| D\eta \| \geqq k/t_{j+1} \quad \text{in } S^1. \tag{4.35}$$

Relations (4.33) and (4.35) imply

$$\left| \frac{\partial \eta^0}{\partial \nu} \right| \leqq \left| \frac{\partial \eta^1}{\partial \nu} \right|, \tag{4.36}$$

which in view of (4.34) and of the ordering property $\mu^0 > \mu^1$ yields

$$\frac{\partial \eta^0}{\partial \nu} = \frac{\partial \eta^1}{\partial \nu} = 0. \tag{4.37}$$

In turn (4.33), (4.35) and (4.37) imply

$$\left| \frac{\partial \eta}{\partial \tau} \right| = k/t_{j+1} \quad \text{on } \gamma. \tag{4.38}$$

Since $\partial \eta / \partial \tau$ is continuous and piecewise constant on $\gamma$, it is constant and

$$\left| \int_\gamma \frac{\partial \eta}{\partial \tau} d\tau \right| = \frac{k \, length(\gamma)}{t_{j+1}}. \tag{4.39}$$

But either $\gamma$ is a closed curve, or its end points belong to $\partial S$. Since $\eta = 0$ on $\partial S$, we conclude that, in both cases,

$$\int_\gamma \frac{\partial \eta}{\partial \tau} d\tau = 0 \tag{4.40}$$

which contradicts (4.39). Consequently, there cannot exist a smooth classical stable solution as soon as $t > t_m$. The proof of the proposition is complete.

Remark 4.4. — A similar conclusion is implicit in the problem of optimal design (evoked in Remark 4.1) in [M & T, 1985]. A conclusion of the same type is explicitly stated in the opposite problem in Remark 41 of [M & T, 1985]. Note however that the underlying theorem used there [Serrin, 1971] appeals to the strong maximum principle.

### 4.3. Case of a circular section

It is assumed in this Subsection that the cross section S is a disk of center (0, 0) and radius R. Polar coordinates $(r, \theta)$ are used throughout. We also set

$$(4.41)\qquad t_c=\frac{1}{\mathrm{R}}\sqrt{\frac{2\kappa}{\mu^0-\mu^1}},\qquad k=\sqrt{\mu^0\mu^1}\,\mathrm{R}\,t_c,\qquad t_c^0=\sqrt{m}\,t_c,\qquad t_c^1=t_c/\sqrt{m}$$

where $m$ denotes the ratio of the post and pre-damage shear moduli, *i.e.*,

$$(4.42)\qquad m=\frac{\mu^1}{\mu^0}.$$

Because of the symmetry of the section S, the damage problem is explicitly solvable. The following theorem is obtained:

THEOREM 4.4. — *If $t$ is the final time of an increasing torsion loading history starting at time* 0 *when the body is assumed undamaged, the relaxed damage problem admits as unique damage field at the final time $t$ the axisymmetric field $\alpha(t)$ defined by*

$$(4.43)\qquad \alpha(t, r)=0 \quad \textit{for all } r, \quad \textit{when } 0\leqq t\leqq t_c^0;$$

$$(4.44)\qquad \alpha(t, r)=\begin{cases} 0 & \textit{if}\quad 0\leqq rt\leqq \mathrm{R}\,t_c^0 \\ (rt/\mathrm{R}\,t_c^0-1)\dfrac{m}{1-m} & \textit{if}\quad \mathrm{R}\,t_c^0\leqq rt\leqq \mathrm{R}\,t, \end{cases}\quad \textit{when } t_c^0\leqq t\leqq t_c^1;$$

$$(4.45)\qquad \alpha(t, r)=\begin{cases} 0 & \textit{if}\quad 0\leqq rt\leqq \mathrm{R}\,t_c^0 \\ (rt/\mathrm{R}\,t_c^0-1)\dfrac{m}{1-m} & \textit{if}\quad \mathrm{R}\,t_c^0\leqq rt\leqq \mathrm{R}\,t_c^1, \\ 1 & \textit{if}\quad \mathrm{R}\,t_c^1\leqq rt\leqq \mathrm{R}\,t \end{cases}\quad \textit{when } t\geqq t_c^1.$$

*In particular $\alpha(t)$ is independent of the choice of the time discretization.*

