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OSCILLATIONS AND ENERGY DENSITIES IN THE WAVE EQUATION

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0. INTRODUCTION.

In a previous paper [2], we addressed the homogenization of the wave equation in the general setting of H-convergence. Specifically we considered the following wave equation with Dirichlet boundary conditions:

$$\begin{cases} \rho^\varepsilon \dfrac{\partial^2 u^\varepsilon}{\partial t^2} - div(A^\varepsilon \operatorname{grad} u^\varepsilon) = f & \text{in } \Omega \times (0,T), \\ u^\varepsilon = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u^\varepsilon(0) = a^\varepsilon \quad \text{in } \Omega, \\ \dfrac{\partial u^\varepsilon}{\partial t}(0) = b^\varepsilon \quad \text{in } \Omega, \end{cases}$$

where u^ε is the unknown field while the other quantities are given. We proved that, under "minimal" assumptions on the various data, it was possible to partition the solution u^ε into

$$u^\varepsilon = \tilde{u}^\varepsilon + v^\varepsilon,$$

where $\tilde{u}^\varepsilon$ can be explicitly described from the only knowledge of the weak limit field u of u^ε while v^ε converges weakly-$*$ to 0 in the appropriate topology as ε tends to zero.

The field v^ε contains a wealth of information which is lost in the limit process. In particular the energy density associated to v^ε, *i.e.*,

$$d^\varepsilon = \frac{1}{2}\left\{\rho^\varepsilon\left(\frac{\partial v^\varepsilon}{\partial t}\right)^2 + A^\varepsilon \operatorname{grad} v^\varepsilon \operatorname{grad} v^\varepsilon\right\}$$

was found to satisfy

$$H^\varepsilon \overset{\text{def}}{\equiv} \int_\Omega d^\varepsilon(x,t)dx \quad \text{is independent of } t,$$

$$H^\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} H,$$

where H is a positive (and in general strictly positive) constant.

Our goal in the present paper is to further describe the local behaviour of d^ε as ε tends to zero; for example we will aim at computing the measure limit d^0 of d^ε. From a more physical standpoint we strive to understand the space-time localization properties of the part of the elastic energy that remains trapped during the homogenization process. Such a task is at present beyond our capabilities in the general setting briefly evoked in this introduction. It is merely performed here in the (much simpler) case where A^ε and ρ^ε are independent of ε, the only oscillations being those introduced by the initial data a^ε and b^ε.

Section 1 is devoted to a precise setting of the problem and to a brief review of the available results. Section 2 addresses the constant coefficient case and uses the method of geometrical optics. Through geometrical optics a detailed description of not only d^ε but also v^ε is provided. Section 3 is devoted to the case of smooth coefficients and uses the concept of H-measures introduced by L. TARTAR (cf. [15]) or of microlocal defect-measures introduced by P. GERARD (cf. [5]-[7]). The H-measure associated to the sequence $(\partial v^\varepsilon/\partial t\ ,\ \operatorname{grad} v^\varepsilon)$ is characterized and it provides in turn a description of d^0. Note that the intricate problems attached to the presence of a boundary are not broached in this study (see the relevant remark at the end of Subsection 1.3).

A more detailed preview of the study is given at the end of Section 1 (Subsection 1.3).

The content of the paper is the following:

CONTENT

1. SETTING OF THE PROBLEM.

This section is divided into three subsections. Subsection 1.1 is devoted to a detailed description of the problem under consideration and to a recall of the results previously obtained in [2]. Subsection 1.2 addresses the "purely periodic" case and demonstrates that, although the field v^ε is perfectly determined in that setting, any attempt to depart from "pure periodicity"

-by allowing slow spatial modulations for example- leads to a much more intricate problem. Subsection 1.3 is a continuation of the introduction and offers a detailed preview of the results derived in the paper.

1.1. Formulation of the problem and recall of previous results.

Consider the following wave equation with Dirichlet boundary conditions:

$$\rho^\varepsilon \frac{\partial^2 u^\varepsilon}{\partial t^2} - div(A^\varepsilon \operatorname{grad} u^\varepsilon) = f \quad \text{in } \Omega \times (0,T), \tag{1.1}$$

$$u^\varepsilon = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{1.2}$$

$$\begin{cases} u^\varepsilon(0) = a^\varepsilon & \text{in } \Omega, \\ \dfrac{\partial u^\varepsilon}{\partial t}(0) = b^\varepsilon & \text{in } \Omega. \end{cases} \tag{1.3}$$

In (1.1)-(1.3), Ω is an open bounded subset of $\mathbb{R}^N$ and u^ε is a scalar valued unknown field whereas

$$\begin{cases} \rho^\varepsilon(x) = \rho(x, \dfrac{x}{\varepsilon}), \\ A^\varepsilon(x) = A(x, \dfrac{x}{\varepsilon}), \\ a^\varepsilon(x) = a^0(x) + \varepsilon\, a(x, \dfrac{x}{\varepsilon}), \\ b^\varepsilon(x) = b(x, \dfrac{x}{\varepsilon}), \end{cases} \tag{1.4}$$

almost everywhere in Ω. The coefficients $\rho(x,y)$ and $a_{ij}(x,y)$ (the $(i,j)^{\text{th}}$ coefficient of $A(x,y)$) are assumed to be smooth functions defined on $\Omega \times \mathcal{T}$ ($\mathcal{T}$ is the unit torus of $\mathbb{R}^N$) and to satisfy, for every x in Ω and every y in $\mathcal{T}$,

$$\begin{cases} \lambda_1 \le \rho(x,y) \le \lambda_2, \\ a_{ij}(x,y) = a_{ji}(x,y) \quad , \quad 1 \le i,j \le N, \\ \lambda_1 |\xi|^2 \le a_{ij}(x,y)\xi_i \xi_j \le \lambda_2 |\xi|^2 \quad , \quad \xi \in \mathbb{R}^N. \end{cases}$$

(Einstein's summation convention for repeated indices will be used throughout). The loading f satisfies

$$f \in L^2(0,T;L^2(\Omega)).$$

The initial conditions a^ε and b^ε are such that $a(x,y)$ and $b(x,y)$ are smooth on $\Omega \times \mathcal{T}$ while

$$\begin{aligned} &a^0 \in C_0^\infty(\Omega), \\ &a(x,y) = 0 \quad , \quad \text{for } x \text{ in a neighbourhood of } \partial\Omega. \end{aligned} \tag{1.5}$$

In Sections 2 and 3, a^0 will be assumed to be 0 and b will satisfy

$$\int_{\mathcal{T}} b(x,y)dy = 0,$$

while in Section 2, ρ and A will be constant coefficients and, in Section 3, ρ and A will be independent of y, i.e.,

$$\begin{cases} \rho^\varepsilon(x) = \rho(x), \\ A^\varepsilon(x) = A(x). \end{cases}$$

Remark 1.1. The smoothness assumptions on the various data entering (1.1)-(1.3) are much too strong; a careful study of reasonable regularity assumptions was performed in [2]. Our current purpose here is not to discuss minimum regularity but rather to avoid questions pertaining to regularity, focusing instead on the oscillating behaviour of the solution field u^ε. □

Under the various above listed hypotheses problem (1.1)-(1.3) is known to yield a unique solution u^ε with

$$u^\varepsilon \in C^0([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)).$$

We are concerned with the intimate behaviour of u^ε as ε tends to zero. To this effect we recall the definition of the homogenization limit (H-limit) A^0 of A^ε which is explicity computable in the present quasi-periodic setting (cf. e.g. [1], Ch.1, Theorem 6.1). Define $\chi_i(x,y)$ as the (smooth) solution in $\Omega \times \mathcal{T}$, unique up to a constant, of

$$-\mathrm{div}_y(A(x,y)\,\mathrm{grad}_y(\chi_i(x,y) + e_i)) = 0,$$

where e_i is the i^{th} basis vector. Then A^0 is defined as the matrix whose $(i,j)^{\text{th}}$ coefficients is

$$a_{ij}^0(x) = \int_T \{a_{ij}(x,y) + a_{ik}(x,y)\,\frac{\partial \chi_j}{\partial y_k}(x,y)\}dy,$$

and the solution u^ε satisfies (cf. e.g. [2], Theorem 3.2)

$$u^\varepsilon \rightharpoonup u \text{ weak-}* \text{ in } L^\infty(0,T;H_0^1(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega)), \tag{1.6}$$

where u is the unique solution in $\mathcal{C}^0([0,T];H_0^1(\Omega)) \cap \mathcal{C}^1([0,T];L^2(\Omega))$ of

$$\overline{\rho}\,\frac{\partial^2 u}{\partial t^2} - \operatorname{div}(A^0 \operatorname{grad} u) = f \quad \text{in } \Omega \times (0,T), \tag{1.7}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{1.8}$$

$$\begin{cases} u(0) = a^0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial t}(0) = b^0 & \text{in } \Omega. \end{cases} \tag{1.9}$$

The only undefined quantities in (1.7) and (1.9) are $\overline{\rho}$ and b^0, which are respectively given by

$$\begin{cases} \overline{\rho}(x) = \displaystyle\int_T \rho(x,y)dy, \\ b^0(x) = \dfrac{1}{\overline{\rho}(x)} \displaystyle\int_T \rho(x,y)\,b(x,y)dy. \end{cases}$$

The convergence (1.6) of u^ε to u can be further explicited upon partitioning u^ε into

$$u^\varepsilon = \tilde{u}^\varepsilon + \tilde{v}^\varepsilon, \tag{1.10}$$

where $\tilde{u}^\varepsilon$ and $\tilde{v}^\varepsilon$ are respectively the unique solutions in $\mathcal{C}^0([0,T];H_0^1(\Omega)) \cap \mathcal{C}^1([0,T];L^2(\Omega))$ of

$$\rho^\varepsilon\,\frac{\partial^2 \tilde{u}^\varepsilon}{\partial t^2} - \operatorname{div}(A^\varepsilon \operatorname{grad} \tilde{u}^\varepsilon) = f \quad \text{in } \Omega \times (0,T), \tag{1.11}$$

(1.12) $$\tilde{u}^\varepsilon = 0 \quad \text{on } \partial\Omega \times (0,T),$$

(1.13) $$\begin{cases} \tilde{u}^\varepsilon(0) = \tilde{a}^\varepsilon & \text{in } \Omega, \\ \dfrac{\partial \tilde{u}^\varepsilon}{\partial t}(0) = b^0 & \text{in } \Omega, \end{cases}$$

and

(1.14) $$\rho^\varepsilon \frac{\partial^2 \tilde{v}^\varepsilon}{\partial t^2} - \operatorname{div}(A^\varepsilon \operatorname{grad} \tilde{v}^\varepsilon) = 0 \quad \text{in } \Omega \times (0,T),$$

(1.15) $$\tilde{v}^\varepsilon = 0 \quad \text{on } \partial\Omega \times (0,T),$$

(1.16) $$\begin{cases} \tilde{v}^\varepsilon(0) = a^\varepsilon - \tilde{a}^\varepsilon & \text{in } \Omega, \\ \dfrac{\partial \tilde{v}^\varepsilon}{\partial t}(0) = b^\varepsilon - b^0 & \text{in } \Omega. \end{cases}$$

The only undefined quantity in (1.11)-(1.16) is $\tilde{a}^\varepsilon$, which is defined as the solution, unique in $H_0^1(\Omega)$, of

(1.17) $$-\operatorname{div}(A^\varepsilon \operatorname{div} \tilde{a}^\varepsilon) = -\operatorname{div}(A^0 \operatorname{grad} a^0) \quad \text{in } \Omega,$$

(1.18) $$\tilde{a}^\varepsilon = 0 \quad \text{on } \partial\Omega.$$

The results obtained in [2] (cf. in particular Theorems 4.1-4.4 of [2]) lead to the following statements of convergence, as ε tends to zero:

(1.19) $$\begin{cases} \dfrac{\partial \tilde{u}^\varepsilon}{\partial t} - \dfrac{\partial u}{\partial t} \to 0 \text{ strongly in } C^0([0,T]; L^2(\Omega)), \\ \operatorname{grad} \tilde{u}^\varepsilon - \operatorname{grad} u - (\operatorname{grad}_y \chi_i)(x, \dfrac{x}{\varepsilon}) \dfrac{\partial u}{\partial x_i} \to 0 \\ \qquad\qquad \text{strongly in } C^0([0,T]; [L^2(\Omega)]^N), \end{cases}$$

(1.20) $$\tilde{v}^\varepsilon \rightharpoonup 0 \quad \text{weak-}* \text{ in } L^\infty(0,T; H_0^1(\Omega)) \cap W^{1,\infty}(0,T; L^2(\Omega)).$$

(Note that the field $\tilde{v}^\varepsilon$ is denoted by v^ε in [2]).

Convergence (1.19) essentially states that the field $\tilde{u}^\varepsilon$ can be recovered from the only knowledge of the limit field u and of the auxiliary functions $\chi_i(x,y)$. From now onward our attention is focused on the field $\tilde{v}^\varepsilon$.

Since (cf. (1.5)) a^0 belongs to $H_0^2(\Omega)$ the following corrector's result holds true for the solution $\tilde{a}^\varepsilon$ of (1.17)-(1.18) (cf. e.g. [1], Ch.1, Theorem 6.2):

$$\tilde{a}^\varepsilon(x) = a^0(x) + \varepsilon \frac{\partial a^0}{\partial x_i}(x)\,\chi_i(x, \frac{x}{\varepsilon}) + r^\varepsilon(x) \quad \text{a.e. in } \Omega, \tag{1.21}$$

where

$$r^\varepsilon \overset{\varepsilon \to 0}{\longrightarrow} 0 \quad \text{strongly in } H_0^1(\Omega). \tag{1.22}$$

Let us emphasize that in the absence of hypothesis (1.5), a somewhat weaker corrector's result is known to hold true (cf. [14], see also [2] Section 2).

In view of (1.21) and of the specific form of $a^\varepsilon(x), b^\varepsilon(x)$ (cf. (1.4)), (1.16) reads as

$$\begin{cases} \tilde{v}^\varepsilon(0) = \varepsilon\, \alpha(x, \frac{x}{\varepsilon}) - r^\varepsilon(x) \quad \text{a.e. in } \Omega, \\ \frac{\partial \tilde{v}^\varepsilon}{\partial t}(0) = \beta(x, \frac{x}{\varepsilon}) \quad \text{a.e. in } \Omega, \end{cases} \tag{1.23}$$

where $\alpha(x, y)$ and $\beta(x, y)$ are respectively defined as

$$\alpha(x, y) = a(x, y) - \frac{\partial a^0}{\partial x_i}(x)\,\chi_i(x, y),$$
$$\beta(x, y) = b(x, y) - b^0(x),$$

for every (x, y) in $\Omega \times \mathcal{T}$.

The form (1.23) of the initial conditions for $\tilde{v}^\varepsilon$ suggests to further partition $\tilde{v}^\varepsilon$ into

$$\tilde{v}^\varepsilon = v^\varepsilon + \tilde{r}^\varepsilon,$$

where v^ε and $\tilde{r}^\varepsilon$ are the unique solutions in $C^0([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ of

$$\rho^\varepsilon \frac{\partial^2 v^\varepsilon}{\partial t^2} - \operatorname{div}(A^\varepsilon \operatorname{grad} v^\varepsilon) = 0 \quad \text{in } \Omega \times (0, T), \tag{1.24}$$

$$v^\varepsilon = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{1.25}$$

$$\begin{cases} v^\varepsilon(0) = \varepsilon\, \alpha(x, \frac{x}{\varepsilon}) & \text{in } \Omega, \\ \frac{\partial v^\varepsilon}{\partial t}(0) = \beta(x, \frac{x}{\varepsilon}) & \text{in } \Omega, \end{cases} \tag{1.26}$$

and

$$\rho^\varepsilon \frac{\partial^2 \tilde{r}^\varepsilon}{\partial t^2} - \operatorname{div}(A^\varepsilon \operatorname{grad} \tilde{r}^\varepsilon) = 0 \quad \text{in } \Omega \times (0,T), \tag{1.27}$$

$$\tilde{r}^\varepsilon = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{1.28}$$

$$\begin{cases} \tilde{r}^\varepsilon(0) = r^\varepsilon & \text{in } \Omega, \\ \frac{\partial \tilde{r}^\varepsilon}{\partial t}(0) = 0 & \text{in } \Omega. \end{cases} \tag{1.29}$$

In view of (1.22), (1.27)-(1.29), the field $\tilde{r}^\varepsilon$ satisfies

$$\tilde{r}^\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} 0 \quad \text{strongly in } C^0([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)). \tag{1.30}$$

Thus the only non trivial contribution to $\tilde{v}^\varepsilon$ is that of v^ε which in view of (1.20), (1.30) satisfies

$$v^\varepsilon \overset{\varepsilon\to 0}{\rightharpoonup} 0 \quad \text{weak-}* \text{ in } L^\infty(0,T; H_0^1(\Omega)) \cap W^{1,\infty}(0,T; L^2(\Omega)). \tag{1.31}$$

It is immediately deduced from (1.30) and Theorem 4.3 of [2] that

$$\frac{1}{2}\int_\Omega \rho^\varepsilon \Big(\frac{\partial v^\varepsilon}{\partial t}\Big)^2 dx \overset{\varepsilon\to 0}{\longrightarrow} \frac{1}{2} H \quad \text{weak-}* \text{ in } L^\infty(0,T),$$
$$\frac{1}{2}\int_\Omega A^\varepsilon \operatorname{grad} v^\varepsilon \operatorname{grad} v^\varepsilon \, dx \overset{\varepsilon\to 0}{\longrightarrow} \frac{1}{2} H \quad \text{weak-}* \text{ in } L^\infty(0,T),$$

where H is defined as

$$H = \frac{1}{2}\int_{\Omega\times T} \{\rho\, \beta^2 + A \operatorname{grad}_y \alpha \operatorname{grad}_y \alpha\}(x,y)\, dx\, dy.$$

Thus the energy $\eta^\varepsilon(t)$ associated with (1.24)-(1.26), namely

$$\eta^\varepsilon(t) = \frac{1}{2}\int_\Omega \Big(\rho^\varepsilon \Big(\frac{\partial v^\varepsilon}{\partial t}\Big)^2 + A^\varepsilon \operatorname{grad} v^\varepsilon \operatorname{grad} v^\varepsilon\Big)(x,t)\, dx$$

satisfies

$$\eta^\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} H \quad \text{weak-}* \text{ in } L^\infty(0,T). \tag{1.32}$$

Remark 1.2. Theorem 4.3 of [2] and (1.30) also imply that if

$$\begin{aligned} e^\varepsilon(t) &= \frac{1}{2}\int_\Omega (\rho^\varepsilon \Big(\frac{\partial u^\varepsilon}{\partial t}\Big)^2 + A^\varepsilon \operatorname{grad} u^\varepsilon \operatorname{grad} u^\varepsilon)(x,t)dx,\\ \tilde{e}^\varepsilon(t) &= \frac{1}{2}\int_\Omega (\rho^\varepsilon \Big(\frac{\partial \tilde{u}^\varepsilon}{\partial t}\Big)^2 + A^\varepsilon \operatorname{grad} \tilde{u}^\varepsilon \operatorname{grad} \tilde{u}^\varepsilon)(x,t)dx,\\ e^0(t) &= \frac{1}{2}\int_\Omega (\overline{\rho}\Big(\frac{\partial u}{\partial t}\Big)^2 + A^0 \operatorname{grad} u \operatorname{grad} u)(x,t)dx, \end{aligned}$$

denote the energies associated with (1.1)-(1.3), (1.11)-(1.13) and (1.7)-(1.9) respectively, then, as ε tends to zero,

$$\begin{aligned} &e^\varepsilon(t) - \tilde{e}^\varepsilon(t) - \eta^\varepsilon(t) \to 0 \quad \text{strongly in } \mathcal{C}^0([0,T]),\\ &\tilde{e}^\varepsilon(t) \to e^0(t) \quad \text{strongly in } \mathcal{C}^0([0,T]), \end{aligned}$$

and thus, by virtue of (1.32),

$$e^\varepsilon(t) \overset{\varepsilon\to 0}{\longrightarrow} e(t) \overset{\text{def}}{\equiv} e^0(t) + H \quad \text{weak-}* \text{ in } L^\infty(0,T).$$

Since H is in general strictly positive there is a non trivial contribution of the field v^ε in the limit $e(t)$ of the energy.