PROOF. — Let $(t_i)_{1\leqq i\leqq \mathrm{I}}$ be an arbitrary increasing sequence of discrete times such that $t_\mathrm{I}=t>0$. We will firstly demonstrate the uniqueness and axisymmetric character, at every time $t_i$, of the stable damage field $\alpha^i$ and of the stable stress function $\eta^i$. By hypothesis $\alpha^0=0$. Let us assume that $\alpha^{i-1}$ is unique and axisymmetric. According to Proposition 4.2, $\eta^i$ is unique. Consider $\omega\in[0, 2\pi]$ and $\eta^i_\omega$ the field defined by

$$(4.46)\qquad \eta^i_\omega(r, \theta)=\eta^i(r, \theta+\omega).$$

By virtue of the axisymmetric character of $\alpha^{i-1}$ it is easily verified that

$$(4.47)\qquad \Phi^i_{\mathrm{rel}}(\eta^i_\omega)=\Phi^i_{\mathrm{rel}}(\eta^i)\quad \text{for all } \omega.$$

But since $\eta^i$ is the unique maximizer of $\Phi^i_{\mathrm{rel}}$, $\eta^i_\omega=\eta^i$ for all $\omega$'s which implies that $\eta^i$ is axisymmetric. Accordingly, the optimal condition (4.30) now reads as

$$(4.48)\qquad \frac{\partial}{\partial r}\left(\left(\frac{1-\alpha^i}{\mu^0}+\frac{\alpha^i}{\mu^1}\right)r\frac{d\eta^i}{dr}\right)+2r=0.$$

A first integration of (4.48) gives

$$\left(\frac{1-\alpha^i(r,\theta)}{\mu^0}+\frac{\alpha^i(r,\theta)}{\mu^1}\right)r\frac{d\eta^i}{dr}(r)+r^2=C^i(\theta). \tag{4.49}$$

Since $\eta^i$ belongs to $H_0^1(S)$, $C^i(\theta)=0$ and (4.49) becomes

$$\left(\frac{1}{\mu^1}-\frac{1}{\mu^0}\right)\alpha^i(r,\theta)=-r\Big/\frac{d\eta^i}{dr}(r)-1/\mu^0 \quad \text{for} \quad r\in(0,R), \tag{4.50}$$

which ensures that $\alpha^i$ is both unique and axisymmetric. By induction this property holds true for every $i\in\{1,\ldots,I\}$.

With the help of (4.50) together with the optimality conditions (4.31), we obtain

$$\alpha^i(r)=\begin{cases}\alpha^{i-1}(r) & \text{when } rt_i<\left(\dfrac{1-\alpha^{i-1}(r)}{\mu^0}+\dfrac{\alpha^{i-1}(r)}{\mu^1}\right)k\\[2ex] (rt_i/R\,t_c^0-1)\dfrac{m}{1-m} & \text{when } \left(\dfrac{1-\alpha^{i-1}(r)}{\mu^0}+\dfrac{\alpha^{i-1}(r)}{\mu^1}\right)k\leqq rt_i\leqq\dfrac{k}{\mu^1}\\[2ex] 1 & \text{when } rt_i>\dfrac{k}{\mu^1}.\end{cases} \tag{4.51}$$

with the initial condition $\alpha^0(r)=0$. By induction (4.51) leads to

$$\alpha(t,r)=\begin{cases}0 & \text{if } rt<R\,t_c^0\\[1ex] (rt/R\,t_c^0-1)\dfrac{m}{1-m} & \text{if } R\,t_c^0\leqq rt\leqq R\,t_c^1\\[1ex] 1 & \text{if } R\,t_c^1<rt.\end{cases} \tag{4.52}$$

Since $\alpha(t)$ given by (4.52) does not depend on the intermediate time steps, the damage response of the body will be independent of the time discretization. The proof of Theorem 4.4 is complete.

REMARK 4.5. — Once again the case of a circular cross section was investigated in [M & T, 1985] in the framework of optimal design (*cf.* Remark 4.1) and the axisymmetry of the fields was observed.

REMARK 4.6. — The stable stress field is also unique at time $t$. According to (4.16), (4.49) and by virtue of the axisymmetry its only nonzero component is the orthoradial shear stress

$$\sigma_{\theta 3}(t,r)=-t\frac{d\eta}{dr}(t,r)=\frac{tr}{((1-\alpha(t,r))/\mu^0)+\alpha(t,r)/\mu^1}. \tag{4.53}$$

In a similar manner the stable displacement field is unique (to within an arbitrary axial translation), and its only nonzero component is the orthoradial displacement

$$u_\theta(t, r, x_3) = trx_3, \tag{4.54}$$

which means that the stable axial displacement $w(t)$ is zero.