Furthermore the energy density $d^0(x,t)$ associated to $e^0(t)$ is exactly

$$d^0(x,t) = \frac{1}{2}\Big(\overline{\rho}(\frac{\partial u}{\partial t})^2 + A^0 \operatorname{grad} u \operatorname{grad} u\Big)(x,t) \quad , \quad x\in\Omega \quad , \quad t\in[0,T],$$

but a description of the energy density associated to H has yet to be achieved. □

We define the energy density associated to v^ε as

$$d^\varepsilon(x,t) = \frac{1}{2}\Big(\rho^\varepsilon(\frac{\partial v^\varepsilon}{\partial t})^2 + A^\varepsilon \operatorname{grad} v^\varepsilon \operatorname{grad} v^\varepsilon\Big)(x,t) \quad , \quad x\in\Omega \quad , \quad t\in[0,T]. \tag{1.33}$$

According to (1.32),

$$\int_{\Omega} d^{\varepsilon}(x,t)dx \overset{\varepsilon\to 0}{\longrightarrow} H. \tag{1.34}$$

We propose to try and compute the measure d, limit of the energy density d^{ε} as ε tends to zero, i.e., to spatially localize convergence (1.34). The available information about v^{ε} (i.e., convergence (1.31)) does not permit the computation of the limit of d^{ε}.

We conclude Subsection 1.1 with a recall of the current setting of our problem. We consider v^{ε} unique solution in $C^0([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$ of

$$\rho^{\varepsilon}\frac{\partial^2 v^{\varepsilon}}{\partial t^2} - \operatorname{div} A^{\varepsilon} \operatorname{grad} v^{\varepsilon} = 0 \quad \text{in } \Omega\times(0,T), \tag{1.35}$$

$$v^{\varepsilon} = 0 \quad \text{on } \partial\Omega\times(0,T), \tag{1.36}$$

$$\begin{cases} v^{\varepsilon}(0) = \varepsilon\alpha(x,\dfrac{x}{\varepsilon}) & \text{in } \Omega, \\ \dfrac{\partial v^{\varepsilon}}{\partial t}(0) = \beta(x,\dfrac{x}{\varepsilon}) & \text{in } \Omega, \end{cases} \tag{1.37}$$

with $\rho^{\varepsilon}, A^{\varepsilon}$ of the form $\rho(x,x/\varepsilon), A(x,x/\varepsilon)$ respectively. Further $\rho(x,y)$, $A(x,y), \alpha(x,y), \beta(x,y)$ are assumed to be smooth on $\Omega\times\mathcal{T}, \alpha$ and β are compactly supported in Ω in the variable x and $\overline{\rho\beta}(x) \overset{\text{def}}{\equiv} \int_{\mathcal{T}}\rho(x,y)\beta(x,y)dy = 0$. We know that v^{ε} satisfies (1.31), (1.34) as ε tends to zero, and that, for a subsequence of ε (still denoted by ε),

$$d^{\varepsilon} \rightharpoonup d \quad \text{weak-}* \text{ in } L^{\infty}(0,T;\mathcal{M}(\Omega)). \tag{1.38}$$

Our goal is the computation of d.

1.2. "Purely" periodic setting versus actual setting.

In Remark 4.6 of [2] the system (1.35)-(1.37) is investigated in the case where ρ, A, α and β are assumed to be independent of x, Ω is a parallelepiped with sides of integer lengths and the Dirichlet boundary conditions on v^{ε} are replaced by periodic boundary conditions. In such a case a quasi explicit formula for the field v^{ε} is readily available, namely,

$$v^\varepsilon(x,t) = \varepsilon\, V\Big(\frac{x}{\varepsilon}\,,\,\frac{t}{\varepsilon}\Big) \quad , \quad x \in \Omega \quad , \quad t \in \mathbb{R}_+, \tag{1.39}$$

where V is the solution of

$$\rho(y)\,\frac{\partial^2 V}{\partial s^2} - \mathrm{div}_y(A(y)\mathrm{grad}_y\, V) = 0 \quad \text{in } \mathcal{T} \times \mathbb{R}_+, \tag{1.40}$$

$$\begin{cases} V(y,0) = \alpha(y) & \text{on } \mathcal{T}, \\ \dfrac{\partial V}{\partial s}(y,0) = \beta(y) & \text{on } \mathcal{T}. \end{cases} \tag{1.41}$$

(Recall that $\int_{\mathcal{T}} \rho(y)\beta(y)dy = 0$). Note that the first equation of the above system implicitly contains periodic spatial boundary conditions on V. In contrast V is not periodic with respect to s.

It is tempting to extend (1.39) to our present setting (1.35)-(1.37) (which is a modulation in x of the data α, β and A associated to (1.40)-(1.41)) and to seek a solution v^ε of (1.35)-(1.37) of the form

$$\tilde{v}^\varepsilon(x,t) = \sum_{P=1}^{J} \varepsilon^P\, V_P(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}) + \tilde{r}^\varepsilon(x,t), \tag{1.42}$$

where J is a (hopefully small) integer and V_P satisfies a (possibly non homogeneous) wave equation in $\mathcal{T} \times \mathbb{R}_+$ with appropriately chosen initial conditions and forcing terms, where the boundary conditions are periodic in y but not in s, and where finally $\tilde{r}^\varepsilon$ converges to zero in the energy norm (*i.e.*, in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+; H^1_0(\Omega)) \cap W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}_+; L^2(\Omega))$).

We however claim that a simple argument based on the notion of domain of dependence for the wave equation shows that $\tilde{v}^\varepsilon$ given by (1.42) cannot be a proper ansatz for v^ε. Assume for example that $\rho(x,y)$ is identically 1 and that $A(x,y)$ is the identity matrix. Then v^ε satisfies

$$\frac{\partial^2 v^\varepsilon}{\partial t^2} - \Delta v^\varepsilon = 0, \tag{1.43}$$

$$\begin{cases} v^\varepsilon(0) = \varepsilon\alpha(x,\dfrac{x}{\varepsilon}), \\ \dfrac{\partial v^\varepsilon}{\partial t}(0) = \beta(x,\dfrac{x}{\varepsilon}) \quad \text{with } \displaystyle\int_{\mathcal{T}} \beta(x,y)dy = 0. \end{cases} \tag{1.44}$$

We do not pay much attention here to the boundary conditions for (1.43)-(1.44) (or to the domain on which (1.43)-(1.44) live); assuming that

$$\alpha(x,y) = \beta(x,y) = 0 \quad , \quad |x| > 1,$$

will imply that v^ϵ is identically nul outside the domain of influence of the initial data and thus that if Ω is chosen to be a sufficiently large ball and T a sufficiently small time, the boundary condition

$$v^\epsilon = 0 \quad \text{on } \partial\Omega \times (0,T)$$

is satisfied.

Inserting an ansatz of the form (1.42) in (1.43)-(1.44) leads to the following "natural" choices for the functions V_P:

$$\frac{\partial^2 V_1}{\partial s^2} - \Delta_y V_1 = 0 \quad \text{in } \mathcal{T} \times \mathbb{R}_+,$$

$$\begin{cases} V_1(x,t,y,0) = \alpha(x,y), \\ \dfrac{\partial V_1}{\partial s}(x,t,y,0) = \beta(x,y), \end{cases}$$

and, for $P \geq 2$,

$$\frac{\partial^2 V_P}{\partial s^2} - \Delta_y V_P = -\Big\{\frac{\partial^2 V_{P-2}}{\partial t^2} - \Delta_x V_{P-2} + 2\Big(\frac{\partial^2 V_{P-1}}{\partial t\,\partial s} - \sum_{i=1}^{N} \frac{\partial^2 V_{P-1}}{\partial x_i \partial y_i}\Big)\Big\}$$
$$\text{in } \mathcal{T} \times \mathbb{R}_+,$$

$$\begin{cases} V_P(x,t,y,0) = 0, \\ \dfrac{\partial V_P}{\partial s}(x,t,y,0) = 0, \end{cases}$$

with $V_0 \equiv 0$.

For such a choice of the V_P's, the variable x in ansatz (1.42) is merely a parameter ; consequently, if α and β are taken to be 0 on a given ball $|x - x_0| \leq r$ of $\mathbb{R}^N$, $\tilde{v}^\epsilon$ will be 0 on that ball for every positive time. But the actual field $v^\epsilon(x,t)$ will feel at time t and point x the effect of the initial conditions for all points y in the domain of dependence, *i.e.*, for all points y in the ball $|y - x| \leq t$; it is thus quite unlikely that $v^\epsilon(x,t)$ will remain identically 0 for all times t on the ball $|x - x_0| \leq r$. Thus, on such a ball,

v^ε would coincide with $\tilde{r}^\varepsilon$ and would in general not converge to zero in the energy norm.

This heuristic argument could be easily formalized in a one dimensional setting upon choosing α to be identically zero and β to be of the form

$$\beta(x,y) = \tilde{\beta}(x)\,\varphi(y),$$

with $\tilde{\beta}$ in $C_0^\infty(\mathbb{R}^N)$ and φ an eigenvector for Laplace's equation on $\mathcal{T}$.

The reader is invited to compare the failing ansatz (1.42) to that provided by geometrical optics (see (2.52)-(2.53)), which proves successful in the case of constant coefficients. Both ansatzs look similar. Two reasons however concur in the failure of the proposed expansion. On the one hand, Cauchy data on v^ε should only specify the value of V_P and $\partial V_P/\partial s$ at $t = s = 0$, and not for every $t > 0$ at $s = 0$. On the other hand $V_P (P \geq 2)$ satisfies a non homogeneous wave equation and, unless additional restrictions are met by V_{P-1}, V_{P-2} and their derivatives so that V_P remains bounded in s, the right hand side of (1.42) is not even an asymptotic expansion in L^2_{loc} ; such restrictions might be thought of as filling the gap left open by the absence of initial conditions for $t > 0$ and $s = 0$. In contrast, the geometrical optics ansatz (2.52)-(2.53) provides a valid expansion.

1.3. Overview of the paper.

In the following sections we propose to address setting (1.35)-(1.37) *in the restricted case of y-independent coefficients.*

We will thus assume that

$$\begin{cases} \rho(x,y) = \rho(x) = \rho^\varepsilon(x), \\ A(x,y) = A(x) = A^\varepsilon(x), \end{cases} \tag{1.45}$$

so that the only oscillations in the data will be those of the initial conditions.

Whenever ρ and A do not depend on x, geometrical optics will be a convenient tool and a very precise description of v^ε itself will be achieved. This is the object of Section 2. Specifically a decomposition of v^ε of the form

$$v^\varepsilon = \hat{v}^\varepsilon + \varepsilon\, r^\varepsilon \tag{1.46}$$

is obtained with

$$\|r^\varepsilon\|_{L^\infty(0,T;H^1(\mathbb{R}^N))\cap W^{1,\infty}(0,T;L^2(\mathbb{R}^N))} \le C_T.$$

The function $\hat{v}^\varepsilon$ is explicitly computed through the Fourier decomposition of the initial data ; it is a series indexed by the Fourier modes (cf. Theorems 2.2 and 2.3).

Let us remark that decomposition (1.46) is probably known to the experts in geometrical optics, although we were unable to find a precise statement of resummation over the Fourier modes in the relevant literature (expansions involving a finite number of modes can be found in [1], Ch. 4, p. 549, equation (2.16)). We thus believe that the results of Theorems 2.2 and 2.3 do not duplicate any previously known work.

From the expansion (1.46) we will proceed to recover (c.f. Theorem 2.4) the weak limit d of the energy density d^ε which was our original task.

When ρ and A do depend on x, geometrical optics is a much more delicate tool (well beyond our expertise) and we will follow a totally different path. Our goal will merely be an accurate description of the measure limit d of the energy density d^ε. To this effect we will invoke the theory of H-measures recently introduced by L. Tartar and by P. Gerard (cf. [15]-[17], [5]-[7]) for studying the limits of quadratic products of weakly converging quantities. This is the object of Section 3.

Our specific goal will be a complete characterization of the H-measure associated to the weakly converging sequence $(\partial v^\varepsilon/\partial t, \operatorname{grad} v^\varepsilon)$ (cf. (1.31), (1.35)-(1.37), (1.45)) ; this will be achieved in Theorem 3.4. From the knowledge of the H-measure we will be in a position to compute d (cf. Theorem 3.5). In the case of constant coefficients d will be explicitly computed and the obtained value will be checked to coincide with that derived from the expansion of v^ε in Section 2 (cf. Theorem 2.4, Corollary 3.3 and Remark 3.19).

A last word regarding the boundary conditions. Our purpose in the present study is to understand the impact of the oscillations of the initial data on the field v^ε. We would thus like to avoid a detailed treatment of the effect of the boundary and boundary conditions. In all fairness we also feel somewhat awed at the complexity of that latter task. We will therefore assume that $\Omega = \mathbb{R}^N$ while our initial conditions will be essentially compactly supported.

Remark 1.3. In this setting, the natural energy space becomes $L^\infty(0,T;\mathcal{D}^{1,2}(\mathbb{R}^N)) \cap W^{1,\infty}(0,T;L^2(\mathbb{R}^N))$, for any fixed $T > 0$, where $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is the completion of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ for the Dirichlet norm $\|\operatorname{grad}\varphi\|^2_{L^2(\mathbb{R}^N)}$ (cf. [3]), and for a fixed ε the solution v^ε to (1.35)-(1.37) (with Ω replaced by $\mathbb{R}^N$) satisfies

$$(1.47) \qquad \|\frac{\partial v^\varepsilon}{\partial t}\|_{L^\infty(0,T;L^2(\mathbb{R}^N))} + \|\operatorname{grad} v^\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}^N))} \le \mathcal{C} < +\infty.$$

If however the initial condition $v^\varepsilon(0)$ belongs to $L^2(\mathbb{R}^N)$, the estimate

$$\|v^\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}^N))} \le \|v^\varepsilon(0)\|_{L^2(\mathbb{R}^N)} + T\|\frac{\partial v^\varepsilon}{\partial t}\|_{L^\infty(0,T;L^2(\mathbb{R}^N))}$$

together with (1.47) provides an estimate for $v^\varepsilon(t)$ in $L^\infty(0,T;H^1(\mathbb{R}^N))$, and permits to keep $L^\infty(0,T;H^1(\mathbb{R}^N)) \cap W^{1,\infty}(0,T;L^2(\mathbb{R}^N))$ as energy space.□

Remark 1.4. Note that the solution v^ε to (1.35)-(1.37) (with Ω replaced by $\mathbb{R}^N$) still satisfies

$$v^\varepsilon \overset{\varepsilon\to 0}{\rightharpoonup} 0,$$

weak-$*$ in the energy space (cf. Remark 1.3). □

As a final note of caution, the field v^ε as well as the data associated to the initial values, *i.e.*, α and β, are assumed to be complex-valued while all coefficients, *i.e.*, ρ and A, are real valued. Such a framework will facilitate the subsequent analysis.

2. THE CASE OF CONSTANT COEFFICIENTS: GEOMETRICAL OPTICS.

This section is divided into three subsections. The first subsection is devoted to the monochromatic case whose fundamentals can be found in any work pertaining to geometrical optics (cf. e.g. [1], Ch. 4, Section 2 or [9], Section 7). Subsection 2.2 addresses the actual setting (1.35)-(1.37), (1.45) and essentially aims at summing the expansion obtained in Subsection 2.1 (cf. Theorem 2.1) over all frequencies. This is the object of Theorems 2.2

and 2.3 which only differ as far as the admissible classes of initial data are concerned. In Subsection 2.3 the weak limit d of the energy density d^ε is computed in the settings of both previous theorems (cf. Theorem 2.4).

2.1. The monochromatic case.

In this subsection we specialize the setting (1.35)-(1.37) (with $\Omega = \mathbb{R}^N$), (1.45), to the constant coefficient and monochromatic case. We thus consider

$$\rho \frac{\partial^2 v^\varepsilon}{\partial t^2} - a_{ij} \frac{\partial^2 v^\varepsilon}{\partial x_i \partial x_j} = 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \tag{2.1}$$

$$\begin{cases} v^\varepsilon(0) = \varepsilon\, \alpha(x)\, e^{2i\pi k.x/\varepsilon} & \text{on } \mathbb{R}^N, \\ \dfrac{\partial v^\varepsilon}{\partial t}(0) = \beta(x)\, e^{i2\pi k.x/\varepsilon} & \text{on } \mathbb{R}^N, \end{cases} \tag{2.2}$$

where ρ is strictly positive, and the $N \times N$ matrix A with entries a_{ij} is definite positive ($A\xi\xi \geq \lambda_1|\xi|^2, \xi \in \mathbb{R}^N$). Further we assume that

$$\alpha(x) \quad , \quad \beta(x) \quad \in H^2(\mathbb{R}^N; \mathbb{C}), \tag{2.3}$$

while

$$\beta(x) = 0 \quad \text{if } k = 0. \tag{2.4}$$

(The reader is invited to think of α, β and v^ε as α_k, β_k and v^ε_k, *i.e.*, the k[th] Fourier coefficient in the decomposition of $\alpha(x, y)$ and $\beta(x, y)$ as far as α_k and β_k are concerned, and the k[th] coefficient in the expansion of v^ε as far as v^ε_k is concerned. The k indices have been dropped in Subsection 2.1 for notational convenience.)

Remark 2.1. In the context of Remarks 1.3, 1.4 the natural energy space is now $L^\infty(\mathbb{R}_+; H^1(\mathbb{R}^N)) \cap W^{1,\infty}(\mathbb{R}_+; L^2(\mathbb{R}^N))$ and v^ε satisfies

$$v^\varepsilon \overset{\varepsilon \to 0}{\rightharpoonup} 0,$$

weakly-$*$ in that space as ε tends to zero. Note that these results remain valid although α and β do not have compact support. □

The geometrical optics treatment is classical (cf. e.g. [1], Ch. 4 or [9], Section 7). A solution of (2.1)-(2.2) of the form

$$v^\varepsilon(x,t) = \varepsilon\, e^{i\,S(x,t)/\varepsilon}\, v(x,t) + \varepsilon\, r^\varepsilon(x,t) \quad , \quad x \in \mathbb{R}^N \quad , \quad t \geq 0, \tag{2.5}$$

is being sought.

Remark 2.2. The actual expression for v^ε will be of the form

$$v^\varepsilon(x,t) = \varepsilon \sum_{\pm} e^{i\,S^\pm(x,t)/\varepsilon}\, v^\pm(x,t) + \varepsilon\, r^\varepsilon(x,t), \tag{2.6}$$

but the $\pm$ dependence will only be introduced when needed. (*) □

We propose to insert ansatz (2.5) into (2.1)-(2.2). To this effect we will repeatedly need the following derivatives:

$$\begin{cases} \dfrac{\partial}{\partial t}(\varepsilon\, e^{i\,S/\varepsilon} v) = (\varepsilon\, \dfrac{\partial v}{\partial t} + i\, \dfrac{\partial S}{\partial t}\, v) e^{i\,S/\varepsilon}, \\ \dfrac{\partial}{\partial x_m}(\varepsilon\, e^{i\,S/\varepsilon} v) = (\varepsilon\, \dfrac{\partial v}{\partial x_m} + i\, \dfrac{\partial S}{\partial x_m}\, v) e^{i\,S/\varepsilon} \quad , \quad 1 \leq m \leq N. \end{cases}$$

Equation (2.1) becomes

$$\begin{aligned} &-\varepsilon^{-1}[\rho(\frac{\partial S}{\partial t})^2 - a_{ij}\, \frac{\partial S}{\partial x_i}\, \frac{\partial S}{\partial x_j}] e^{i\,S/\varepsilon}\, v \\ &+ \varepsilon^0[(\rho\, \frac{\partial^2 S}{\partial t^2} - a_{ij}\, \frac{\partial^2 S}{\partial x_i \partial x_j}) v + 2(\rho\, \frac{\partial S}{\partial t}\, \frac{\partial v}{\partial t} - a_{ij}\, \frac{\partial S}{\partial x_i}\, \frac{\partial v}{\partial x_j}] i\, e^{iS/\varepsilon} \\ &+ \varepsilon[(\rho\, \frac{\partial^2 v}{\partial t^2} - a_{ij}\, \frac{\partial v}{\partial x_i}\, \frac{\partial v}{\partial x_j}) e^{iS/\varepsilon} + (\rho\, \frac{\partial^2 r^\varepsilon}{\partial t^2} - a_{ij}\, \frac{\partial^2 r^\varepsilon}{\partial x_i \partial x_j})] = 0, \end{aligned} \tag{2.7}$$

while the initial conditions (2.2) read as

(*) Throughout the text, the symbol $\sum_\pm$ implies that the contributions of the "+ terms" and of the "− terms" are to be added to each other, *i.e.*, $\sum_\pm a_\pm = a_+ + a_-, \sum_\pm \pm a_\pm = a_+ - a_-$.

$$\tag{2.8} \begin{cases} \alpha(x)e^{2i\pi k.x/\varepsilon} = e^{iS(x,0)/\varepsilon}\, v(x,0) + r^\varepsilon(x,0), \\ \beta(x)e^{2i\pi k.x/\varepsilon} = \Big(i\,\dfrac{\partial S}{\partial t}(x,0) + \varepsilon\,\dfrac{\partial v}{\partial t}(x,0)\Big)e^{iS(x,0)/\varepsilon} + \varepsilon\,\dfrac{\partial r^\varepsilon}{\partial t}(x,0). \end{cases}$$

The terms of order $\varepsilon^{-1}, \varepsilon^0$ and ε are successively set to zero in equality (2.7) yielding

$$\tag{2.9} \rho\Big(\frac{\partial S}{\partial t}\Big)^2 - a_{ij}\,\frac{\partial S}{\partial x_i}\,\frac{\partial S}{\partial x_j} = 0,$$

$$\tag{2.10} \rho\,\frac{\partial S}{\partial t}\,\frac{\partial v}{\partial t} - a_{ij}\,\frac{\partial S}{\partial x_i}\,\frac{\partial v}{\partial x_j} = -\frac{1}{2}\Big(\rho\,\frac{\partial^2 S}{\partial t^2} - a_{ij}\,\frac{\partial^2 S}{\partial x_i \partial x_j}\Big)v,$$

$$\tag{2.11} \rho\,\frac{\partial^2 r^\varepsilon}{\partial t^2} - a_{ij}\,\frac{\partial^2 r^\varepsilon}{\partial x_i \partial x_j} = -\Big(\rho\,\frac{\partial^2 v}{\partial t^2} - a_{ij}\frac{\partial v}{\partial x_i}\,\frac{\partial v}{\partial x_j}\Big)e^{iS/\varepsilon}.$$

Equation (2.9) reduces to

$$\tag{2.12} \rho^{1/2}\,\frac{\partial S}{\partial t} \pm \sqrt{a_{ij}\,\frac{\partial S}{\partial x_i}\,\frac{\partial S}{\partial x_j}} = 0,$$

which is precisely the eiconal equation. In view of the first equation of (2.8) a natural choice for the initial condition on S is

$$\tag{2.13} S(x,0) = 2\pi k \cdot x.$$

The solution to (2.12), (2.13) is

$$\tag{2.14} S^{\pm}(x,t) = 2\pi[k \cdot x \mp \Big(\frac{Ak \cdot k}{\rho}\Big)^{1/2} t].$$

The phases S^+ and S^- correspond to a plane wave situation.

Remark 2.3. In the case where ρ, A depend on x, an ansatz similar to (2.5) may be proposed. The eiconal equation then becomes

$$\tag{2.15} \frac{\partial S}{\partial t} + \omega(x,S) = 0,$$

where

$$\omega(x,k) = \pm\Big(\frac{A(x)k\cdot k}{\rho(x)}\Big)^{1/2},$$

with the same initial condition, namely (2.13).

The solution to (2.13)-(2.15) is well known (cf. e.g. [1], Ch. 4, Section 2) to be (locally) the transported value of (2.13) along the rays which are in turn the projections in space (in x) of the bicharacteristics defined as the integral curves of the following Hamiltonian system:

$$\begin{cases} \dfrac{d\overline{x}_p}{dt} = \dfrac{\partial\omega}{\partial k_p}(\overline{x},\overline{k}) \quad , \quad \overline{x}_p(0) = x_p, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 \le p \le N, \\ \dfrac{d\overline{k}_p}{dt} = -\dfrac{\partial\omega}{\partial x_p}(\overline{x},\overline{k}) \quad , \quad \overline{k}_p(0) = 2\pi k_p, \end{cases}$$

The bicharacteristics are well defined for all times as can be readily verified once it is remarked that

$$\omega(\overline{x}(t),\overline{k}(t)) = \omega(\overline{x}(0),\overline{k}(0)) \quad , \quad t \ge 0.$$

Their projections in physical space might however intersect at a point (x,t) ; at such a point $S(x,t)$ is not well defined as a transported value along the rays and the geometrical optics ansatz becomes more delicate to handle. The reader is referred to Remark 2.7, as well as to the comments at the end of Paragraphe 3.2.1. □

The existence of two different phase functions S^+ and S^- (cf. (2.14)) prompt us to consider a solution v^ε of the form (2.6) as anounced in Remark 2.2. Then (2.8) has to be changed into

$$\begin{cases} \alpha(x)\, e^{2i\pi k\cdot x/\varepsilon} = (v^+(x,0) + v^-(x,0))e^{2i\pi k\cdot x/\varepsilon} + r^\varepsilon(x,0), \\ \beta(x)\, e^{2i\pi k\cdot x/\varepsilon} = [-2i\pi(\dfrac{Ak\cdot k}{\rho})^{1/2}(v^+(x,0) - v^-(x,0)) \\ \qquad\qquad + \varepsilon(\dfrac{\partial v^+}{\partial t}(x,0) + \dfrac{\partial v^-}{\partial t}(x,0))]e^{2i\pi k\cdot x/\varepsilon} + \varepsilon\,\dfrac{\partial r^\varepsilon}{\partial t}(x,0). \end{cases}$$

Upon defining

$$V^\pm(x) \overset{\text{def}}{\equiv} \frac{1}{2}\{\alpha(x) \pm \frac{i}{2\pi}\Big(\frac{\rho}{Ak\cdot k}\Big)^{1/2}\beta(x)\}, \tag{2.16}$$

we are thus led to set

$$\begin{cases} v^+(x,0) = V^+(x), \\ v^-(x,0) = V^-(x), \end{cases} \tag{2.17}$$

$$\begin{cases} r^\varepsilon(x,0) = 0, \\ \dfrac{\partial r^\varepsilon}{\partial t}(x,0) = -(\dfrac{\partial v^+}{\partial t}(x,0) + \dfrac{\partial v^-}{\partial t}(x,0))e^{2i\pi k\cdot x/\varepsilon}, \end{cases} \tag{2.18}$$

and (2.8) is identically satisfied.

Equation (2.10) becomes

$$\frac{\partial v^\pm}{\partial t} \pm \frac{Ak}{(\rho Ak\cdot k)^{1/2}} \operatorname{grad} v^\pm = 0, \tag{2.19}$$

which are transport equations for the fields v^+ and v^-. In view of (2.17), the solutions to (2.19) are

$$v^\pm(x,t) = V^\pm\Big(x \mp \frac{Ak}{(\rho\, Ak\cdot k)^{1/2}}t\Big). \tag{2.20}$$

Recalling (2.6), (2.14), (2.20) we have thus established that system (2.1)-(2.2) is solved by v^ε defined upon setting

$$\hat{v}^\varepsilon(x,t) = \varepsilon \sum_{\pm} e^{2i\pi[k\cdot x \mp (\frac{Ak\cdot k}{\rho})^{1/2}t]/\varepsilon} V^\pm\Big(x \mp \frac{Ak}{(\rho\, Ak\cdot k)^{1/2}}t\Big), \tag{2.21}$$

and

$$v^\varepsilon(x,t) = \hat{v}^\varepsilon(x,t) + \varepsilon\, r^\varepsilon(x,t). \tag{2.22}$$

In (2.21) V^+ and V^- are given by (2.16) and in (2.22) r^ε satisfies

$$\rho \frac{\partial^2 r^\varepsilon}{\partial t^2} - a_{ij} \frac{\partial^2 r^\varepsilon}{\partial x_i \partial x_j} = -R^\varepsilon, \tag{2.23}$$

$$\begin{cases} r^\varepsilon(x,0) = 0, \\ \dfrac{\partial r^\varepsilon}{\partial t}(x,0) = T^\varepsilon, \end{cases} \tag{2.24}$$

with

$$(2.25)\qquad \begin{cases} R^\varepsilon(x,t) = \sum_{\pm} e^{2i\pi[k\cdot x \mp (\frac{Ak\cdot k}{\rho})^{1/2} t]/\varepsilon}\Big(\frac{(Ak)_i\,(Ak)_j}{Ak\cdot k}\Big) - a_{ij}\Big) \\ \qquad\qquad \frac{\partial^2 V^\pm}{\partial x_i \partial x_j}\Big(x \mp \frac{Ak}{(\rho\, Ak\cdot k)^{1/2}} t\Big), \\ T^\varepsilon(x) = \sum_{\pm} \pm e^{2i\pi\, k\cdot x/\varepsilon}\, \frac{Ak\cdot \operatorname{grad} V^\pm(x)}{(\rho\, Ak\cdot k)^{1/2}}. \end{cases}$$

Remark 2.4. The following expressions for the derivatives of $\hat{v}^\varepsilon$ will be helpful at various stages of Section 2:

$$\frac{\partial \hat{v}^\varepsilon}{\partial t} = \sum_{\pm} e^{iS^\pm(x,t)/\varepsilon}\{\mp 2i\pi\Big(\frac{Ak\cdot k}{\rho}\Big)^{1/2} V^\pm$$

$$\mp\varepsilon\frac{Ak}{(\rho\, Ak\cdot k)^{1/2}}\operatorname{grad} V^\pm\}\Big(x \mp \frac{Ak\, t}{(\rho\, Ak\cdot k)^{1/2}}\Big),$$

$$\operatorname{grad}\hat{v}^\varepsilon = \sum_{\pm} e^{iS^\pm(x,t)/\varepsilon}\{2i\pi k\, V^\pm + \varepsilon \operatorname{grad} V^\pm\}\Big(x \mp \frac{Ak\, t}{(\rho\, Ak\cdot k)^{1/2}}\Big).$$

□

We now propose to derive energy estimates on $\hat{v}^\varepsilon$ and r^ε. The computation of the energy norm of $\hat{v}^\varepsilon$ is straightforward. Recalling Remark 2.4 we have, for any $T > 0$,

$$(2.26)\qquad \begin{aligned} &\Big\|\frac{\partial \hat{v}^\varepsilon}{\partial t}\Big\|_{L^\infty(0,T;L^2(\mathbb{R}^N))} + \|\operatorname{grad}\hat{v}^\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}^N))} \\ &\quad \le \mathcal{C}\{|k|\,\|V^\pm\|_{L^2(\mathbb{R}^N)} + \varepsilon\,\|\nabla V^\pm\|_{L^2(\mathbb{R}^N)}\}, \end{aligned}$$

where $\mathcal{C}$ is a constant that does not depend on T or k.

Define ω as

$$\begin{aligned} \omega =& (|k|\,\|\alpha\|_{L^2(\mathbb{R}^N)} + \|\operatorname{grad}\alpha\|_{L^2(\mathbb{R}^N)}) \\ &+ (\|\beta\|_{L^2(\mathbb{R}^N)} + \frac{1}{|k|}\|\operatorname{grad}\beta\|_{L^2(\mathbb{R}^N)}) \quad , \quad k \neq 0, \\ \omega =& (|k|\,\|\alpha\|_{L^2(\mathbb{R}^N)} + \|\operatorname{grad}\alpha\|_{L^2(\mathbb{R}^N)}) \quad , \quad k = 0. \end{aligned}$$

(The reader should think of ω as ω_k associated to k, α_k, β_k). Then, in view of (2.16), (2.26) is bounded above by $C\omega$ where C only depends on ρ, and A, *i.e.*,

$$\|\frac{\partial \hat{v}^\varepsilon}{\partial t}\|_{L^\infty(0,T;L^2(\mathbb{R}^N))} + \|\mathrm{grad}\hat{v}^\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}^N))} \leq C\,\omega. \tag{2.27}$$

The energy estimate on r^ε is obtained through multiplication of (2.23) by $\partial r^\varepsilon/\partial t$. For any $T > 0$ and any $0 \leq t \leq T$, the usual steps lead to

$$\begin{aligned}
&\int_{\mathbb{R}^N} (\rho\Big|\frac{\partial r^\varepsilon}{\partial t}\Big|^2 + A\,\mathrm{grad}\,r^\varepsilon \cdot \mathrm{grad}\,r^\varepsilon)(x,t)dx \\
&= \int_{\mathbb{R}^N} \rho|T^\varepsilon(x)|^2 dx - \int_0^t \int_{\mathbb{R}^N} (R^\varepsilon \frac{\partial r^\varepsilon}{\partial t}\,(x,s)dx\,ds \\
&\leq \rho\|T^\varepsilon\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{2\rho}\|R^\varepsilon\|^2_{L^2(0,T;L^2(\mathbb{R}^N))} + \frac{1}{2}\int_0^t \int_{\mathbb{R}^N} \rho\Big|\frac{\partial r^\varepsilon}{\partial t}\Big|^2(x,s)dx\,ds
\end{aligned}$$

for $0 \leq t \leq T$.

Gronwall's lemma implies in turn that

$$\begin{aligned}
&\rho\|\frac{\partial r^\varepsilon}{\partial t}\|^2_{L^\infty(0,T;L^2(\mathbb{R}^N))} + \lambda_1\|\mathrm{grad}\,r^\varepsilon\|^2_{L^\infty(0,T;L^2(\mathbb{R}^N))} \\
&\qquad \leq \{\rho\|T^\varepsilon\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{2\rho}\,\|R^\varepsilon\|^2_{L^2(0,T;L^2(\mathbb{R}^N))}\}e^{T/2}.
\end{aligned} \tag{2.28}$$

The right-hand side of (2.28) can be bounded above, with the help of (2.25), by

$$C\,e^{T/2}\{\|\mathrm{grad}\,V^\pm\|^2_{L^2(\mathbb{R}^N)} + T\|\nabla^2 V^\pm\|^2_{L^2(\mathbb{R}^N)}\}, \tag{2.29}$$

where $\nabla^2 V^\pm$ denotes the Hessian matrix of $V^\pm$. Note that the constant C depends on ρ, A, but not on T or k.

Define ω' as

$$\begin{aligned}
\omega' =& \Big(\|\mathrm{grad}\,\alpha\|_{L^2(\mathbb{R}^N)} + \|\nabla^2\alpha\|_{L^2(\mathbb{R}^N)}\Big) \\
&+ \frac{1}{|k|}\Big(\|\mathrm{grad}\,\beta\|_{L^2(\mathbb{R}^N)} + \|\nabla^2\beta\|_{L^2(\mathbb{R}^N)}\Big)\,,\ k \neq 0, \\
\omega' =& (\|\mathrm{grad}\,\alpha\|_{L^2(\mathbb{R}^N)} + \|\nabla^2\alpha\|_{L^2(\mathbb{R}^N)})\,,\ k = 0,
\end{aligned}$$

(The reader should think of ω' as ω'_k associated to k, α_k, β_k). Then, in view of (2.16), (2.29) is bounded above by $C'e^T\omega'^2$ where C' only depends on ρ and A (Note that $CTe^{T/2} \leq C'e^T$). Recalling (2.28) finally yields

$$\|\frac{\partial r^\varepsilon}{\partial t}\|_{L^\infty(0,T;L^2(\mathbb{R}^N))} + \|\operatorname{grad} r^\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}^N))} \leq C'e^{T/2}\omega', \tag{2.30}$$

where again C only depends on ρ, A (and λ_1).

Expansion (2.21) together with estimates (2.27), (2.30) provide a complete description (of order ε in the energy) of the solution v^ε to (2.1)-(2.2). Specifically we have proved the following theorem where all k indices are reintroduced:

Theorem 2.1. *The solution v_k^ε of*

$$\rho\frac{\partial^2 v_k^\varepsilon}{\partial t} - a_{ij}\frac{\partial^2 v_k^\varepsilon}{\partial x_i \partial x_j} = 0 \quad in\, \mathbb{R}^N \times \mathbb{R}_+, \tag{2.31}$$

$$\begin{cases} v_k^\varepsilon(0) = \varepsilon\, \alpha_k(x)\, e^{2i\pi k\cdot x/\varepsilon} & on\, \mathbb{R}^N, \\ \dfrac{\partial v_k^\varepsilon}{\partial t}(0) = \beta_k(x)\, e^{2i\pi k\cdot x/\varepsilon} & on\, \mathbb{R}^N, \end{cases} \tag{2.32}$$

with

$$\begin{cases} \alpha_k(x)\,,\ \beta_k(x) \in H^2(\mathbb{R}^N;\mathbb{C}), \\ \beta_k(x) = 0 \quad if\, k = 0, \end{cases} \tag{2.33}$$

is given by

$$v_k^\varepsilon(x,t) = \hat{v}_k^\varepsilon(x,t) + \varepsilon\, r_k^\varepsilon(x,t), \tag{2.34}$$

with

$$\hat{v}_k^\varepsilon(x,t) = \varepsilon \sum_{\pm} e^{2i\pi[k\cdot x \mp (\frac{Ak\cdot k}{\rho})^{1/2}t]/\varepsilon} V_k^\pm\Big(x \mp (\frac{Ak}{(\rho\, Ak\cdot k)^{1/2}}t\Big), \tag{2.35}$$

where

$$V_k^\pm(x) = \frac{1}{2}\{\alpha_k(x) \pm \frac{i}{2\pi}(\frac{\rho}{Ak\cdot k})^{1/2}\beta_k(x)\}. \tag{2.36}$$

Further $\hat{v}_k^\varepsilon$ and r_k^ε satisfy, for any $0 \leq T < +\infty$,

$$\|\hat{v}_k^\varepsilon\|_{L^\infty(0,T;H^1(\mathbb{R}^N))\cap W^{1,\infty}(0,T;L^2(\mathbb{R}^N))} \le C\,\omega_k, \tag{2.37}$$

$$\|r_k^\varepsilon\|_{L^\infty(0,T;H^1(\mathbb{R}^N))\cap W^{1,\infty}(0,T;L^2(\mathbb{R}^N))} \le C'\,e^{T/2}\,\omega_k', \tag{2.38}$$

with

$$\begin{cases} \omega_k = (|k|\,\|\alpha_k\|_{L^2(\mathbb{R}^N)} + \|\mathrm{grad}\,\alpha_k\|_{L^2(\mathbb{R}^N)}) \\ \qquad + \begin{cases} 0 & \text{if } k = 0, \\ \|\beta_k\|_{L^2(\mathbb{R}^N)} + \dfrac{1}{|k|}\,\|\mathrm{grad}\,\beta_k\|_{L^2(\mathbb{R}^N)} & \text{if } k \neq 0, \end{cases} \\ \omega_k' = (\|\mathrm{grad}\,\alpha_k\|_{L^2(\mathbb{R}^N)} + \|\nabla^2\alpha_k\|_{L^2(\mathbb{R}^N)}) \\ \qquad + \begin{cases} 0 & \text{if } k = 0, \\ \dfrac{1}{|k|}\,(\|\mathrm{grad}\,\beta_k\|_{L^2(\mathbb{R}^N)} + \|\nabla^2\beta_k\|_{L^2(\mathbb{R}^N)}) & \text{if } k \neq 0, \end{cases} \end{cases} \tag{2.39}$$

and C, C' are constants which do not depend on T or k. □

Define, for any w in $L^\infty(0,T;H^1(\mathbb{R}^N)) \cap W^{1,\infty}(0,T;L^2(\mathbb{R}^N))$,

$$d(w) = \frac{1}{2}(\rho|\frac{\partial w}{\partial t}|^2 + A\,\mathrm{grad}\,w\,\overline{\mathrm{grad}\,w}). \tag{2.40}$$

The weak limit (in ε) of $d(v_k^\varepsilon) = d^\varepsilon$ can be immediately computed with the help of expansion (2.34) and of estimates (2.37)-(2.38). We obtain the following

Corollary 2.1. *In the context of Theorem 2.1, the sequence of energy densities $d(v_k^\varepsilon)$ (cf. (2.40)) converges weak-$*$ in $L^\infty(0,T;\mathcal{M}(\mathbb{R}^N))$ $(0 < T < +\infty)$ to*

(2.41)

$$d_k = \frac{1}{4}\sum_{\pm}\Big\{\rho|\beta_k\Big(x \pm \frac{Ak\,t}{(\rho\,Ak\cdot k)^{1/2}}\Big)|^2 + 4\pi^2(Ak\cdot k)|\alpha_k\Big(x \pm \frac{Ak\,t}{\rho\,Ak\cdot k)^{1/2}}\Big)|^2$$
$$\pm\, 4\pi(\rho\,Ak\cdot k)^{1/2} Im\Big[\beta_k\Big(x \pm \frac{Ak\,t}{(\rho\,Ak\cdot k)^{1/2}}\Big)\overline{\alpha}_k\Big(x \pm \frac{Ak\,t}{(\rho\,Ak\cdot k)^{1/2}}\Big)\Big]\Big\}.$$

Proof of Corollary 2.1. In view of (2.34) we have

$$d(v_k^\varepsilon) = d(\hat{v}_k^\varepsilon) + \rho_k^\varepsilon,$$

where

$$\rho_k^\varepsilon = \varepsilon^2\, d(r_k^\varepsilon) + 2\varepsilon Re\{\rho \frac{\partial r_k^\varepsilon}{\partial t}\,\overline{\frac{\partial \hat{v}_k^\varepsilon}{\partial t}} + A\,\mathrm{grad}\, r_k^\varepsilon\, \overline{\mathrm{grad}\,\hat{v}_k^\varepsilon}\}.$$