At the close of this study of the torsion problem for a circular section, we will compare the stable generalized solution to a classical but unstable solution of the full three-dimensional incremental problem. The reader is invited to verify that the pair $(u(t), \chi(t))$, where $u(t)$ is the stable displacement field and $\chi(t)$ is the axisymmetric characteristic function given by

$$\chi(t, r) = \begin{cases} 0 & \text{if } 0 \leqq t \leqq t_c \text{ or } 0 \leqq r < \mathrm{R}\dfrac{t_c}{t} \\ 1 & \text{if } t > t_c \text{ and } \mathrm{R}\dfrac{t_c}{t} < r < \mathrm{R}, \end{cases} \tag{4.55}$$

is solution of problem $(\mathscr{P}_1)$. While the displacement field is the same for both solutions, the evolution of damage is quite different. The onset of damage appears respectively at $t_c$ for $\chi$ and $\sqrt{m}\, t_c$ for $\alpha$. There are *two stages* in the evolution of $\chi$: firstly, when $0 \leqq t \leqq t_c$, the cylinder remains undamaged; then, when $t \geqq t_c$, the cylinder progressively undergoes damage from the outer surface to the center. There are *three stages* in the evolution of $\alpha$: firstly, when $0 \leqq t \leqq \sqrt{m}\, t_c$, the cylinder remains undamaged; secondly, when $\sqrt{m}\, t_c \leqq t \leqq t_c/\sqrt{m}$, partial damage occurs at the outer surface, increases and propagates to the center; thirdly, when $t \geqq t_c/\sqrt{m}$, total damage appears at the outer surface and propagates to the center. It is worth pointing out that there always exists at this third stage an intermediate zone where the material is partially damaged (as in the second stage). The radii of the sound zone and of the partially damaged zone decrease with time like $1/t$. These results are in agreement with those of Proposition 4.3 once it has been remarked that in the case of a circular section of radius R $\bar{\eta}(r) = (\mathrm{R}^2 - r^2)/2$ and hence $t_m = t_c^0$.

Let us compare the corresponding torsion torques $\Gamma_\chi(t)$ and $\Gamma_\alpha(t)$ necessary to sustain the torsion angle $t$ of the section $\mathrm{S} \times \{\mathrm{L}\}$. Since in both cases the stress field is axisymmetric and independent of $z$, we have

$$\Gamma(t) = 2\pi \int_0^{\mathrm{R}} \sigma_{\theta 3}(t, r)\, r^3\, dr. \tag{4.56}$$

For the unstable classical solution, we obtain:

$$\Gamma_\chi(\tau) = \begin{cases} \tau \Gamma_c & \text{when } 0 \leqq \tau \leqq 1 \\ \left(m\tau + (1-m)\dfrac{1}{\tau^3}\right)\Gamma_c & \text{when } \tau \geqq 1, \end{cases} \tag{4.57}$$

while for the stable generalized one we have

$$(4.58)\qquad \Gamma_\alpha(\tau)=\begin{cases} \tau\,\Gamma_c & \text{when } 0\leqq\tau\leqq\sqrt{m} \\ \left(\dfrac{4\sqrt{m}}{3}-\dfrac{m^2}{3}\dfrac{1}{\tau^3}\right)\Gamma_c & \text{when } \sqrt{m}\leqq\tau\leqq\dfrac{1}{\sqrt{m}} \\ \left(m\tau+\dfrac{(1-m^3)}{3m}\dfrac{1}{\tau^3}\right)\Gamma_c & \text{when } \tau\geqq\dfrac{1}{\sqrt{m}} \end{cases}$$

where $\tau=t/t_c$, $\Gamma_c=\pi R^4/2\mu^0 t_c$, $m=\mu^1/\mu^0$. The two damage stages of the unstable solution and the three of the stable one appear in the expression of the torque. Moreover, if $m\in[3/4, 1)$, $\Gamma_\chi$ is an increasing function of $t$, but if $m\in(0, 3/4)$, $\Gamma_\chi$ is not monotone, it decreases for $\tau\in[1, (3(1-m)/m)^{1/4})$. On the other hand, $\Gamma_\alpha$ is a strictly increasing function of $t$ (it is even of class $C^1$). Finally, when $t$ tends to infinity, both $\Gamma_\chi$ and $\Gamma_\alpha$ are asymptotic to a line whose slope corresponds to the torsional rigidity modulus of the entirely damaged section.

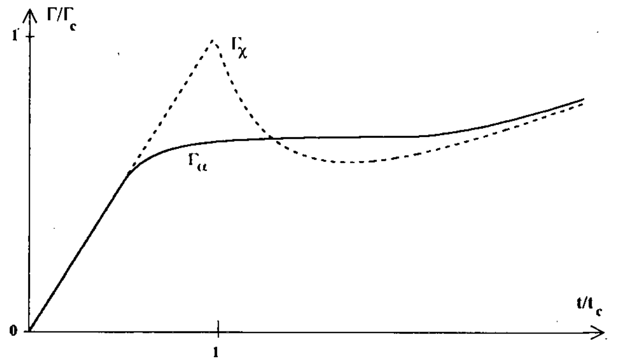


Fig. 4. — Torsion torque versus torsion angle for the unstable classical solution and the stable generalized solution. — Stable response --- Unstable response.

When plotting the graphs of $\Gamma_\chi/\Gamma_c$ or $\Gamma_\alpha/\Gamma_c$ versus $\tau=t/t_c$, we obtain curves that only depend on the ratio $m$ of the two shear moduli of the material. There is no scale effect, the response is independent of the ratio R/L. The graphs are plotted in Figure 4 for $m=1/4$.

## 5. Concluding remarks

Let us recapitulate our hypotheses and conclusions. Three constitutive ingredients are required:

1. A linear elastic material that undergoes brutal partial damage, the rigidity tensor decreasing from $A^0$ to $A^1$ with $A^0>A^1>0$.

2. An energetic damage yield criterion, namely $(A^0-A^1)\varepsilon.\varepsilon>2\kappa$, $\kappa>0$.

3. A stability criterion which consists in minimizing the sum of the potential energy and the dissipated energy of the body with respect to kinematically admissible displacements and admissible damage arrangements within the body.

Under these hypotheses, we show the existence of (possibly several) stable solutions in the large. Local fine mixtures of the sound and the damaged material are required. They give rise to effective stiffness tensors and a relaxed problem appears. It involves the set of all possible binary mixtures of the two damage states of the material in a prescribed volume fraction and it can be viewed as an evolution problem arising from a partial progressive damage model with a scalar parameter $\alpha$ – the volume fraction of the damaged material – increasing from 0 to 1 and with a rigidity tensor decreasing progressively from $A^0$ to $A^1$. Let us emphasize that it is not an arbitrary progressive model but one that is uniquely and unambiguously determined from the knowledge of $A^0$, $A^1$ and $\kappa$. One could retort that a progressive model could have been chosen at step 1. In that case a relaxation phenomenon would still take place but the computation of the relaxed energy density would become untractable because of the paucity of results pertaining to bounds on mixtures of more than two materials.

Of course our choice of the constitutive ingredients is subject to discussion. In particular the assumption that $A^1>0$ is fundamental. Without it, the evolution problem admits no stable solution, even generalized ones. The two examples treated in [F & M, 1991] should convince the dubious reader. In the case of a one-dimensional traction problem, the Maxwell force vanishes when $A^1=0$. In the case of the torsion problem, the torque $\Gamma_\alpha(t)$ corresponding to the stable solution vanishes at any $t$ if the damaged shear modulus $\mu^1$ is equal to zero. These results can be generalized because, when $A^1=0$, the relaxed elastic energy $\varphi^*$ introduced in Section 2 vanishes. The stringent restriction that $A^1$ remain positive is to be seen as a blatant confession of our ignorance of the principles that could preside over the occurrence of total damage.

As far as the second and the third ingredients are concerned, namely the choices of the yield and the stability criteria, our choice seems the most natural, but its consequences are important. The choice of an energetic criterion is essential in that it permits the introduction of the Lagrangian $\mathscr{L}$. The stability criterion requires the search of an (absolute) minimum for $\mathscr{L}$. A less severe criterion would be to enlarge the search to include relative minima.