The term ρ_k^ε does not contribute to the limit energy density since, by virtue of (2.37)-(2.38),

$$\text{(2.42)}\qquad \int_0^T \int_{\mathbb{R}^N} |\rho_k^\varepsilon| dx\, dt \le CT\{\varepsilon^2 \|r_k^\varepsilon\|^2_{en(T)} + \varepsilon \|r_k^\varepsilon\|_{en(T)} \|v_k^\varepsilon\|_{en(T)}\} \le CT\{\varepsilon^2\, e^T (\omega_k')^2 + \varepsilon\, e^{T/2}\, \omega_k\, \omega_k'\},$$

(*) where C is a constant that does not depend on T or k. Because of estimate (2.42) it suffices to compute the limit of $d(\hat{v}_k^\varepsilon)$ which is a sum of terms of the form (cf. Remark 2.4)

(2.43)

$$d(\alpha, k, \pm, (\pm)') = \varepsilon^\alpha\, e^{i\, S_k^\pm(x,t)/\varepsilon}\, e^{-i\, S_k^{(\pm)'}(x,t)/\varepsilon}\, \psi_{k,\pm,(\pm)',\alpha}(x,t)\ ,\ 0 \le \alpha \le 2,$$

where $\psi_{k,\pm,(\pm)',\alpha}(x,t)$ does not depend on ε and belongs to $L^1(0, T \times \mathbb{R}^N)$. If $(\pm) \ne (\pm)'$ a convenient change of variable transforms, for any φ in $C_0^\infty((0,T) \times \mathbb{R}^N)$,

$$\int_0^T \int_{\mathbb{R}^N} d(\alpha, k, , \pm, (\pm)')(x,t) \varphi(x,t) dx\, dt$$

into

$$\int_0^T \int_{\mathbb{R}^N} \varepsilon^\alpha\, e^{i\, y_1/\varepsilon}\, e^{-i\, y_2/\varepsilon}\, \tilde{\psi}(y_1, y_2, \dots, y_{n+1}) \tilde{\varphi}(y_1, \dots, y_{N+1}) dy,$$

which converges to zero as ε tends to zero, *even if* $\alpha = 0$. Thus the only terms that will contribute to the weak limit of $d(\hat{v}_k^\varepsilon)$ will be the square terms associated with $\alpha = 0$. Upon simple inspection of the expressions for $\partial \hat{v}_k^\varepsilon / \partial t$ and grad $\hat{v}^\varepsilon$ in Remark 2.4 we thus obtain

$$d_k(x,t) = 4\pi^2 (Ak \cdot k) \sum_{\pm} \left| V^\pm (x \mp \frac{Ak\, t}{(\rho\, Ak \cdot k)^{1/2}}) \right|^2,$$

(*) The notation $\|\ \|_{en(T)}$ will be used throughout the text as a shorthand notation for $L^\infty(0, T; H^1(\mathbb{R}^N)) \cap W^{1,\infty}(0, T; L^2(\mathbb{R}^N))$.

which is precisely (2.41) once $V^{\pm}$ have been replaced by their expressions (2.36). □

2.2. The polychromatic case.

In this subsection we return to the actual setting of our problem which involves the entire Fourier spectrum of α and β (cf. (1.35)-(1.37), (1.45)). If all Fourier frequencies are excited, that is if the initial conditions α^ε and β^ε are of the form

$$\left\{\begin{aligned} \alpha^\varepsilon(x) &= \sum_{k\in Z^N} \alpha_k(x)\, e^{2i\pi k\cdot x/\varepsilon}, \\ \beta^\varepsilon(x) &= \sum_{k\in Z^N-\{0\}} \beta_k(x)\, e^{2i\pi k\cdot x/\varepsilon}, \end{aligned}\right. \tag{2.44}$$

then the ansatz proposed in Theorem 2.1 will become

$$v^\varepsilon(x,t) = \hat{v}^\varepsilon(x,t) + \varepsilon\, r^\varepsilon(x,t),$$

with

$$\left\{\begin{aligned} \hat{v}^\varepsilon_{(x,t)} &= \sum_{k\in Z^N} \hat{v}^\varepsilon_k(x,t) \\ &= \varepsilon \sum_{k\in Z^N} \Big\{\sum_{\pm} e^{2i\pi[k\cdot x \mp (\frac{Ak\cdot k}{\rho})^{1/2} t]/\varepsilon} V_k^{\pm}\big(x \mp \frac{Ak\, t}{(\rho\, Ak\cdot k)^{1/2}}\big)\Big\}, \\ r^\varepsilon_{(x,t)} &= \sum_{k\in Z^N} r^\varepsilon_k(x,t), \end{aligned}\right.$$

which is for now a formal expression because it has yet to be proved that all relevant series converge. This latter task is however rendered obvious with the estimates obtained in Theorem 2.1. In order for $\hat{v}^\varepsilon$ to be well defined it is sufficient, by virtue of (2.37), to assume that

$$\sum_{k\in Z^N} \omega_k < +\infty, \tag{2.45}$$

which will make the series $\sum_{k\in Z^N} \hat{v}^\varepsilon_k$ absolutely convergent in the energy norm. By the same token r^ε will be well defined if

(2.46) $$\sum_{k\in Z^N} \omega'_k < +\infty,$$

which will make $\varepsilon\, r^\varepsilon$ a remainder of order ε in the energy norm.

Note that in view of (2.39), (2.45) implies that α^ε defined in the first equation of (2.44) is well defined as an absolutely converging series in $H^1(\mathbb{R}^N)$ while β^ε defined in the second equation of (2.44) is well defined as an absolutely converging series in $L^2(\mathbb{R}^N)$.

These considerations prompt us to state the following theorem whose proof is contained in the preceeding discussion:

THEOREM 2.2. *Let α_k, β_k satisfy, for all k in Z^N,*

(2.47) $$\alpha_k(x,), \beta_k(x) \in H^2(\mathbb{R}^N; \mathbb{C})\ ,\ \beta_k(x) = 0 \text{ if } k = 0,$$

(2.48) $$\sum_{k\in Z^N} \omega_k \overset{\text{def}}{\equiv} \sigma < +\infty \quad , \quad \sum_{k\in Z^N} \omega'_k \overset{\text{def}}{\equiv} \sigma' < +\infty,$$

where ω_k and ω'_k are defined in (2.39). Consider v^ε the solution to

(2.49) $$\rho\frac{\partial^2 v^\varepsilon}{\partial t^2} - a_{ij}\frac{\partial^2 v^\varepsilon}{\partial x_i \partial x_j} = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^N,$$

(2.50) $$\begin{cases} v^\varepsilon(0) = \varepsilon\, \alpha^\varepsilon(x) & \text{on } \mathbb{R}^N, \\ \dfrac{\partial v^\varepsilon}{\partial t}(0) = \beta^\varepsilon(x) & \text{on } \mathbb{R}^N, \end{cases}$$

where

(2.51) $$\begin{cases} \alpha^\varepsilon = \displaystyle\sum_{k\in Z^N} \alpha_k(x)\, e^{2i\pi k\cdot x/\varepsilon}, \\ \beta^\varepsilon = \displaystyle\sum_{k\in Z^N-\{0\}} \beta_k(x)\, e^{2i\pi k\cdot x/\varepsilon}. \end{cases}$$

Then α^ε and β^ε are well defined at least as elements of $H^1(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$ respectively and v^ε converges weak-$*$ to zero in $L^\infty(0,T;H^1(\mathbb{R}^N)) \cap W^{1,\infty}(0,T;L^2(\mathbb{R}^N))$ for any $0 < T < +\infty$. Furthermore, (2.52) $$v^\varepsilon_{(x,t)} = \hat{v}^\varepsilon_{(x,t)} + \varepsilon\, r^\varepsilon(x,t),$$

with

(2.53) $$\hat{v}^\varepsilon(x,t) = \varepsilon \sum_{k\in Z^N} \Big\{ \sum_{\pm} e^{2i\pi[k\cdot x \mp (\frac{Ak\cdot k}{\rho})^{1/2} t]/\varepsilon} V_k^{\pm}(x \mp \frac{Ak\,t}{(\rho\, Ak\cdot k)^{1/2}}) \Big\},$$

where

(2.54) $$V_k^{\pm}(x) = \frac{1}{2}\{\alpha_k(x) \pm \frac{i}{2\pi}(\frac{\rho}{Ak\cdot k})^{1/2}\beta_k(x)\}.$$

Finally

(2.55) $$\begin{cases} \|\hat{v}^\varepsilon\|_{L^\infty(0,T;H^1(\mathbb{R}^N))\cap W^{1,\infty}(0,T;L^2(\mathbb{R}^N))} \le \mathcal{C}\,\sigma, \\ \|r^\varepsilon\|_{L^\infty(0,T;H^1(\mathbb{R}^N))\cap W^{1,\infty}(0,T;L^2(\mathbb{R}^N))} \le \mathcal{C}'\, e^{T/2}\,\sigma'. \end{cases}$$

Remark 2.5. The reader will not fail to notice that we never defined α^ε or β^ε as $\alpha(x, x/\varepsilon)$ or $\beta(x, x/\varepsilon)$ but rather as explicit (absolutely converging) series (cf. (2.51)). If $\alpha(x,y)$ and $\beta(x,y)$ are smooth functions on $\mathbb{R}^N \times \mathcal{T}$ ($\mathcal{T}$ is the unit torus) then upon setting

$$\begin{cases} \alpha(x,y) = \sum_{k\in Z^N} \alpha_k(x)\, e^{2i\pi k\cdot y}, \\ \beta(x,y) = \sum_{k\in Z^N-\{0\}} \beta_k(x)\, e^{2i\pi k\cdot y}, \end{cases}$$

it is immediately seen that

(2.56) $$\begin{cases} \alpha^\varepsilon(x) = \alpha(x, \frac{x}{\varepsilon}), \\ \beta^\varepsilon(x) = \beta(x, \frac{x}{\varepsilon}), \end{cases}$$

and this remark will give rise to Theorem 2.3.

If however $\alpha(x,y)$ and $\beta(x,y)$ are less regular (not continuous) then the definition of $\alpha(x, x/\varepsilon)$ or $\beta(x, x/\varepsilon)$ becomes technical or even impossible whereas one is always at liberty to consider α^ε and β^ε defined through (2.51). □

Remark 2.6. In a context similar to that of the previous remark sufficient conditions for $\alpha(x,y)$ and $\beta(x,y)$ to be such that their (x-dependent) Fourier coefficients satisfy (2.48) are easily derived. Specifically consider $s \in \mathbb{R}$; then any element f in $L^2(\mathbb{R}_x^N; H_y^s(\mathcal{T}))$ expands as the following absolutely converging sequence in $L^2(\mathbb{R}_x^N; H_y^s(\mathcal{T}))$:

$$f = \sum_{k \in Z^N} f_k(x) e^{2i\pi k \cdot y},$$

with

$$\sum_{k \in Z^N} \|f_k\|^2_{L^2(\mathbb{R}^N)} (1 + |k|^{2s}) < +\infty.$$

Thus, if $s > 3/2$, then for example

$$\sum_{k \in Z^N} |k| \|f_k\|_{L^2(\mathbb{R}^N)} = \Big(\sum_{k \in Z^N - \{0\}} |k|^{2s} \|f_k\|^2_{L^2(\mathbb{R}^N)} \Big)^{1/2} \Big(\sum_{k \in Z^N - \{0\}} |k|^{2-2s} \Big)^{1/2}$$
$$\leq \mathcal{C}_s \|f\|_{L^2(\mathbb{R}^N; H^s(\mathcal{T}))},$$

where $\mathcal{C}_s \stackrel{\text{def}}{\equiv} \left(\sum_{k \in Z^N - \{0\}} |k|^{2-2s} \right)^{1/2}$.

In view of (2.39), (2.48) is satisfied if and only if

$$\text{(2.57)} \quad \sum_{k \in Z^N} \Big(|k| \, \|\alpha_k\|_{L^2(\mathbb{R}^N)} + \|\text{grad}\, \alpha_k\|_{L^2(\mathbb{R}^N)} + \|\nabla^2 \alpha_k\|_{L^2(\mathbb{R}^N)} \Big) < +\infty,$$

(2.58)
$$\sum_{k \in Z^N - \{0\}} \Big(\|\beta_k\|_{L^2(\mathbb{R}^N)} + \frac{1}{|k|} \|\text{grad}\, \beta_k\|_{L^2(\mathbb{R}^N)} + \frac{1}{|k|} \|\nabla^2 \beta_k\|_{L^2(\mathbb{R}^N)} \Big) < +\infty.$$

But, by virtue of the above remarks, (2.57) is satisfied if

$$\text{(2.59)} \qquad \alpha \in L^2(\mathbb{R}_x^N; L^2(\mathcal{T})) \cap H^2(\mathbb{R}_x^N; L^2(\mathcal{T})) \,, \; s > 3/2,$$

while (2.58) is satisfied if

$$\text{(2.60)} \qquad \begin{cases} \beta \in L^2(\mathbb{R}_x^N; L^2(\mathcal{T})) \cap H^2(\mathbb{R}_x^N; H_y^s(\mathcal{T})) \,, \; s > -1/2, \\ \displaystyle\int_{\mathcal{T}} \beta(x,y) dy = 0. \end{cases}$$

Note also that (2.47) is then satisfied.

The conclusions of Theorem 2.1 thus remain valid when (2.47)-(2.48) are replaced by (2.59)-(2.60). □

By virtue of Remarks 2.5 and 2.6 we conclude that if $\alpha(x,y)$ and $\beta(x,y)$ are elements of $\mathcal{C}_0^\infty(\mathbb{R}^N \times \mathcal{T})$ with $\int_{\mathcal{T}} \beta(x,y)dy = 0$ then (2.56) holds true and (2.59)-(2.60) (hence (2.47)-(2.48)) are satisfied, and we obtain the following

THEOREM 2.3. *Let α and β be elements of $\mathcal{C}_0^\infty(\mathbb{R}^N \times \mathcal{T})$ such that*

$$\int_{\mathcal{T}} \beta(x,y)dy = 0. \tag{2.61}$$

Then the solution v^ε to

$$\rho \frac{\partial^2 v^\varepsilon}{\partial t^2} - a_{ij} \frac{\partial^2 v^\varepsilon}{\partial x_i \partial x_j} = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^N, \tag{2.62}$$

$$\begin{cases} v^\varepsilon(0) = \varepsilon\, \alpha(x, \dfrac{x}{\varepsilon}) & \text{on } \mathbb{R}^N, \\ \dfrac{\partial v^\varepsilon}{\partial t}(0) = \beta(x, \dfrac{x}{\varepsilon}) & \text{on } \mathbb{R}^N, \end{cases} \tag{2.63}$$

satisfies all the conclusions of Theorem 2.2 (especially (2.52)-(2.55)) upon setting, for any k in Z^N,

$$\begin{cases} \alpha_k(x) = \displaystyle\int_{\mathcal{T}} \alpha(x,y)\, e^{-2i\pi k\cdot y} dy, \\ \beta_k(x) = \displaystyle\int_{\mathcal{T}} \beta(x,y) e^{-2i\pi k\cdot y} dy. \end{cases}$$

□

Remark 2.7. Theorems 2.2 and 2.3 provide an accurate description of the field v^ε on any finite interval $[0,T]$ for a quite reasonable class of initial data. Although it is somewhat hidden in the argument the key component in the success of expansion (2.52)-(2.53) is the ability to solve (2.12)-(2.13), *i.e.*, to unambiguously transport the initial value of the phase $(2\pi k \cdot x)$ along the rays. As previously noted in Remark 2.3 when the coefficients ρ and A depend on x, the rays may intersect (although the bicharacteristics never do) thus preventing a meaningul definition of the phase S in a smooth enough space (*i.e.*, for example $W^{2,\infty}$) for the geometrical optics ansatz to be successfull.

A non-zero non-intersecting time t_0 may be shown to exist for the rays (independently of the frequency k). However at $t = t_0$ the solution will not look like a superposition of plane waves and will have to be redecomposed into plane waves before one can proceed any further. The expansion then becomes untractable. □

2.3. The limit energy density.

In Corollary 2.1 we derived the weak limit d_k of the energy density $d(v_k^\varepsilon)$ associated to the propagation of a monochromatic oscillating wave. We will show that the polychromatic case produces a limit energy density d which is merely the sum over all frequencies of the individual limit energy densities associated to each frequency. This is the object of the following

Theorem 2.4. *In the contexts of Theorems 2.2 or 2.3 the sequence of energy densities $d(v^\varepsilon)$ (cf. (2.40)) converges weak-$*$ in $L^\infty(0,T;\mathcal{M}(\mathbb{R}^N))$ $(0 < T < +\infty)$ to*

(2.64)

$$\begin{aligned} d =&\frac{1}{4}\sum_{k\in Z^N}\sum_{\pm}\Big\{(\rho|\beta_k\Big(x \pm \frac{Akt}{(\rho Ak\cdot k)^{1/2}}\Big)|^2 \\ &+ 4\pi^2(Ak\cdot k)|\alpha_k\Big(x \pm \frac{Akt}{(\rho Ak\cdot k)^{1/2}}\Big)|^2 \\ &\pm 4\pi((\rho Ak\cdot k)^{1/2} Im\Big[\beta_k\Big(x \pm \frac{Akt}{(\rho Ak\cdot k)^{1/2}}\Big)\overline{\alpha_k}\Big(x \pm \frac{Akt}{(\rho Ak\cdot k)^{1/2}}\Big)\Big]\Big\}. \end{aligned}$$

Furthermore d is an element of $L^1((0,T)\times\mathbb{R}^N)$. □

Remark 2.8. The weak limit of any quadratic form in $(\partial v^\varepsilon/\partial t, \operatorname{grad} v^\varepsilon)$ could also be computed. In particular the weak limits of the kinetic energy $(1/2\,\rho\,|\partial v^\varepsilon/\partial t|^2)$ and of the potential energy $(1/2\,A \operatorname{grad} v^\varepsilon \cdot \operatorname{grad} v^\varepsilon)$ can be determined. By virtue of the principle of equipartition of the energy densities (cf. (5.19) in [2]) the limit kinetic and potential energy densities are actually both equal to $1/2\,d$. □

Proof of Theorem 2.4. In view of (2.52)-(2.53) the solution field v^ε is decomposed as follows, for any K in Z_+,

$$v^\varepsilon(x,t) = \sum_{\substack{|k|\leq K\\ k\in Z^N}} \hat{v}_k^\varepsilon(x,t) + \sum_{\substack{|k|>K\\ k\in Z^N}} \hat{v}^\varepsilon(x,t) + \varepsilon\, r^\varepsilon(x,t),$$

while

$$\Big\| \sum_{\substack{|k|>K\\ k\in Z^N}} \hat{v}_k^\varepsilon \Big\|_{en(T)} \leq \sum_{\substack{|k|>K\\ k\in Z^N}} \|\hat{v}_k^\varepsilon\|_{en(T)} \leq \mathcal{C} \sum_{\substack{|k|>K\\ k\in Z^N}} \omega_k,$$
$$\leq \mathcal{C}\,\omega''(K),$$

where

$$\omega''(K) = \sum_{\substack{|k|>K\\ k\in Z^N}} \omega_k \overset{K\to+\infty}{\longrightarrow} 0. \tag{2.65}$$

Thus, in the spirit of (2.42), it is easily verified that

$$d(v^\varepsilon) = d\Big(\sum_{\substack{|k|\leq K\\ k\in Z^N}} \hat{v}_k^\varepsilon\Big) + \rho_k^\varepsilon,$$

where

$$\int_0^T \int_{\mathbb{R}^N} |\rho_k^\varepsilon| dx\, dt \leq \omega''(\varepsilon, K), \tag{2.66}$$

with

$$\omega''(\varepsilon,K) \overset{\text{def}}{\equiv} \mathcal{C}T\{\varepsilon^2\, e^T \sigma'^2 + \varepsilon\, e^{T/2}\sigma\sigma' + \sigma\omega''(K)\}, \tag{2.67}$$

where $\mathcal{C}$ is a constant that does not depend on T or k. Thus for any φ in $\mathcal{C}_0^\infty((0,T)\times\mathbb{R}^N)$,

$$\Big|\int_0^T \int_{\mathbb{R}^N} \rho_K^\varepsilon\, \varphi\, dx\, dt\Big| \leq \omega''(\varepsilon,K)\|\varphi\|_{L^\infty((0,T)\times\mathbb{R}^N)}. \tag{2.68}$$

We now propose to compute the limit of $d(\sum_{|k|\leq K, k\in Z^N} \hat{v}_k^\varepsilon)$. An argument identical to that which led to the weak convergence to zero of terms of the form (2.43) with $(\pm) \neq (\pm)'$ demonstrates that the only terms that will contribute to the weak limit of $d(\sum_{|k|\leq K, k\in Z^N} \hat{v}_k^\varepsilon)$ are the square terms originating in the 0^{th} order (in ε) terms in the expressions for $\partial\hat{v}_k^\varepsilon/\partial t$ and grad $\hat{v}_k^\varepsilon$ (cf. Remark 2.4.). We thus obtain

$$\lim_{\varepsilon \to 0} \int_0^T \int_{\mathbb{R}^N} d\Big(\sum_{\substack{|k| \leq K \\ k \in Z^N}} \hat{v}_k^\varepsilon \Big) \varphi \, dx \, dt = \int_0^T \int_{\mathbb{R}^N} \Big(\sum_{\substack{|k| \leq K \\ k \in Z^N}} d_k \Big) \varphi \, dx \, dt, \tag{2.69}$$

where d_k is given in (2.41).