As a final remark we venture to claim the originality of our work in the framework of damage mechanics although it is loosely reminiscent of the numerical study performed in [S, 1988]. In his paper, Silling studies trilinear nonmonotone elastic media and (numerically) finds solutions that can be interpreted as fine mixtures of the first and the third "phases" of the material.

**Acknowledgement**

Part of this work was performed while the first author was enjoying the hospitality of Professor G. W. Milton at the Courant Institute under the financial support of the Packard Foundation.

## REFERENCES


ALLAIRE G., KOHN R. V., 1991, Explicit optimal bounds on the elastic energy of a two-phase composite in two space dimensions, *Quarterly Appl. Math.* (to appear).

ANDRIEUX S., BAMBERGER Y., MARIGO J. J., 1986, Un modèle de matériau microfissuré pour les bétons et les roches, *J. Mec. Theor. Appl.*, **5**, 3, 471-513.

BALL J., JAMES R., 1987, Fine phase mixtures as minimizers of energy, *Arch. Rat. Mech. Analysis*, **100**, 13-52.

BENALLAL A., BILLARDON R., GEYMONAT G., 1989, Some mathematical aspects of the damage softening rate problem, in *Cracking and Damage*, Mazars J. and Bazant Z. Eds., Elsevier Applied Science, Amsterdam, 247-258.

BENDSOE M., KIKUSHI N., 1988, Generating optimal topologies in structural design using a homogenization method, *Comput. Meth. Appl. Mech. Engng.*, **71**, 197-224.

DAL MASO G., KOHN R. V., *The local character of* G-*closure*, in preparation.

DE BORST R., 1989, Analysis of spurious kinematic modes in finite element analysis of strain-softening solids, in *Cracking and Damage*, Mazars J. and Bazant Z. Eds., Elsevier Applied Science, Amsterdam, 335-345.

EHRLACHER A., FEDELICH B., 1989, Stability and bifurcation of simple dissipative systems: application to brutal damage, in *Cracking and Damage*, Mazars J. and Bazant Z. Eds., Elsevier Applied Science, Amsterdam, 217-227.

EKELAND I., TEMAM R., 1974, *Analyse convexe et problèmes variationnels*, Gauthier-Villars, Paris.

FRANCFORT G., MARIGO J. J., 1991, Mathematical analysis of damage evolution in a brittle damaging continuous medium, in *Mécanique, Modélisation Numérique et Dynamique des matériaux*, Publications du L.M.A. n° 124, C.N.R.S., France, 245-276.

FRANCFORT G., MURAT F., 1986, Homogenization and optimal bounds in linear elasticity, *Arch. Rat. Mech. Anal.*, **94**, 307-334.

GERMAIN P., 1973, *Cours de mécanique des milieux continus*, I. *Théorie générale*, Masson, Paris.

HASHIN Z., SHTRIKMAN S., 1963, A variational approach to the theory of the elastic behavior of multiphase materials, *J. Mech. Phys. Solids*, **11**, 127-140.

HILL R., 1950, *The mathematical theory of plasticity*, Clarendon Press, Oxford.

JAMES R. D., KINDERLHERER D., 1989, Theory of diffusionless phase transformations, in *Partial Differential Equations and Continuum Models of Phase Transitions*, Rascle M. *et al.* Eds., Lecture Notes in Physics, 344, Springer Verlag, Berlin, 51-84.

KACHANOV L. M., 1958, Time of the rupture process under creep conditions, *Izv. Akad. Nauk. S.S.S.R., Otd. Tech. Nauk.*, **8**, 26-31.

KACHANOV L., 1975, *Éléments de la théorie de la plasticité*, Éditions MIR, Moscou.

KAWOHL B., STARA J., WITTUM G., *Analysis and numerical studies of a shape design problem* (to appear).

KIKUSHI N., SUZUKI K., Structural optimization of a linearly elastic structure using the homogenization theory, in *Composite Media and Homogenization Theory*, Dal Maso G. *et al.* Eds. (to appear).

KOHN R. V., 1988, Recent progress in the mathematical modeling of composite materials, in *Composite Material Response: Constitutive Relations and Damage Mechanisms*, Sih G. *et al.* Eds., Elsevier, New York, 155-177.

KOHN R. V., 1990, The relationship between linear and nonlinear variational models of coherent phase transitions, in *Trans. 7th Army Conf. on Appl. Math. and Computing*, *ARO Report* n° 90-1.