Further

$$\sum_{k \in Z^N} d_k \quad \text{is an absolutely converging series in } L^1((0,T) \times \mathbb{R}^N). \tag{2.70}$$

Indeed, by virtue of (2.41),
(2.71)

$$0 \leq \int_0^T \int_{\mathbb{R}^N} \sum_{k \in Z^N} d_k \, dx \, dt \leq \mathcal{C}T \sum_{k \in Z^N} \{ \|\beta_k\|^2_{L^2(\mathbb{R}^N)} + |k|^2 \, \|\alpha_k\|^2_{L^2(\mathbb{R}^N)} \},$$

where $\mathcal{C}$ is a constant which does not depend on T or k.

Upon recalling (2.57)-(2.58) we have

$$\sum_{k \in Z^N} |k| \, \|\alpha_k\|_{L^2(\mathbb{R}^N)} + \sum_{k \in Z^N - \{0\}} \|\beta_k\|_{L^2(\mathbb{R}^N)} < +\infty,$$

which implies in turn that

$$\sum_{k \in Z^N} |k|^2 \, \|\alpha_k\|^2_{L^2(\mathbb{R}^N)} + \sum_{k \in Z^N - \{0\}} \|\beta_k\|^2_{L^2(\mathbb{R}^N)} < +\infty, \tag{2.72}$$

since $\ell^1 \subset \ell^2$ (small L^p-spaces).

The absolute convergence of $\sum_{k \in Z^N} d_k$ is obtained by inserting estimate (2.72) into inequality (2.71).

Collecting (2.67)-(2.69), letting ε tend to zero, then choosing K large enough and appealing to (2.65) and (2.70) yields the desired result. □

3. THE CASE OF SMOOTH COEFFICIENTS: H-MEASURES.

This section is devoted to the investigation of the wave equation with oscillating initial data and non oscillating coefficients that are smooth (C^∞) functions of x, the space variable.

Specifically we consider the system

$$\begin{cases} \rho(x)\dfrac{\partial^2 v^\varepsilon}{\partial t^2} - \operatorname{div}(A(x)\operatorname{grad} v^\varepsilon) = 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}, \\ v^\varepsilon(0) = \gamma^\varepsilon(x) \quad \text{in } \mathbb{R}^N, \\ \dfrac{\partial v^\varepsilon}{\partial t}(0) = \beta^\varepsilon(x) \quad \text{in } \mathbb{R}^N. \end{cases} \tag{3.1}$$

We assume that $\rho(x)$ and $A(x)$ are smooth functions on $\mathbb{R}^N$ satisfying uniform boundedness and coercivity properties and that, for any N-tuple α,

$$|D^\alpha a_{ij}(x)| \quad \text{is uniformly bounded on } \mathbb{R}^N. \tag{3.2}$$

The quantities γ^ε and β^ε are smooth functions on $\mathbb{R}^N$, compactly supported with a common compact support K^0. They further satisfy

$$\begin{cases} \gamma^\varepsilon \rightharpoonup 0 \quad \text{weakly in } H^1(\mathbb{R}^N), \\ \beta^\varepsilon \rightharpoonup 0 \quad \text{weakly in } L^2(\mathbb{R}^N), \end{cases} \tag{3.3}$$

as ε tends to zero, which in turn easily implies that

$$v^\varepsilon \rightharpoonup 0 \quad \text{in } L^\infty(\mathbb{R}; H^1(\mathbb{R}^N)) \cap W^{1,\infty}(\mathbb{R}; L^2(\mathbb{R}^N)), \tag{3.4}$$

as ε tends to zero.

Remark 3.1. The initial conditions will be further specialized in Subsection 3.3 to be of the form

$$\begin{cases} \gamma^\varepsilon(x) = \varepsilon\, \alpha(x, \dfrac{x}{\varepsilon}), \\ \beta^\varepsilon(x) = \beta(x, \dfrac{x}{\varepsilon}), \end{cases} \tag{3.5}$$

where $\alpha(x,y)$ and $\beta(x,y)$ are smooth on $\mathbb{R}^N \times \mathcal{T}$ ($\mathcal{T}$ is the unit torus), compactly supported and β further satisfies

$$\int_{\mathcal{T}} \beta(x,y)dy = 0. \tag{3.6}$$

Such initial conditions will permit to compare the results of Subsection 3.2 -results which only depend on hypothesis (3.3)- to those of Section 2 whenever $A(x)$ and $\rho(x)$ are assumed to be independent of x. □

As mentioned before in Section 1 and in Remarks 2.3, 2.7, geometrical optics is a much more delicate tool to implement in the present setting. We will follow a different path and obtain an accurate description of the measure-limit of the energy density

$$d^\varepsilon(x,t) = \frac{1}{2}[\rho(x)\Big(\frac{\partial v^\varepsilon}{\partial t}\Big)^2 + A(x)\operatorname{grad} v^\varepsilon \operatorname{grad} v^\varepsilon](x,t),$$

as well as of the measure-limits of all quadratic quantities in $(\operatorname{grad} v^\varepsilon, \partial v^\varepsilon/\partial t)$.

Let us emphasize however that we will not be in a position to provide a description of the field v^ε itself, in striking contast with the outcome of the geometrical optics method (cf. Theorems 2.2-2.3). Thus the employed method, although definitely more flexible, is less precise: it only pertains to the "second moments" of the field v^ε.

We propose to resort, for the fulfillment of our task, to the theory of H-measures introduced by L. TARTAR (cf. [15]-[17]) or to that, very similar, of microlocal defect measures introduced by P. GERARD (cf. [5]-[7]). The first subsection of Section 3 is devoted to a brief and basic review of the fundamentals of H-measure theory. The second subsection applies that theory to our specific setting (*i.e.*, (3.1)-(3.3)) while the third subsection specializes the results obtained in the previous subsection to the setting of Remark 3.1 and compares them with those of Section 2 in the constant coefficient case.

3.1. H-measures.

This subsection is almost entirely borrowed from [15] with some reference to [5]-[7]. All results quoted here are due to the above mentioned authors.

H-measures are concerned with sequences of $\mathbb{R}^M$-valued functions that weakly converge to zero in L^2. Because minimal regularity hypotheses are not foremost in our study we adopt P. GERARD's definition over L. TARTAR's.

To this effect we recall the most basic facts about pseudo-differential operators, referring the readers to [8], [18] for example for further details.

Firstly the space $S^m(\mathbb{R}^Q, \mathbb{R}^{M^2})$ of symbols of order m on R^Q is defined as the set of all elements $p(y,\xi)$ in $\mathcal{C}^\infty(\mathbb{R}^Q_y \times \mathbb{R}^Q_\xi; \mathbb{R}^{M^2})$ such that for every compact subset K of $\mathbb{R}^Q$ and for every pair of n-tuples γ, δ there exists a constant $\mathcal{C}_{\gamma,\delta}(K)$ such that

$$|D^\gamma_y D^\delta_\xi p(y,\xi)| \leq \mathcal{C}_{\gamma,\delta}(K)(1+|\xi|)^{m-|\delta|}\ ,\ y \in K\ ,\ \xi \in \mathbb{R}^Q.$$

A standard pseudo-differential operator P of order m is defined, for every $\mathbb{C}^M$-valued u in $[\mathcal{C}^\infty_0(\mathbb{R}^Q)]^M$, as

$$\begin{aligned} Pu(y) &= \frac{1}{(2\pi)^Q} \int_{\mathbb{R}^Q} e^{iy\cdot\xi} p(y,\xi)\hat{u}(\xi) d\xi \\ &= (p(y,\cdot)\hat{u}(\cdot))\check{}(y), \end{aligned}$$

where $\hat{}$ and $\check{}$ denote the Fourier and inverse Fourier transformations, up to normalization constants. As such P extends to a continuous mapping from $[H^t(\mathbb{R}^Q)]^M$ to $[H^{t-m}_{\text{loc}}(\mathbb{R}^Q)]^M$ for every real number t. If further there exists a constant $\mathcal{C}_{\gamma,\delta}$ independent of K such that

$$(3.7) \qquad |D^\gamma_y D^\delta_\xi p(y,\xi)| \leq \mathcal{C}_{\gamma,\delta}(1+|\xi|)^{m-|\delta|}\ ,\ y \in \mathbb{R}^Q\ ,\ \xi \in \mathbb{R}^Q,$$

then P extends to a continuous mapping from $[H^t(\mathbb{R}^Q)]^M$ to $[H^{t-m}(\mathbb{R}^Q)]^M$.

We further specialize $p(y,\xi)$ to be of the form

(3.8)
$$\begin{cases} p(y,\xi) = p^m(y,\xi)\,\chi(\xi) + p^{m-1}(y,\xi), \\ p^m(y,\xi) \quad \text{is homogeneous of degree } m \text{ in } \xi, \\ p^{m-1}(y,\xi) \in S^{m-1}(\mathbb{R}^Q;\mathbb{R}^{M^2})\ ,\ p^{m-1} \text{ satisfies (3.7)}, \\ \chi(\xi) \in \mathcal{C}^\infty(\mathbb{R}^Q)\ ,\ 0 \leq \chi(\xi) \leq 1\ ,\ \chi(\xi) \equiv 0 \text{ in a neighbourhood of } \xi = 0, \\ \qquad \chi(\xi) \equiv 1, \text{ for } |\xi| \text{ large enough.} \end{cases}$$

The set of operators P associated to symbols $p(y,\xi)$ of the form (3.8) is denoted by $\psi^m(\mathbb{R}^Q;\mathbb{R}^M)$; p^m is called the principal symbol. The subset of $\psi^m(\mathbb{R}^Q;\mathbb{R}^M)$ associated to compactly supported p^m's and p^{m-1}'s (in y) is denoted by $\psi^m_c(\mathbb{R}^Q;\mathbb{R}^M)$. Note that changing the cut-off function χ only modifies P by a smoothing operator (*i.e.*, with a symbol in $\cap_m S^m(\mathbb{R}^Q;\mathbb{R}^M)$).

The following existence theorem for H-measure is then derived ([15], Theorem 1.1 and Corollary 1.2, or [5], Theorem 1):

THEOREM 3.1. *Let V^ε be a sequence of $[L^2(\mathbb{R}^Q)]^M$ that converges weakly to zero in that space. There exists a subsequence of V^ε and a $\mathbb{R}^M \times \mathbb{R}^M$ -valued Radon measure μ on $\mathbb{R}^Q_y \times S^{Q-1}_\xi$ such that, for every element P of $\psi^0_c(\mathbb{R}^Q;\mathbb{R}^M)$,*

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^Q_y}(PV^\varepsilon)\cdot\overline{V}^\varepsilon\,dy &= <\mu,p^0>\\ &=\int_{\mathbb{R}^Q_y\times S^{Q-1}_\xi} tr\{p^0(y,\xi)\mu(dy\,d\xi)\}\\ &=\sum_{i,j=1}^M\int_{\mathbb{R}^Q_y\times S^{Q-1}_\xi} p^0_{ij}(y,\xi)\mu_{ij}(dy\,d\xi).\end{aligned}\tag{3.9}$$

Further the measure μ is hermitian semi-positive in the following sense

$$\begin{cases}\mu_{ij}=\overline{\mu}_{ji}\quad, & 1\le i,j\le M,\\ \displaystyle\sum_{i,j=1}^M \mu_{ij}\,h_i\,\overline{h}_j & \text{is a positive Radon measure, } h\in\mathbb{C}^M.\end{cases}$$

Finally if $V^\varepsilon\equiv 0$ outside a closed set K of $\mathbb{R}^Q$, then supp $\mu\subset K\times S^{Q-1}_\xi$. The measure μ is the H-measure associated to the subsequence V^ε. □

Example 3.1. Theorem 3.1 permits in particular the identification of the measure limit of $|V^\varepsilon|^2$. To this effect p^0 is taken to be of the form

$$p^0_{ij}(y,\xi)=\varphi(y)\delta_{ij}\quad,\quad 1\le i,j\le M\quad,\quad \varphi\in C^\infty_0(\mathbb{R}^Q).$$

Application of (3.9) yields

$$\lim_{\varepsilon\to 0}|V^\varepsilon|^2=\sum_{i=1}^M\int_{S^{Q-1}_\xi}\mu_{ii}(dy\,d\xi).$$

□

H-measures behave remarkably well with respect to localization as demonstrated by the following result due to L. TARTAR ([15], Theorem 1.6)).

THEOREM 3.2. *Consider a sequence V^ε which converges weakly to zero in $[L^2(\mathbb{R}^Q)]^M$ and which defines a H-measure μ. If V^ε further satisfies*

$$\sum_{k=1}^{Q}\sum_{i=1}^{M}\frac{\partial}{\partial y_k}(C_k^i(y)V_i^\varepsilon)\to 0 \quad \textit{strongly in } H^{-1}_{\text{loc}}(\mathbb{R}^Q),$$

where $C_k^i(y)$ $(1\le k\le Q, 1\le i\le M)$ is continuous on $\mathbb{R}_y^Q$, then

$$\sum_{k=1}^{Q}\sum_{i=1}^{M} C_k^i(y)\xi_k\,\mu_{ij}=0 \quad , \quad 1\le j\le M, \tag{3.10}$$

for every (y,ξ) in $\mathbb{R}_y^Q\times S_\xi^{Q-1}$. □

Note that a similar theorem can be proved when the differential relations

$$\sum_{k=1}^{Q}\sum_{i=1}^{M}\frac{\partial}{\partial y_k}(C_k^i(y)V_i^\varepsilon)\to 0 \quad \text{strongly in } H^{-1}_{\text{loc}}(\mathbb{R}^Q),$$

are replaced by

$$PV^\varepsilon\to 0 \quad \text{strongly in } H^{-1}_{\text{loc}}(\mathbb{R}^Q),$$

where P is an element of $\psi^1(\mathbb{R}^Q;\mathbb{R}^M)$. Relations (3.10) then become

$$p_i^1(y,\xi)\mu_{ij}=0 \quad , \quad 1\le j\le M.$$

Equality (3.10) is actually a set of M-relations that the H-measure μ must satisfy. Property (3.10) is refered to as localization since it localizes the support of μ.

Finally the following lemma about symbolic calculus for pseudo-differential operators is needed (cf. e.g. [18], Theorems 4.2, 4.3 and Corollary 4.2 or [15], Lemma 3.2 or [6], Corollary 1.3):

LEMMA 3.1. *Let P and Q be two scalar-valued pseudo-differential operators in $\psi^m(\mathbb{R}^Q;\mathbb{R})$ and $\psi^{m'}(\mathbb{R}^Q;\mathbb{R})$ respectively whose principal symbols* *are respectively p^m and $q^{m'}$. Then P^* (the adjoint of P), PQ and $[P,Q] \stackrel{\text{def}}{\equiv} PQ - QP$ are scalar valued pseudo-differential operators which belong to $\psi^m(\mathbb{R}^Q;\mathbb{R}), \psi^{m+m'}(\mathbb{R}^Q;\mathbb{R})$ and $\psi^{m+m'-1}(\mathbb{R}^Q;\mathbb{R})$ respectively. Their principal symbol $p^{*m}, (pq)^{m+m'}, \{p^m, q^{m'}\}$ are respectively given by*

$$\begin{cases} p^{*m} = \overline{p^m}, \\ (pq)^{m+m'} = p^m\, q^{m'}, \end{cases} \tag{3.11}$$

$$\begin{aligned} \{p^m, q^{m'}\} &= \frac{1}{i}\Big(\frac{\partial p^m}{\partial \xi}\cdot\frac{\partial q^{m'}}{\partial y} - \frac{\partial q^{m'}}{\partial \xi}\cdot\frac{\partial p^m}{\partial y}\Big) \\ &= \frac{1}{i}\sum_{k=1}^{Q}\Big(\frac{\partial p^m}{\partial \xi_k}\frac{\partial q^{m'}}{\partial y_k} - \frac{\partial q^{m'}}{\partial \xi_k}\frac{\partial p^m}{\partial y_k}\Big). \end{aligned} \tag{3.12}$$

□

We have now at our disposal the main ingredients necessary to a successful pursuit of our analysis of (3.1)-(3.3). A few other results pertaining to H-measures will be needed in the sequel and called forth whenever deemed appropriate.

3.2. The H-measure associated to the solution of the wave equation.

As noted in (3.4) the solution v^ε of (3.1), (3.3) converges weakly to zero in the energy norm. Thus the $\mathbb{R}^{N+1}$-valued field V^ε defined, *for any given* $T > 0$, as

(3.13)
$$\begin{cases} V_0^\varepsilon(x,t) = \theta(t)\dfrac{\partial v^\varepsilon}{\partial t}(x,t) \quad , \quad V_i^\varepsilon(x,t) = \theta(t)\dfrac{\partial v^\varepsilon}{\partial x_i}(x,t) \quad , \quad 1 \le i \le N, \\ \theta(t) = 1 \quad \text{on } [0,T] \quad , \quad \theta \in C_0^\infty(\mathbb{R}), \end{cases}$$

is a weakly converging sequence in $L^2(\mathbb{R}^{N+1})$ with weak limit zero. The time truncation performed in (3.13) is needed since the solution v^ε of (3.1), (3.3) lies in $L^\infty(\mathbb{R}_t; H^1(\mathbb{R}^N)) \cap W^{1,\infty}(\mathbb{R}_t; L^2(\mathbb{R}^N))$ and not in $L^2(\mathbb{R}_t; H^1(\mathbb{R}^N)) \cap H^1(\mathbb{R}_t; L^2(\mathbb{R}^N))$, the natural space from the stand point of H-measure theory.

Setting $Q = N+1$ and $M = N+1$ places us in the setting of the first subsection and permits application of Theorem 3.1 to a subsequence of V^ε (still indexed by ε). We conclude to the existence of a H-measure μ which is a $(N+1) \times (N+1)$ matrix of (scalar valued) Radon measures μ_{ij} on $\mathbb{R}_t \times \mathbb{R}_x^N \times S_\xi^N$.

In the remainder of Section 3 the following notation is adopted:

$$\begin{cases} y = (y_0, y_1, ..., y_N) = (t, x), \\ \xi = (\xi_0, \xi_1, ..., \xi_N) = (\tau, \eta), \end{cases}$$

with

$$\begin{cases} y_0 = t \quad , \quad y_i = x_i \quad , \quad 1 \le i \le N, \\ \xi_0 = \tau \quad , \quad \xi = \eta_i \quad , \quad 1 \le i \le N. \end{cases}$$

Then μ_{ij} is a measure of the form $\mu_{ij}(dy\, d\xi)$, with $0 \le i, j \le N$ and all sums of the form $\sum_{i=1}^{Q}, \sum_{j=1}^{M}$ that appear in Subsection 3.1 are to be replaced by $\sum_{i=0}^{N}, \sum_{j=0}^{N}$, since all indices run from 0 to N (and not from 1 to $N+1$).

Our ultimate goal is to pass to the limit in the energy density d^ε, *i.e.*, to compute for any φ in $C_0^\infty((0,T) \times \mathbb{R}^N)$ the limit, as ε tends to zero, of

(3.14)

$$D_\varepsilon \overset{\text{def}}{\equiv} \int_{(0,T)\times\mathbb{R}^N} d^\varepsilon\, \varphi\, dt\, dx = \frac{1}{2} \int_{\mathbb{R}^{N+1}} [\rho(x)(V_0^\varepsilon)^2 + \sum_{i,j=1}^{N} a_{ij}(x) V_j^\varepsilon V_i^\varepsilon] \varphi\, dx\, dt.$$

Note that the fact that $\theta \equiv 1$ on $[0, T]$ has been implicitly used in the second equality of (3.14).