KOHN R. V., LIPTON R., 1988, Optimal bounds for the effective energy of a mixture of isotropic, incompressible, elastic materials, *Arch. Rat. Mech. Anal.*, **102**, 331-350.

KOHN R. V., MILTON G. W., 1986, On bounding the effective conductivity of anisotropic composites, in *Homogenization and Effective Moduli of Materials and Media*, I.M.A. Volumes in Mathematics and its Applications, 1, Ericksen *et al.* Eds., Springer, New York, 97-125.

KOHN R. V., STRANG G., 1986, Optimal design and relaxation of variational problems I-III, *Comm. Pure Appl. Math.*, **39**, 113-137, 139-182, 353-377.

LEMAITRE J., 1978, Théorie mécanique de l'endommagement isotrope appliquée à la fatigue des métaux, in *Structures sous chargements cycliques*, ENPC, Paris.

LEMAITRE J., CHABOCHE J. L., 1978, Aspect phénoménologique de la rupture par endommagement, *J. Méc. Appl.*, **2**, 317-365.

LEMAITRE J., CHABOCHE J. L., 1985, *Mécanique des matériaux solides*, Dunod, Paris.

LENE F., LEGUILLON D., 1982, Homogenized constitutive law for a partially cohesive composite material, *Int. J. Solids Struct.*, **18**, 443-458.

LURIE K., CHERKAEV A., 1986, The effective properties of composites and problems of optimal design of constructions, *Uspekhi Mekhaniki*, **9.2**, 1-81 (in Russian).

MARIGO J. J., 1985, Modelling of brittle and fatigue damage for elastic materials by growth of microvoids, *Eng. Fract. Mech.*, **21.4**, 861-874.

MARIGO J. J., 1989, Constitutive relations in Plasticity, Damage and Fracture Mechanics based on a Work Property, *Nuclear Eng. Design*, **114**, 249-272.

MARIGO J. J., PIDERI C., 1987, Decohesion criteria in fibred composites and reinforced beams or plates, in *Mechanical behaviour of adhesive joints*, Pluralis, 415-431.

MILTON G. W., KOHN R. V., 1988, Variational bounds on the effective moduli of anisotropic composites, *J. Mech. Phys. Solids*, **36**, 597-629.

MURAT F., 1978, H-convergence, *Séminaire d'analyse fonctionnelle et numérique de l'Université d'Alger* (multigraphed).

MURAT F., TARTAR L., 1985, Calcul des variations et homogénéisation, in *Les Méthodes d'homogénéisation: théorie et applications en Physique*, Coll. de la Direction des Études et Recherches d'E.D.F., Eyrolles, Paris, 319-369.

NGUYEN Q. S., 1984, Bifurcation et stabilité des systèmes irréversibles obéissant au principe du travail maximal, *J. Méc.*, **3**, 41-61.

NGUYEN Q. S., 1987, Bifurcation and postbifurcation analysis in plasticity and brittle fracture, *J. Mech. Phys. Solids*, **35**, 303-324.

SERRIN J., 1971, A symmetry problem in potential theory, *Arch. Rat. Mech. Anal.*, **43**, 304-318.

SILLING S. A., 1988, Numerical studies of loss of ellipticity near singularities in an elastic material, *J. Elasticity*, **19**, 213.

SPAGNOLO S., 1968, Sulla convergenza di soluzioni di equazioni paraboliche ed ellitiche, *Ann. Sci. Norm. Sup. Pisa*, **22**, 577-597.

STOLZ C., 1989, On some aspects of stability and bifurcation in fracture and damage mechanics, in *Cracking and Damage*, Mazars J. and Bazant Z. Eds., Elsevier Applied Science, Amsterdam, 207-216.

SUQUET P., 1981, Approche par homogénéisation d'une forme d'endommagement, *C.R. Acad. Sci. Paris*, **292**, Series I, 809-812.

TARTAR L., 1977, *Cours Peccot*, Collège de France.

TARTAR L., 1985, Estimations fines de coefficients homogénéisés, in *Ennio De Giorgi Colloqium*, Kree P. Ed., Pitman Research Notes in Mathematics, **125**, Boston, 168-187.

(Manuscript received January 28, 1991,
accepted March 5, 1992.)