Defining $p^0(y, \xi)$ as the ξ-independent matrix with coefficients

$$p_{ij}^0(y,\xi)(= p_{ij}^0(y)) = \begin{cases} \rho(x)\varphi(t,x) \text{ if } i = j = 0, \\ \quad 0 \qquad \text{if } i = 0, 1 \le j \le N \text{ or } j = 0, 1 \le i \le N, \\ a_{ij}(x)\varphi(t,x) \text{ if } 1 \le i, j \le N, \end{cases}$$

permits to rewrite (3.14) as

$$D_\varepsilon = \frac{1}{2} \int_{\mathbb{R}^{N+1}} P V^\varepsilon \overline{V}^\varepsilon\, dy,$$

where P is a pseudo-differential operator with p^0 as principal symbol. Application of Theorem 3.1 yields

$$\lim_{\varepsilon \to 0} D_\varepsilon = \frac{1}{2} < \mu, P^0 > = \frac{1}{2} \int_{\mathbb{R}_y^{N+1}} tr[p^0(y)(\int_{S_\xi^N} \mu(dy\, d\xi)]dy. \tag{3.15}$$

Thus the identification of the measure limit of d^ε will be attained once μ is identified. Such a problem was addressed in [15] (Section 3.3). Lemma 3.10 and Theorem 3.12 of [15] read as follows:

THEOREM 3.3. *The H-measure μ has the form*

$$\mu = (\xi \otimes \xi)\nu, \text{ i.e., } \mu_{ij} = \xi_i \xi_j \nu, \ 0 \le i,j \le N, \tag{3.16}$$

where $\xi \in S^N$ and $\nu(dy\, d\xi)$ is a non negative scalar-valued measure satisfying

$$Q(x,\xi)\nu = 0, \tag{3.17}$$

$$< \xi_0 \nu, \{\phi, Q\} > = 0. \tag{3.18}$$

In (3.17), (3.18),

$$Q(x,\xi) = \frac{1}{2}[\rho(x)\tau^2 - A(x)\eta \cdot \eta] = \frac{1}{2}[\rho(x)\xi_0^2 - \sum_{i,j=1}^{N} a_{ij}(x)\xi_i \xi_j], \tag{3.19}$$

and $\phi(t,x,\xi)$ is any smooth compactly supported function of $(0,T) \times \mathbb{R}_x^N \times S_\xi^N$. Finally $\{\ \}$ denotes the Poisson bracket (cf. (3.12) in Lemma 3.1). □

Remark 3.2. The proof of Theorem 3.3 will not be given here but a proof of (3.18) would be easily recovered from the proof of (3.97) in Theorem 3.4 below.

Equations (3.16) and (3.17) are simple applications of the localization principle (*i.e.*, Theorem 3.2) to the sequence $V^\varepsilon = (\partial v^\varepsilon/\partial t, \operatorname{grad} v^\varepsilon)$; indeed V^ε has zero time-space curl (which yields (3.16)) while the sequence $(\rho(x)V_0^\varepsilon, -\sum_{j=1}^N a_{ij}(x)V_j^\varepsilon, 1 \le i \le N)$ has zero divergence (which yields (3.17)). □

Remark 3.3. Note that in Theorem (3.12) of [15] equation (3.18) is stated as

$$< \nu, \{\phi, Q\} > = 0. \tag{3.18bis}$$

It is our belief that (3.18) should be stated as is, as witnessed by the discussion of the role of the initial conditions and Theorem 3.4 below. In any case (3.18) and (3.18bis) are equivalent. Indeed choosing ϕ of the form $\xi_0\tilde{\phi}(y,\xi)$ in (3.18bis) yields (3.18) since

$$\{\xi_0\tilde{\phi}, Q\} = \xi_0\{\tilde{\phi}, Q\}.$$

Conversely choosing ϕ of the form $1/\xi_0\psi(\xi)\tilde{\phi}(y,\xi)$ in (3.18), with

$$\begin{cases} \psi \in C^\infty(S_\xi^N), \\ \psi(\xi) = 0 \text{ in a neighbourhood of } \xi_0 = 0, \\ \psi(\xi) \equiv 1 \text{ outside a neighbourhood (in } S_\xi^N) \text{ of } \xi_0 = 0, \end{cases}$$

and using the information (3.20) below on the support of ν yields (3.18bis) since

$$< \xi_0\nu, \{\frac{1}{\xi_0}\psi\tilde{\phi}, Q\} >=< \nu, \{\psi\tilde{\phi}, Q\} >=< \nu, \{\tilde{\phi}, Q\} > .$$

□

Remark 3.4. It is important to observe that

$$\begin{cases} (\eta,\tau) \notin \text{supp } \nu \text{ for } (\eta,\tau) \text{ in a neighbourhood (in } S_\xi^N) \\ \text{of the points where } \tau = 0, 1 \text{ or } -1. \end{cases} \tag{3.20}$$

Indeed (3.17) reads as

$$\begin{cases} \rho(x)\tau^2 = \sum_{i,j=1}^{N} a_{ij}(x)\eta_i\eta_j, \\ \\ \tau^2 + |\eta|^2 = 1, \end{cases} \qquad \text{on} \quad \text{supp } \nu.$$

The coercivity property of $A(x)$ and positivity property of $\rho(x)$ (assumed throughout this study) then imply (3.20). □

In the sequel we seek an interpretation of (3.18) as a transport equation. This is the object of Paragraph 3.2.1. It is proved there that ν is transported along specific integral curves of a system of ordinary differential equations (see Lemma 3.2). These integral curves are in turn interpreted in terms of the Hamiltonian Q in Remark 3.9. The measure ν is then determined by "its initial conditions" which need to be recovered. This is the object of Paragraph 3.2.2 and specifically of Theorem 3.4. Paragraph 3.2.3 is very short and devoted to the limit energy density.

3.2.1. Interpreting (3.18).

To avoid technicalities it is assumed in this paragraph that ν is represented by a density

$$\nu(dy\, d\xi) = \nu(y,\xi)dy\, d\xi.$$

The argument could be carried through in the case of a general measure ν because all functions that are applied to ν are $\mathcal{C}_0^\infty$ and all computations could thus be performed in a distributional sense. Lemma 3.2 in Paragraph 3.2.1 is consequently stated for an arbitrary measure ν satisfying (3.18).

Our first task is to integrate equation (3.18) by parts so as to evidence a transport equation (albeit in weak form). Equation (3.18) reads as

$$\begin{aligned} &\int_{\mathbb{R}^{N+1}} dy \int_{S^N} (\xi_0\nu)(y,\xi)\Big(\sum_{i=0}^N \frac{\partial\phi}{\partial\xi_i}\frac{\partial Q}{\partial y_i}\Big)d\xi \\ &- \int_{\mathbb{R}^{N+1}} dy \int_{S^N} (\xi_0\nu)(y,\xi)\Big(\sum_{i=0}^N \frac{\partial Q}{\partial\xi_i}\frac{\partial\phi}{\partial y_i}\Big)d\xi = 0. \end{aligned} \tag{3.21}$$

Since ϕ has compact support in y the last integral in (3.21) reads as

$$-\int_{\mathbb{R}^{N+1}} dy \int_{S^N} \Big(\sum_{i=0}^N \Big(\frac{\partial(\xi_0\nu)}{\partial y_i}\frac{\partial Q}{\partial \xi_i} + (\xi_0\nu)\frac{\partial^2 Q}{\partial y_i\,\partial\xi_i}\Big)\Big)\phi\, d\xi. \tag{3.22}$$

The handling of the first integral of (3.21) is more delicate because the integration over the frequency variable ξ takes place on the sphere S^N. Upon setting

$$\theta_i = (\xi_0\nu)\frac{\partial Q}{\partial y_i} \quad , \quad 0 \le i \le N,$$

our task is reduced to that of performing an integration by parts on an integral of the form

$$\int_{S^N} \sum_{i=0}^{N} \theta_i \frac{\partial \phi}{\partial \xi_i}\, d\xi,$$

for which the following holds true (cf. e.g. [12], Lemma 4.9):

$$\begin{aligned}
&\int_{S^N} \Big\{ \sum_{i=0}^{N} \theta_i \frac{\partial \phi}{\partial \xi_i} - \Big(\sum_{i=0}^{N} \theta_i n_i \Big) \frac{\partial \phi}{\partial n} \Big\} d\xi \\
&+ \int_{S^N} \phi \Big(\sum_{i=0}^{N} \frac{\partial \theta_i}{\partial \xi_i} - \sum_{i,j=0}^{N} \frac{\partial \theta_i}{\partial \xi_j} n_i n_j \Big) d\xi \qquad (3.23)\\
&= \int_{S^N} H \Big(\sum_{i=0}^{N} \theta_i n_i \Big) \phi \, d\xi,
\end{aligned}$$

where n stands for the outwardly directed unit normal to S^N and H denotes the mean curvature. Since ϕ, which is defined on S^N, is extended to $\mathbb{R}^{N+1}$ in (3.23) by homogeneity of degree zero and since $n = \xi$ on S^N,

$$\frac{\partial \phi}{\partial n} = \sum_{i=0}^{N} \frac{\partial \phi}{\partial \xi_i} \xi_i = 0. \qquad (3.24)$$

Further the mean curvature H of the sphere S^N is

$$H = N. \qquad (3.25)$$

Thus recalling (3.23)-(3.25) the first integral of (3.21) reads as

(3.26)

$$\begin{aligned}
&\int_{\mathbb{R}^{N+1}} dy \int_{S^N} \phi \{ N(\xi_0 \nu) \Big(\sum_{i=0}^{N} \frac{\partial Q}{\partial y_i} \xi_i \Big) \\
&\quad - \sum_{i=0}^{N} \frac{\partial}{\partial \xi_i} \Big((\xi_0 \nu) \frac{\partial Q}{\partial y_i} \Big) + \sum_{i,j=0}^{N} \frac{\partial}{\partial \xi_j} \Big((\xi_0 \nu) \frac{\partial Q}{\partial y_i} \Big) \xi_i \xi_j \} \\
&= \int_{\mathbb{R}^{N+1}} dy \int_{S^N} \phi \Big\{ (\xi_0 \nu) \sum_{i=0}^{N} \Big(N \frac{\partial Q}{\partial y_i} \xi_i - \frac{\partial^2 Q}{\partial y_i \partial \xi_i} + \Big(\sum_{j=0}^{N} \frac{\partial^2 Q}{\partial y_i \partial \xi_j} \xi_j \Big) \xi_i \Big) \\
&\quad - \sum_{i=0}^{N} \frac{\partial (\xi_0 \nu)}{\partial \xi_i} \Big(\frac{\partial Q}{\partial y_i} - \Big(\sum_{j=0}^{N} \frac{\partial Q}{\partial y_j} \xi_j \Big) \xi_i \Big) \Big\} d\xi.
\end{aligned}$$

Subtracting (3.22) from (3.26), recalling the homogeneous character of degree 2 of Q, hence of $\partial Q / \partial y_i$, $0 \le i \le N$, yields in place of (3.21)

(3.27)
$$\int_{\mathbb{R}^{N+1}} dy \int_{S^N} \phi\Big\{\sum_{i=0}^{N}\Big[\frac{\partial(\xi_0\nu)}{\partial y_i}\frac{\partial Q}{\partial \xi_i} - \frac{\partial(\xi_0\nu)}{\partial \xi_i}\Big(\frac{\partial Q}{\partial y_i} - \Big(\sum_{j=0}^{N}\frac{\partial Q}{\partial y_j}\xi_j\Big)\xi_i\Big)\Big] + (N+2)(\xi_0\nu)\Big(\sum_{i=0}^{N}\frac{\partial Q}{\partial y_i}\xi_i\Big)\Big\}d\xi = 0.$$

The test function ϕ is an arbitrary smooth (compactly supported) function on $(0,T)\times\mathbb{R}^N_x\times S^N_\xi$; thus (3.27) implies

(3.28)
$$\sum_{i=0}^{N}\frac{\partial(\xi_0\nu)}{\partial y_i}\frac{\partial Q}{\partial \xi_i} - \frac{\partial(\xi_0\nu)}{\partial \xi_i}\Big(\frac{\partial Q}{\partial y_i} - \Big(\sum_{j=0}^{N}\frac{\partial Q}{\partial y_j}\,\xi_j\Big)\xi_i\Big) + (N+2)(\xi_0\nu)\Big(\sum_{i=0}^{N}\frac{\partial Q}{\partial y_i}\xi_i\Big) = 0 \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}^N_x\times S^N_\xi).$$

Differentiation of (3.17) with respect to y_i permits to rewrite (3.28) as

(3.29)
$$\sum_{i=0}^{N}\frac{\partial(\xi_0\nu)}{\partial y_i}\Big(\frac{\partial Q}{\partial \xi_i} - (N+2)Q\,\xi_i\Big) - \frac{\partial(\xi_0\nu)}{\partial \xi_i}\Big(\frac{\partial Q}{\partial y_i} - \Big(\sum_{j=0}^{N}\frac{\partial Q}{\partial y_j}\,\xi_j\Big)\xi_i\Big) = 0.$$

The above computation has enabled us to obtain a transport equation for the measure ν, namely (3.29). Further analysis of that equation will be pursued momentarily.

Remark 3.5. This remark pertains to the definition of a trace for the quantity $\partial Q/\partial\tau(\tau\nu)$ at $t=0$.

The scalar valued measure ν is a non negative Radon measure and can thus be represented, by virtue of Riesz representation theorem, by a regular Borel measure $d\nu$. We are thus at liberty to consider

$$\int_{\mathbb{R}^{N+1}_y}\int_{S^N_\xi} r(x,t,\xi)\chi_{0,\infty}(t)d\nu,$$

where $\chi_{0,\infty}$ is the characteristic function of the open intervall $(0,\infty)$ and r is an element of $C_0^\infty([0,\infty)\times\mathbb{R}^N_x\times S^N_\xi)$; we set, for such r's,

(3.30)
$$\ll \nu, r \gg = \int_{\mathbb{R}^{N+1}_y}\int_{S^N_\xi} r\,\chi_{(0,\infty)}d\nu.$$

Note that

$$\ll \nu, r \gg = < \nu, r > \quad \text{for } r \text{ in } C_0^\infty((0,\infty)\times\mathbb{R}^N_x\times S^N_\xi).$$

When ϕ lies in $C_0^\infty([0,\infty)\times \mathbb{R}_x^N \times S_\xi^N), r = \tau\{\phi, Q\}$ belongs to the same set which permits to define $\ll \tau\nu, \{\phi, Q\} \gg$. We will sketch below a proof to the effect that $\ll \tau\nu, \{\phi, Q\} \gg$ only depends on $\phi(t=0, x, \xi)$; *the previous expression defines the trace of* $\partial Q/\partial\tau\ \tau\nu$ *at time* $t=0$. Indeed, for ϕ in $C_0^\infty([0,T) \times R_x^N \times S_\xi^N)$

$$\ll \tau\nu, \{\phi, Q\} \gg = \int_0^T dt \int_{\mathbb{R}_x^N} dx \int_{S_\xi^N} d\xi \sum_{i=0}^N \Big(\frac{\partial\phi}{\partial\xi_i}\frac{\partial Q}{\partial y_i} - \frac{\partial\phi}{\partial y_i}\frac{\partial Q}{\partial \xi_i}\Big)\xi_0 \nu(x,t,\xi).$$

Performing various integrations by parts in a manner analogous to that which led to the transport equation (3.29) and taking into account the boundary terms at time $t=0$ (ϕ is not zero at $t=0$) enables us to rewrite the above equality as

$$\begin{aligned} \ll \tau\nu, \{\phi, Q\} \gg &= \int_0^T \int_{\mathbb{R}_x^N} \int_{S_\xi^N} \{\text{left hand side of (3.29)}\}\ \phi\, d\xi\, dx\, dt \\ &+ \int_{\mathbb{R}_x^N} dx \int_{S_\xi^N} d\xi \Big[\frac{\partial Q}{\partial \tau}(\tau\nu)\Big]_{t=0} \phi_{(t=0,x,\xi)}. \end{aligned} \tag{3.31}$$

In view of (3.29), the right hand side of (3.31) reduces to its second term and thus $\ll \tau\nu, \{\phi, Q\} \gg$ defines the trace of $\partial Q/\partial\tau\ \tau\nu$ at $t=0$ when $\nu(x,t,\xi)$ is a (smooth) density for ν.

When ν does not have the previously assumed regularity, $\ll \tau\nu, \{\phi, Q\} \gg$ can be proved to only depend on $\phi(t=0)$. To this effect a sequence ψ_δ of smooth functions on $\mathbb{R}$ is defined by

$$\psi_\delta(t) = \psi(\frac{t}{\delta}), \tag{3.32}$$

with ψ a non negative non decreasing function such that

$$\begin{cases} \psi(t) = 0 & \text{if } t \le \dfrac{1}{2}, \\ \psi(t) = 1 & \text{if } t \ge 1. \end{cases}$$

Consider for any ϕ in $C_0^\infty([0,T) \times \mathbb{R}_x^N \times S_\xi^N)$ the function $\phi\psi_\delta$ which is an admissible test function in (3.21). Singling out in (3.21) the term which involves $\phi\psi_\delta'$ leads to

$$< \xi_0\nu, \psi_\delta \sum_{i=0}^{N}\Big(\frac{\partial\phi}{\partial\xi_i}\frac{\partial Q}{\partial y_i} - \frac{\partial\phi}{\partial y_i}\frac{\partial Q}{\partial\xi_i}\Big) > = < \xi_0\nu, \frac{\partial Q}{\partial\xi_0}\phi\psi'_\delta > .$$

As δ tends to zero the left hand side in the above equality tends to $\ll \tau\nu, \{\phi, Q\} \gg$. Defining

$$\tilde{\phi}(t,x,\xi) = \phi(t,x,\xi) - \phi(0,x,\xi)$$

permits to rewrite the right hand side in the above equality as

$$< \tau\nu, \frac{\partial Q}{\partial\tau}\phi(0)\psi'_\delta > + \int_{\mathbb{R}_y^{N+1}}\int_{S_\xi^N}\tilde{\phi}(t,x,\xi)\psi'_\delta(t)\tau\frac{\partial Q}{\partial\tau}d\nu.$$

Since ψ'_δ vanishes outside $(\delta/2,\delta)$ and is of the order of $1/\delta$ inside $(\delta/2,\delta)$ while $\tilde{\phi}$ is of the order of δ on $(\delta/2,\delta)$, the second term is bounded by

$$C\int_{\mathbb{R}_y^{N+1}}\int_{S_\xi^N}\chi_{(\delta/2,\delta)}(t)d\nu,$$

which tends to zero with δ. The remaining term, which only depends on $\phi(0)$ thus tends to $\ll \tau\nu, \{\phi, Q\} \gg$ as δ tends to zero which proves in turn that the latter only depends on $\phi(0)$. □

Remark 3.6. In the context of Remark 3.5, it can be further deduced that

(3.33) $\qquad \nu$ has a trace at $t = 0$.

Property (3.33) is a direct consequence of (3.20) in Remark 3.4 once it is noted that by virtue of the definition (3.17) of Q

$$\frac{\partial Q}{\partial\tau}\tau = \rho(x)\tau^2,$$

and that $\rho(x)$ is bounded away from zero. □

We now return to (3.29) and propose to further analyze its structure as a transport equation.

It is tempting to introduce the following system of ordinary differential equations:

$$
(3.34)\qquad \begin{cases} \dfrac{d\overline{y}_i}{ds} = \dfrac{\partial Q}{\partial \xi_i}(\overline{y},\overline{\xi}) - (N+2)Q(\overline{y},\overline{\xi})\xi_i, \\ \\ \dfrac{d\overline{\xi}_i}{ds} = -\dfrac{\partial Q}{\partial y_i}(\overline{y},\overline{\xi}) + \Big(\displaystyle\sum_{j=0}^{N} \dfrac{\partial Q}{\partial y_j}(\overline{y},\overline{\xi})\overline{\xi}_j\Big)\overline{\xi}_i, \end{cases} \qquad 0 \le i \le N,
$$

together with its initial conditions

$$
(3.35)\qquad \overline{y}(s=0) = y^* \quad , \quad \overline{\xi}(s=0) = \xi^*.
$$

Since the right hand side of (3.34) is a locally Lipschitz functions of $(\overline{y},\overline{\xi})$ on $\mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$ (note that Q is a quadratic polynomial in $\overline{\xi}$) the system (3.34)-(3.35) admits a unique local (in s) solution. We are exclusively interested in frequencies that belong to the unit sphere S^N and thus constrain ξ^* to be an element of S^N. In such a case multiplication of the second equation of (3.34) by $\overline{\xi}_i$ and summation over i $(0 \le i \le N)$ implies that the local solution $(\overline{y}(x), \overline{\xi}(s))$ satisfies

$$
|\overline{\xi}(x)|^2 = 1
$$

over its interval of existence. Consideration of the first equation of (3.34) implies in turn that $|\overline{y}(x)|$ remains bounded over the interval of existence since the functions of y (of x actually) that enter the expression of Q are uniformly bounded. Thus both $|\overline{\xi}(s)|$ and $|\overline{y}(s)|$ remain bounded over the interval of existence and the system (3.34)-(3.35) admits a unique global solution (in s) for any set of initial conditions (y^*, ξ^*) with $\xi^* \in S^N$.

Recalling (3.29) we conclude that $\xi_0\nu$ remains constant along the integral curves of (3.34)-(3.35) that live on $\mathbb{R}_y^{N+1} \times S_\xi^N$. We have thus proved the

Lemma 3.2. *In the context of Theorem 3.3, $\xi_0\nu$ remains constant along the integral curves of (3.34)-(3.35) that live on $\mathbb{R}_y^{N+1} \times S_\xi^N$.* □

Remark 3.7. Because Remark 3.4 states that $\xi_0 = \tau$ is not zero on the support of ν Lemma 3.2 actually provides a description of ν itself at any point of the integral curves of (3.34)-(3.35) with $|\xi^*| = 1$. □

Remark 3.8. If the initial conditions y^* and ξ^* of (3.34) are such that

$$Q(y^*,\xi^*) = 0$$

it is easily proved through the computation of $d/ds\ (Q(\overline{y}(s),\overline{\xi}(s)))$ that

$$Q(\overline{x}(s),\overline{\xi}(s)) = 0, \tag{3.36}$$

for the solution $(\overline{x}(s),\overline{\xi}(s))$ of (3.34)-(3.35). □

Remark 3.9. In the context of Remark 3.8, this long remark is aimed at proving that the integral curves of (3.34)-(3.35) that satisfy

$$\begin{cases} |\xi^*| = 1, \\ Q(y^*,\xi^*) = 0, \end{cases} \tag{3.37}$$

can be interpreted as the projections on $\mathbb{R}_y^{N+1} \times S_\xi^N$ of the integral curves of some Hamiltonian system on $\mathbb{R}_y^{N+1} \times \mathbb{R}_\xi^{N+1}$ with the same initial conditions.

To this effect the following system of ordinary differential equations is considered:

$$\begin{cases} \dfrac{d\tilde{y}_i}{ds} = \dfrac{\partial}{\partial \xi_i}\left(\dfrac{Q}{|\zeta|}\right)(\tilde{y},\tilde{\zeta}), \\ \\ \dfrac{d\tilde{\zeta}_i}{ds} = -\dfrac{\partial}{\partial y_i}\left(\dfrac{Q}{|\zeta|}\right)(\tilde{y},\tilde{\zeta}), \end{cases} \qquad 0 \le i \le N, \tag{3.38}$$

together with its initial conditions

$$\tilde{y}(s=0) = y^\sharp \quad , \quad \tilde{\zeta}(s=0) = \zeta^\sharp. \tag{3.39}$$

The Lipschitz character of $Q/|\zeta|$ on $\mathbb{R}_y^{N+1} \times \mathbb{R}_\zeta^{N+1}$ implies global existence and uniqueness of the solution of (3.38)-(3.39) for any set of initial conditions $(y^\sharp,\zeta^\sharp)$. Note that if $\zeta^\sharp = 0$ then $\tilde{\zeta}(s) = 0$ for any $s > 0$.

Further the associated Hamiltonian $Q/|\zeta|$ remains constant along the integral curves of (3.38)-(3.39). Specifically if the initial conditions $(y^\sharp,\zeta^\sharp)$ are such that

$$\begin{cases} \zeta^\sharp \neq 0, \\ Q(y^\sharp, \zeta^\sharp) = 0, \end{cases} \tag{3.40}$$

then $\tilde{\zeta}(s) \neq 0$ and $Q(\tilde{y}(s), \tilde{\zeta}(s)) = 0$ along the integral curves of (3.38)-(3.39). The homogeneous character of Q in the variable ξ implies that

$$Q(\tilde{y}(s), \frac{\tilde{\zeta}(s)}{|\tilde{\zeta}(s)|}) = 0, \tag{3.41}$$

along the integral curves of (3.38)-(3.39) with initial conditions satisfying (3.40).

Using once again the homogeneous character of degree 2 of Q in ξ easily implies that $(\tilde{y}(s), \tilde{\zeta}(s)/|\tilde{\zeta}(s)| \stackrel{\text{def}}{\equiv} \tilde{\xi}(s))$ satisfies

$$\begin{cases} \dfrac{d\tilde{y}_i}{ds} = \dfrac{\partial Q}{\partial \xi_i}(\tilde{y}, \tilde{\xi}) - Q(\tilde{y}, \tilde{\xi})\tilde{\xi}_i, \\ \\ \dfrac{d\tilde{\xi}_i}{ds} = -\dfrac{\partial Q}{\partial y_i}(\tilde{y}, \tilde{\xi}) + \Big(\displaystyle\sum_{j=0}^{N} \dfrac{\partial Q}{\partial y_j}(\tilde{y}, \tilde{\xi})\tilde{\xi}_j\Big)\tilde{\xi}_i, \end{cases} \qquad 0 \leq i \leq N, \tag{3.42}$$

which by virtue of (3.41) reads as

$$\begin{cases} \dfrac{d\tilde{y}_i}{ds} = \dfrac{\partial Q}{\partial \xi_i}(\tilde{y}, \tilde{\xi}), \\ \\ \dfrac{d\tilde{\xi}_i}{ds} = -\dfrac{\partial Q}{\partial y_i}(\tilde{y}, \tilde{\xi}) + \Big(\displaystyle\sum_{j=0}^{N} \dfrac{\partial Q}{\partial y_j}(\tilde{y}, \tilde{\xi})\tilde{\xi}_j\Big)\tilde{\xi}_i. \end{cases} \qquad 0 \leq i \leq N, \tag{3.43}$$

The initial conditions $(y^\sharp, \xi^\sharp \stackrel{\text{def}}{\equiv} \zeta^\sharp/|\zeta^\sharp|)$ to (3.43) satisfy

$$\begin{cases} \xi^\sharp \in S^N, \\ Q(y^\sharp, \xi^\sharp) = 0. \end{cases}$$

We set

$$\begin{cases} y^* = y^\sharp, \\ \xi^* = \xi^\sharp, \end{cases}$$

recall (3.36) in Remark 3.8, and conclude that

$$\begin{cases} \overline{y}(s) = \tilde{y}(s), \\ \overline{\xi}(s) = \tilde{\xi}(s), \end{cases}$$

whenever (y^*, ξ^*) satisfy (3.37). Note however that systems (3.34) and (3.42) (which both live on $\mathbb{R}_y^{N+1} \times S_\xi^N$) differ by a constant coefficient ($N+2$ versus 1) in the first equation; but the accompanying term vanishes in both (3.34) and (3.42) whenever the initial conditions (y^*, ξ^*) satisfy $Q(y^*, \xi^*) = 0$.

We have thus established that the projections $(\tilde{y}(s), \tilde{\xi}(s))$ on $\mathbb{R}_y^{N+1} \times S_\xi^N$ of the integral curves of (3.38) are precisely the integral curves of (3.34), along which the measure $\xi_0 \nu$ is transported, whenever the initial conditions $(y^\sharp, \zeta^\sharp)$ to (3.38) satisfy $\zeta^\sharp \neq 0$, $Q(y^\sharp, \zeta^\sharp) = 0$.

In the sequel the integral curves of (3.38) will be referred to, somewhat inappropriately, as the "bicharacteristic strips". □

Remark 3.10. This equally long remark is concerned with the initial conditions associated to the transport equation (3.29). It demonstrates that the knowledge of the trace of ν at time $t = 0$ (a meaningful concept according to Remark 3.6) will permit through application of Lemma 3.2 to recover all of the measure ν. In other words any point in the support of ν can be reached by one (and only one) integral curve of (3.34)-(3.35) and that integral curve intersects the hyperplane $y_0 = t = 0$.

Indeed let $(\hat{y}, \hat{\xi})$ be any point in $\mathbb{R}_y^{N+1} \times S_\xi^N$ in the support of ν. Then, according to (3.17),

$$Q(\hat{y}, \hat{\xi}) = 0. \tag{3.44}$$

We claim that any point $(\hat{y}, \hat{\xi})$ satisfying (3.44) can be reached by an integral curve of (3.34)-(3.35), which is obvious, but also that such an integral curve will intersect the hyperplane $y_0 = t = 0$, which is not so obvious. To this effect the unique integral curve to (3.34) with initial condition

$$\overline{y}(s = 0) = \hat{y} \quad , \quad \overline{\xi}(s = 0) = \hat{\xi}, \tag{3.45}$$

is considered. It satisfies, by virtue of Remark 3.8 and (3.44)

$$
(3.46)\quad \begin{cases} \dfrac{d\overline{y}_0}{ds}(s) = \dfrac{\partial Q}{\partial \xi_0}(\overline{y}(s),\overline{\xi}(s)) = \rho(\overline{x}(s))\overline{\xi}_0(s), \\ \dfrac{d\overline{\xi}_0}{ds}(s) = -\dfrac{\partial Q}{\partial y_0}(\overline{y}(s),\overline{\xi}(s)) + \Big(\displaystyle\sum_{j=0}^{N} \dfrac{\partial Q}{\partial y_j}(\overline{y}(s),\overline{\xi}(s))\overline{\xi}_j(s)\Big)\overline{\xi}_0(s) \\ \qquad = \Big(\displaystyle\sum_{j=0}^{N} \dfrac{\partial Q}{\partial y_j}(\overline{y}(s),\overline{\xi}(s))\overline{\xi}_j(s)\Big)\overline{\xi}_0(s), \end{cases}
$$

where $\overline{y}(s) = (\overline{t}(s), \overline{x}(s))$.

The second equation of (3.46) implies that $\overline{\xi}_0(s)$ has a constant sign which implies in turn, upon inspection of the first equation of (3.46), that $\overline{y}_0(s)$ is monotone. We shall prove that

$$
(3.47)\qquad |\overline{\xi}_0(s)| \quad \text{is bounded away from } 0,
$$

which implies that

$$
|\overline{y}_0(s)| \overset{s\to\pm\infty}{\longrightarrow} +\infty.
$$

Thus there exists a value of s for which $\overline{y}_0(s) = \overline{t}(s) = 0$.

It remains to prove (3.47). A sequence s_n with

$$
\begin{cases} |s_n| \to +\infty, \\ |\overline{\xi}_0(s_n)| \to i \overset{\text{def}}{\equiv} \underline{\lim}_{s\in\mathbb{R}} |\xi_0(s)|, \end{cases}
$$

is considered. At the possible expense of extracting a subsequence (still indexed by n) we may assume that

$$
(3.48)\qquad \begin{cases} \overline{\xi}(s_n) \to \xi^\infty & \text{in } S^N, \\ \rho(\overline{x}(s_n)) \to \rho^\infty & \text{in } \mathbb{R}, \\ A(\overline{x}(s_n)) \to A^\infty & \text{in } \mathbb{R}^{N^2}. \end{cases}
$$

Since the integral curve of (3.34)-(3.45) satisfies

$$
Q(\overline{y}(s), \overline{\xi}(s)) = 0,
$$

we are at liberty to pass to the limit in

$$
Q(\overline{y}(s_n), \overline{\xi}(s_n)) = 0.
$$

Recalling the convergences (3.48) we obtain

$$\begin{cases} (\xi_0^\infty)^2 + \sum_{i=1}^{N}(\xi_i^\infty)^2 = 1, \\ \rho^\infty(\xi_0^\infty)^2 - \sum_{i,j=1}^{N} a_{ij}^\infty \xi_i^\infty \xi_j^\infty = 0, \end{cases}$$

which proves that $i = |\xi_0^\infty| \neq 0$. Note that the boundedness properties of $\rho(x)$ and $A(x)$ have been implicitly used in the second and third convergences of (3.48). □

As a conclusion to Paragraph 3.2.1, we would like to emphasize the contrast with geometrical optics, a contrast most striking at this point of the argument in our opinion. To solve the eiconal equation one must transport the value of the initial phase along the projections of the bicharacteristics in physical space (cf. Remarks 2.3, 2.7) whereas the transport equation for the H-measure is solved through a simple transport of the (meaningful) value of the H-measure along the projections of the "bicharacteristics strips" on $\mathbb{R}_y^{N+1} \times S_\xi^N$. The latter projections are well defined for all "times" s. In all fairness, it should be observed that the idea of integrating the eiconal equation and the transport equation in the whole phase space -without projection on the physical space- is not new. It is precisely at the root of Maslov's ansatz (cf. e.g. []). The present approach does not however require the introduction of Maslov's index because only the modulus of the amplitude is needed here.

3.2.2. The initial condition for ν.

It has been established in Remark 3.5 that $\partial Q/\partial\tau\ \tau\nu$ has a trace at $t = 0$ which is defined through $\ll \tau\nu, \{\phi, Q\} \gg$. Our goal in the present paragraph is to compute this trace from the only knowledge of the H-measure associated to the initial conditions for (3.1). Once (and if) such a task is completed, a full description of the H-measure ν will be achieved according to Remark 3.10.

Our method follows premises similar to those of the method undertaken by L. TARTAR in Section 3.4 of [15] in the different case of a scalar transport equation. To this effect we are forced to revisit the proof of Theorem 4 (or more exactly of (3.18)) -proof which can be found in [15]- and to avail ourselves of slightly different test functions. Specifically we consider throughout this paragraph an element P of ψ_0 $(\mathbb{R}^{N+1};\mathbb{R})$ with *y-independent* principal symbol and apply it to (3.1). Its symbol $p(y,\xi)$ is defined as follows

$$\begin{cases} p^0(\xi) = p^0\Big(\frac{\eta}{\sqrt{\tau^2+|\eta|^2}}\,,\,\frac{\tau}{\sqrt{\tau^2+|\eta|^2}}\Big), \\ p(y,\xi) = p^0(\xi)\,\tilde{\chi}(\tau,\eta)(= p^0(\xi)\,\tilde{\chi}(\xi_0,...,\xi_n)), \end{cases}$$

where

$$p^0 \in \mathcal{C}^\infty(S^N), \tag{3.49}$$

and

(3.50)

$$\begin{cases} \tilde{\chi}(\tau,\eta) \in \mathcal{C}^\infty(\mathbb{R}^{N+1}), 0 \le \tilde{\chi}(\tau,\eta) \le 1, \tilde{\chi}(\tau,\eta) = 0 \text{ around } (\tau,\eta) = 0, \\ \qquad \tilde{\chi}(\tau,\eta) = 1 \text{ if } \tau^2 + |\eta|^2 \text{ is large enough}, \\ \chi(s) \in \mathcal{C}_0^\infty(\mathbb{R})\,,\ 0 \le \chi(s) \le 1\,,\ \chi(s) = 1 \text{ around } s = 0. \end{cases}$$

The function $\chi(s)$ will be of use in the proof of Lemma 3.3.

Since the symbol of P does not depend upon $y = (t,x)$ P commutes with the various differentiations, *i.e.*,

$$\frac{\partial}{\partial y_i} Pu = P\Big(\frac{\partial u}{\partial y_i}\Big) \quad , \quad 0 \le i \le N.$$

Applying $P \circ \theta$ (where $\theta(t)$ was defined in (3.13)) to the first equation (3.1) yields

(3.51)

$$\frac{\partial}{\partial t}\Big(\rho(x)P(\theta\frac{\partial v^\varepsilon}{\partial t})\Big) - \sum_{i,j=1}^N \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\,P(\theta\frac{\partial v^\varepsilon}{\partial x_j})\Big)$$
$$- P\Big(\rho\frac{\partial\theta}{\partial t}\,\frac{\partial v^\varepsilon}{\partial t}\Big) + \frac{\partial}{\partial t}\Big([P,\rho](\theta\frac{\partial v^\varepsilon}{\partial t})\Big) - \sum_{i,j=1}^N \frac{\partial}{\partial x_i}\Big([P,a_{ij}](\theta\frac{\partial v^\varepsilon}{\partial x_j})\Big) = 0,$$

where $[P,Q]$, the commutator, has been defined in Lemma 3.1 as $PQ - QP$. Note that, appealing to Lemma 3.1,

$$\begin{cases} K_0 \overset{\text{def}}{\equiv} \dfrac{\partial}{\partial t} \circ [P, \rho] = \dfrac{\partial}{\partial y_0} \circ [P, \rho], \\ K_j \overset{\text{def}}{\equiv} \displaystyle\sum_{i=1}^{N} \dfrac{\partial}{\partial x_i} \circ [P, a_{ij}] = \sum_{i=1}^{N} \dfrac{\partial}{\partial y_i} \circ [P, a_{ij}]\ ,\ 1 \le j \le N, \end{cases}$$

are elements of $\psi^0(\mathbb{R}^{N+1}; \mathbb{R})$ with respective principal symbols $k_0^0(y,\xi)$ and $k_j^0(y,\xi)$ given (by virtue of (3.12)) by

$$\begin{cases} k_0^0(y,\xi) = \xi_0 \displaystyle\sum_{i=1}^{N} \dfrac{\partial p^0}{\partial \xi_i} \dfrac{\partial \rho}{\partial x_i}, \\ k_j^0(y,\xi) = \displaystyle\sum_{i,n=1}^{N} \xi_i \dfrac{\partial p^0}{\partial \xi_n} \dfrac{\partial a_{ij}}{\partial x_n}, \quad 1 \le j \le N. \end{cases} \tag{3.52}$$

Equation (3.51) is in turn multiplied by $\overline{P(\theta \partial v^\varepsilon / \partial t)}$. Notation (3.13) is recalled and a straighforward computation leads to

$$\begin{aligned} &\frac{1}{2} \frac{\partial}{\partial t} \{\rho(x)|PV_0^\varepsilon|^2 + \sum_{i,j=1}^{N} a_{ij}(x) PV_i^\varepsilon\, \overline{PV_j^\varepsilon}\} \\ &- \{P(\rho \frac{\partial \theta}{\partial t} \frac{\partial v^\varepsilon}{\partial t}) \overline{PV_0^\varepsilon} - \sum_{i,j=1}^{N} a_{ij}(x)\, PV_i^\varepsilon\, \overline{P(\frac{\partial \theta}{\partial t} \frac{\partial v^\varepsilon}{\partial x_j})}\} \\ &- Re\{\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}(a_{ij}(x)\, PV_j^\varepsilon\, \overline{PV_0^\varepsilon})\} \\ &- Re\{(-K_0\, V_0^\varepsilon + \sum_{j=1}^{N} K_j\, V_j^\varepsilon) \overline{PV_0^\varepsilon}\} = 0. \end{aligned} \tag{3.53}$$

Multiplication of (3.53) by an arbitrary element φ of $\mathcal{C}_0^\infty(\mathbb{R}_x^N)$, integration over $\mathbb{R}_x^N$ and appropriate integration by parts is performed. Upon defining

$$\mathcal{R}^\varepsilon(t) = \frac{1}{2}\{\int_{\mathbb{R}^N} \rho(x)|PV_0^\varepsilon|^2 + \sum_{i,j=1}^{N} a_{ij}(x)\, PV_i^\varepsilon\, \overline{PV_j^\varepsilon}) \varphi\, dx\}(t), \tag{3.54}$$

we obtain, for every t in $\mathbb{R}$,

(3.55)

$$\frac{d\mathcal{R}^\varepsilon(t)}{dt} + Re\{\int_{\mathbb{R}^N} \Big(\sum_{i,j=1}^{N} a_{ij}(x) PV_j^\varepsilon \overline{PV_0^\varepsilon} \frac{\partial \varphi}{\partial x_j}\Big) dx\}(t)$$

$$- Re\{\int_{R^N}\Big(P(\rho\frac{\partial\theta}{\partial t}\frac{\partial v^\varepsilon}{\partial t})\overline{PV_0^\varepsilon} - \sum_{i,j=1}^N a_{ij}(x)PV_i^\varepsilon\overline{P(\frac{\partial\theta}{\partial t}\frac{\partial v^\varepsilon}{\partial x_j})}\Big)\varphi\, dx\}(t)$$

$$- Re\{\int_{\mathbb{R}^N}(-K_0 V_0^\varepsilon + \sum_{j=1}^N K_j V_j^\varepsilon)\overline{PV_0^\varepsilon}\varphi\, dx\}(t) = 0.$$

All quantities entering (3.55) (including $\mathcal{R}^\varepsilon$) are well defined in view of the smoothness of the solution v^ε to (3.1)-(3.3). The various integrations by part leading to (3.55) are also licit as a result of that smoothness.

Remark 3.11. The remark is aimed at proving that, at the possible expense of the extraction of a subsequence,

$$\mathcal{R}^\varepsilon(t) \xrightarrow{\varepsilon\to 0} \mathcal{R}(t), \tag{3.56}$$

uniformly on any compact interval of $\mathbb{R}$.

Indeed define

$$\mathcal{D}^\varepsilon(t) = \frac{1}{2}\{\int_{\mathbb{R}^N}(\rho(x)|PV_0^\varepsilon|^2 + \sum_{i,j=1}^N a_{ij}(x)PV_i^\varepsilon\,\overline{PV_j^\varepsilon}\, dx\}(t).$$

The energy estimate (3.4), the compactness of the support of $\theta(t)$, and the boundedness properties of the various elements of $\psi^0(\mathbb{R}^{N+1};\mathbb{R})$ entering formula (3.53) easily imply, upon integration of (3.53) over $\mathbb{R}^N_x$, that

$$\frac{d\mathcal{D}^\varepsilon}{dt}(t) \quad \text{is bounded in } L^1(\mathbb{R}),$$

independently of ε. Furthermore $\mathcal{D}^\varepsilon$ is bounded in $L^1(\mathbb{R})$ by its very definition. Thus $\mathcal{D}^\varepsilon$ is a bounded sequence of absolutely continuous functions, which implies that

$$\mathcal{D}^\varepsilon \quad \text{is bounded in } L^\infty(\mathbb{R}),$$

independently of ε ; therefore

$$PV_i^\varepsilon(t) \quad (0 \le i \le N) \quad \text{is bounded in } L^\infty(\mathbb{R}; L^2(\mathbb{R}^N_x)), \tag{3.57}$$

independently of ε.

By virtue of estimate (3.57), an argument similar to that used above for $\mathcal{D}^\varepsilon$ demonstrates, upon consideration of (3.55), that

$$\mathcal{R}^\varepsilon \quad \text{is bounded in } H^1(\mathbb{R}),$$

independently of ε, which yields (3.56). □

We now multiply (3.55) by an arbitrary function $\psi(t)$ in $C_0^\infty([0,T))$ and perform the integration by parts on $[0,T]$. We obtain

(3.58)

$$\begin{aligned}
&-\psi(0)\mathcal{R}^\varepsilon(0) - \frac{1}{2}\int_0^T \int_{\mathbb{R}^N} \Big(\rho(x)|PV_0^\varepsilon|^2 + \sum_{i,j=1}^N a_{ij}(x) PV_i^\varepsilon\, \overline{PV_j^\varepsilon}\Big)\varphi \frac{\partial \psi}{\partial t} dx\, dt \\
&+ Re\{\int_0^T \int_{\mathbb{R}^N} \Big(\sum_{i,j=1}^N a_{ij}(x) PV_j^\varepsilon\, \overline{PV_0^\varepsilon} \frac{\partial \varphi}{\partial x_i}\psi\Big) dx\, dt\} \\
&+ Re\{\int_0^T \int_{\mathbb{R}^N} \Big(P(\rho \frac{\partial \theta}{\partial t}\, \frac{\partial v^\varepsilon}{\partial t})\overline{PV_0^\varepsilon} - \sum_{i;j=1}^N a_{ij}(x) PV_i^\varepsilon\, \overline{P(\frac{\partial \theta}{\partial t}\, \frac{\partial v^\varepsilon}{\partial x_j})}\Big)\varphi\, \psi\, dx\, dt\} \\
&- Re\{\int_0^T \int_{\mathbb{R}^N} (- V\, V_0^\varepsilon + \sum_{j=1}^N K_j\, V_j^\varepsilon)\overline{PV_0^\varepsilon}\, \varphi\, \psi\, dx\, dt\} = 0.
\end{aligned}$$

We now propose to pass to the limit in each of the terms entering (3.58) as ε tends to zero. At this point of the argument the difference with the proof of (3.18) in Theorem 3.3 lies in the non compact character of the support of ψ in $(0,T)$, which explains the presence of the non zero term $\psi(0)\mathcal{R}^\varepsilon(0)$.

By virtue of (3.56) the first term of (3.58) tends to $-\psi(0)\mathcal{R}(0)$.

As ε tends to zero the fourth term in (3.58) tends to zero. Indeed, for example,

$$P\Big(\rho \frac{\partial \theta}{\partial t}\, \frac{\partial v^\varepsilon}{\partial t}\Big) = \frac{\partial \theta}{\partial t}\, P\Big(\rho \frac{\partial v^\varepsilon}{\partial t}\Big) + [P, \frac{\partial \theta}{\partial t}]\Big(\rho \frac{\partial v^\varepsilon}{\partial t}\Big).$$

Since $\partial\theta/\partial t$ is identically zero on $[0,T]$ the first term in the above formula does not contribute to the computation. The second term in that formula converges strongly to zero in $L^2(\mathbb{R}^{N+1})$ because it is the result of the application of the commutator $[P, \partial\theta/\partial t]$ -an element of $\psi^{-1}(\mathbb{R}^{N+1};\mathbb{R})$ according to Lemma 3.1 -to a bounded sequence in $L^2(\mathbb{R}^{N+1})$. The reader will undoubtedly object that $\rho\, \partial v^\varepsilon/\partial t$ is only bounded in $L^\infty(\mathbb{R}; L^2(\mathbb{R}^N))$ but this latter obstacle is alleviated through a convenient rewriting of $(\partial\theta/\partial t)$ $(\partial v^\varepsilon/\partial t)$ as

$(\partial\theta/\partial t)\ \hat{\theta}\ (\partial v^\varepsilon/\partial t)$, with $\hat{\theta}$ in $C_0^\infty(\mathbb{R})$ and $\hat{\theta} \equiv 1$ on the support of $\partial\theta/\partial t$, in the previous formula.

The three remaining terms in (3.58) will be handled in an identical manner. We illustrate the process on the first part of the second term, namely,

$$\int_0^T \int_{\mathbb{R}^N} \rho(x)|PV_0^\varepsilon|^2 \varphi \frac{\partial\psi}{\partial t}\, dx\, dt,$$

which, upon appealing to the cut-off function ψ_δ introduced in (3.32), reads as

$$\int_0^T \int_{\mathbb{R}^N} \rho(x)\,|PV_0^\varepsilon|^2 \varphi \frac{\partial\psi}{\partial t}\,\psi_\delta\, dx\, dt + \int_0^T \int_{\mathbb{R}^N} \rho(x)|PV_0^\varepsilon|^2\, \varphi \frac{\partial\psi}{\partial t}(1-\psi_\delta) dx\, dt.$$

In view of estimate (3.57) and since $1-\psi_\delta$ vanishes identically outside $(0,\delta)$ the second term in the above expression is at most of the order of δ independently of ε.

For a fixed δ, $\varphi\,\psi_\delta$ is compactly supported in $(0,T)\times\mathbb{R}_x^N\times S_\xi^N$ and a direct application of Theorem 3.1 and of Lemma 3.1 (especially (3.11)) yields

$$< \nu, \rho(x)\tau^2 |p^0(\tau,\eta)|^2\, \varphi \frac{\partial\psi}{\partial t}\,\psi_\delta >$$

as limit of the first term when ε tends to zero. But, recalling (3.30) in Remark 3.5,

$$\lim_{\delta\to 0} < \nu, \rho(x)\tau^2 |p^0(\tau,\eta)|^2\, \varphi \frac{\partial\psi}{\partial t}\,\psi_\delta > = \ll \nu, \rho(x)\tau^2 |p^0(\tau,\eta)|^2\, \varphi \frac{\partial\psi}{\partial t} \gg,$$

and thus

$$\lim_{\delta\to 0} \int_0^T \int_{\mathbb{R}^N} \rho(x)|PV_0^\varepsilon|^2\, \varphi \frac{\partial\psi}{\partial t}\, dx\, dt$$
$$= \ll \nu, \rho(x)\tau^2 |p^0(\tau,\eta)|^2\, \varphi \frac{\partial\psi}{\partial t} \gg .$$

Upon passing to the limit in the remaining terms of (3.58) and collecting the resulting expressions we obtain

$$-\psi(0)\mathcal{R}(0) - \frac{1}{2} \ll \nu, (\rho\tau^2 + \sum_{i,j=1}^N a_{ij}(x)\eta_i\,\eta_j)\,|p^0(\tau,\eta)|^2\, \varphi \frac{\partial\psi}{\partial t} \gg$$

$$+ \ll \nu, \Big(\sum_{i,j=1}^N a_{ij}(x)\eta_j\,\psi \frac{\partial\varphi}{\partial x_i}\Big)\tau\,|p^0(\tau,\eta)|^2 \gg \tag{3.59}$$

$$- \ll \nu, (-\tau\, k_0^0 + \sum_{j=1}^{N} \eta_j\, k_j^0)\tau\, p^0(\tau,\eta)\varphi\psi \gg = 0.$$

By virtue of (3.17), (3.19), we have

$$< \nu, \rho(x)\tau^2 |p^0(\tau,\eta)|^2\, \varphi\, \frac{\partial\psi}{\partial t}\, \psi_\delta > \doteq < \nu, \sum_{i,j=1}^{N} a_{ij}(x)\eta_i\,\eta_j |p^0(\tau,\eta)|^2 \varphi\, \frac{\partial\psi}{\partial t}\psi_\delta >,$$

which yields

$$\ll \nu, \rho(x)\tau^2 |p^0(\tau,\eta)|^2\, \varphi\, \frac{\partial\psi}{\partial t} \gg = \ll \nu, \sum_{i,j=1}^{N} a_{ij}(x)\eta_i\eta_j |p^0(\tau,\eta)|^2\, \varphi\, \frac{\partial\psi}{\partial t} \gg,$$

as δ tends to zero. Using the above identity together with (3.52) as far as the last term of (3.59) is concerned we obtain

$$\begin{aligned} & - \psi(0)\mathcal{R}(0) - \ll \tau\nu, \rho(x)\tau \frac{\partial}{\partial t}(|p^0(\tau,\eta)|^2\psi\varphi) \gg \\ (3.60) \quad & + \ll \tau\nu, \sum_{i,j=1}^{N} a_{ij}\eta_j \frac{\partial}{\partial x_i}(|p^0(\tau,\eta)|^2\psi\varphi) \gg \\ & + \frac{1}{2} \ll \tau\nu, \sum_{\ell=1}^{N}\Big(\frac{\partial\rho}{\partial x_\ell}\tau^2 - \sum_{i,j=1}^{N} \frac{\partial a_{ij}}{\partial x_\ell}\,\eta_i\,\eta_j\Big)\frac{\partial}{\partial \eta_\ell}(|p^0(\tau,\eta)|^2\psi\varphi) \gg = 0. \end{aligned}$$

In view of the definition (3.19) of Q, (3.60) may be expressed as

$$(3.61) \qquad \psi(0)\mathcal{R}(0) = \ll \tau\nu, \{\phi, Q\} \gg,$$

with

$$(3.62) \qquad \phi(t,x,\xi) = |p^0(\tau,\eta)|^2\psi(t)\varphi(t),$$

(ϕ is defined on $\mathbb{R}_y^{N+1} \times S_\xi^N$ on which $\tau^2 + |\eta|^2 = 1$).

It remains to compute the quantity $\mathcal{R}(0)$. The computation is burdened by technical difficulties. It would be tempting to appeal to (3.3), replace $V_0^\varepsilon, V_i^\varepsilon$ $(1 \le i \le N)$ in the expression (3.54) for $\mathcal{R}^\varepsilon(t)$ taken at $t = 0$ by β^ε and grad γ^ε respectively and pass to the limit in the resulting expression which would then involve a different family of H-measures, namely those associated with the initial data. Such a procedure is however incorrect since the quantities that enter (3.54) are the traces of PV_i^ε $(0 \le i \le N)$ at $t = 0$ and are not equal to the image of the traces of V^ε (*i.e.*, β^ε and grad γ^ε) under P, which is by the way a meaningless notion since P is a pseudo-differential operator acting on functions of $N + 1$ variables.

This difficulty is circumvened through an adequate rewriting of $\mathcal{R}^\varepsilon(t)$ as

$$\mathcal{R}^\varepsilon(t) = \tilde{Q}^\varepsilon(t) + \mathcal{T}^\varepsilon(t), \tag{3.63}$$

with $\tilde{Q}^\varepsilon$ defined in (3.78) below.

Further it will be proved by virtue of (3.73), (3.79) and Remark 3.13 below that

$$\begin{cases} \mathcal{T}^\varepsilon \rightharpoonup 0 \quad \text{in } \mathcal{D}'(\mathbb{R}), \\ \tilde{Q}^\varepsilon_{(t)} \to \tilde{Q}_{(t)} \quad \text{uniformly on any compact interval of } \mathbb{R}, \end{cases} \tag{3.64}$$

as ε tends to zero.

With the help of (3.64), $\mathcal{R}(0)$ will be identified as the limit $\tilde{Q}(0)$ of $\tilde{Q}^\varepsilon(0)$; that limit can in turn be explicitly computed (cf. (3.93)) in terms of the initial conditions β^ε and γ^ε.

To this effect we remark that the ε-independent compactness of the support of the initial conditions together with the property of finite speed of propagation of the solution v^ε to (3.1)-(3.2) (cf. e.g. [11], Theorem 6.10, p. 364) implies that

$$\theta v^\varepsilon \text{ is supported in a compact subset } K \text{ of } \mathbb{R}^{N+1}_{x,t}, \tag{3.65}$$

where θ has been defined in (3.13), and K is of the form

$$K = [T_1, T_2] \times K',$$

where $-\infty < T_1 < T_2 < +\infty$ and K' is a compact subset of $\mathbb{R}^N_x$ (which contains K^0 the support of the initial conditions β^ε and γ^ε).

We then consider ζ to be a non negative element of $C_0^\infty(\mathbb{R}^N_x)$ *with* $\zeta(x) \equiv 1$ *on* K' and introduce the pseudo-differential operator Λ of order 1 (element of $\psi^1_c(\mathbb{R}^N; \mathbb{R})$) with associated principal symbol

$$\lambda(x,\eta) = \zeta(x)\Big(\sum_{i,j=1}^{N} a_{ij}(x)\eta_i\eta_j\Big)^{1/2} = \zeta(x)(A(x)\eta\cdot\eta)^{1/2}.$$

Note that although $\lambda(x,\eta)$ is not $\mathcal{C}^\infty$ at $\eta = 0$, $\lambda(x,\eta)$ can be viewed as a *bona fide* principal symbol (in the sense of (3.8)) since it is to be multiplied by a cut-off function around $\eta = 0$. Note also that Λ maps $H^1(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$. We then set

$$\text{(3.66)}\qquad \begin{cases} v_+^\varepsilon \equiv \zeta(x)\sqrt{\rho(x)}\,\dfrac{\partial v^\varepsilon}{\partial t} - i\Lambda v^\varepsilon, \\ v_-^\varepsilon \equiv \zeta(x)\sqrt{\rho(x)}\,\dfrac{\partial v^\varepsilon}{\partial t} + i\Lambda v^\varepsilon. \end{cases}$$

Let us emphasize that the projection on $\mathbb{R}_x^N$ of the supports of $v_\pm^\varepsilon$ are compact.

Since Λ only acts on the spatial directions the energy estimate (3.4) immediately implies that

$$\text{(3.67)}\qquad v_+^\varepsilon \text{ (resp. } v_-^\varepsilon) \rightharpoonup 0 \quad \text{in } L^\infty(\mathbb{R}_t; L^2(\mathbb{R}_x^N)),$$

as ε tend to zero.

Further it is easily checked through application of Lemma 3.1 that v_+^ε and v_-^ε respectively satisfy

$$\text{(3.68)}\qquad \begin{cases} \zeta(x)\sqrt{\rho(x)}\,\dfrac{\partial v_+^\varepsilon}{\partial t} + i\Lambda v_+^\varepsilon = \mathcal{R}_0\,\dfrac{\partial v^\varepsilon}{\partial t} - \displaystyle\sum_{i=1}^{N} \mathcal{R}_i\,\dfrac{\partial v^\varepsilon}{\partial x_i} + \mathcal{R}' v^\varepsilon, \\ \zeta(x)\sqrt{\rho(x)}\,\dfrac{\partial v_-^\varepsilon}{\partial t} - i\Lambda v_-^\varepsilon = -\mathcal{R}_0\,\dfrac{\partial v^\varepsilon}{\partial t} - \displaystyle\sum_{i=1}^{N} \mathcal{R}_i\,\dfrac{\partial v^\varepsilon}{\partial x_i} + \mathcal{R}' v^\varepsilon, \end{cases}$$

where

$$\mathcal{R}_0 = [\zeta\sqrt{\rho}, -i\Lambda], \mathcal{R}_i = \zeta^2\sum_{j=1}^{N}[a_{ij}, \frac{\partial}{\partial x_j}]\,,\ 1 \le i \le N,$$

are elements of $\psi_c^0(\mathbb{R}_x^N;\mathbb{R})$ and $\mathcal{R}'$ is an element of $\psi_c^1(\mathbb{R}_x^N;\mathbb{R})$. Since once again the $\mathcal{R}_i$'s and $\mathcal{R}'$ only act on spatial directions, the energy estimate (3.4) permits us to rewrite (3.68) as

$$\text{(3.69)}\qquad \begin{cases} \zeta(x)\sqrt{\rho(x)}\,\dfrac{\partial v_+^\varepsilon}{\partial t} + i\Lambda v_+^\varepsilon = r_+^\varepsilon, \\ \zeta(x)\sqrt{\rho(x)}\,\dfrac{\partial v_-^\varepsilon}{\partial t} - i\Lambda v_-^\varepsilon = r_-^\varepsilon, \end{cases}$$

with r_+^ε and r_-^ε bounded in $L^\infty(\mathbb{R}_t; L^2(\mathbb{R}_x^N))$ independently of ε.

Finally upon recalling (3.1) the initial conditions for v_+^ε and v_-^ε are

$$\begin{cases} v_+^\varepsilon(0) = \zeta(x)\sqrt{\rho(x)}\,\dfrac{\partial v^\varepsilon}{\partial t}(0) - i\Lambda(v^\varepsilon(0)) \\ \qquad = \zeta(x)\sqrt{\rho(x)}\beta^\varepsilon - i\Lambda\gamma^\varepsilon \equiv v_{+0}^\varepsilon, \\ v_-^\varepsilon(0) = \zeta(x)\sqrt{\rho(x)}\,\dfrac{\partial v^\varepsilon}{\partial t}(0) + i\Lambda(v^\varepsilon(0)) \\ \qquad = \zeta(x)\sqrt{\rho(x)}\beta^\varepsilon + i\Lambda\gamma^\varepsilon \equiv v_{-0}^\varepsilon, \end{cases} \tag{3.70}$$

and in view of (3.3) v_{+0}^ε and v_{-0}^ε converge weakly in $L^2(\mathbb{R}_x^N)$ to zero as ε tends to zero.

Define for $\theta(t)$ given in (3.13) ($\theta(t) \equiv 1$ on on $[0,T]$) and for the operator P considered at the beginning of Paragraph 3.2.2

$$\begin{cases} Q_+^\varepsilon(t) = \dfrac{1}{4}[\displaystyle\int_{\mathbb{R}^N} (|P(\theta v_+^\varepsilon)|^2\,\varphi\, dx](t), \\ Q_-^\varepsilon(t) = \dfrac{1}{4}[\displaystyle\int_{\mathbb{R}^N} (|P(\theta v_-^\varepsilon)|^2\,\varphi\, dx](t), \\ Q^\varepsilon(t) = Q_+^\varepsilon(t) + Q_-^\varepsilon(t). \end{cases} \tag{3.71}$$

For any smooth function ψ in $C_0^\infty(\mathbb{R})$

$$\begin{aligned} \int_{\mathbb{R}} Q^\varepsilon(t)\,\psi(t)dt &= \frac{1}{4}\int_{\mathbb{R}^{N+1}} (|P(\theta v_+^\varepsilon)|^2 + |P(\theta v_-^\varepsilon)|^2)\varphi(x)\psi(t)dx\,dt \\ &= \frac{1}{4}\int_{\mathbb{R}^{N+1}} \{|((\zeta\sqrt{\rho}\,PV_0^\varepsilon - i\Lambda P(\theta v^\varepsilon)) \\ &\quad - [\zeta\sqrt{\rho}\frac{\partial}{\partial t} - i\Lambda, P](\theta v^\varepsilon) + (\mathcal{R}\circ\theta)v^\varepsilon|^2 \\ &\quad + |((\zeta\sqrt{\rho}PV_0^\varepsilon + i\Lambda P(\theta v^\varepsilon)) - [\zeta\sqrt{\rho}\frac{\partial}{\partial t} + i\Lambda, P](\theta v^\varepsilon) \\ &\quad + (\mathcal{R}'\circ\theta)v^\varepsilon|^2\}\varphi(x)\psi(t)dx\,dt \\ &= \frac{1}{2}\int_{\mathbb{R}^{N+1}} (\rho(x)|PV_0^\varepsilon|^2 + \sum_{i,j=1}^N a_{ij}(x)PV_i^\varepsilon\,\overline{PV_j^\varepsilon})\varphi(x)\psi(t)dx\,dt \\ &\quad + \int_{R^{N+1}} v^\varepsilon\overline{(\mathcal{R}''\circ\theta)v^\varepsilon}\varphi\psi\,dx\,dt, \end{aligned} \tag{3.72}$$

where $\mathcal{R}, \mathcal{R}'$, are elements of $\psi_c^0(\mathbb{R}^{N+1};\mathbb{R})$ and $\mathcal{R}''$ is an element of $\psi^1(\mathbb{R}^{N+1};\mathbb{R})$ obtained through multiple use of Lemma 3.1 in the second and third equalities of (3.72). Note that we have implicitly used the fact that $\zeta \equiv 1$ on the support of $\theta\,\partial v^\varepsilon/\partial y_i$ $(0 \le i \le N)$ in the last equality of

(3.72). The reader should also be aware that commutators such as $[-i\Lambda, P]$ are not totally meaningful since Λ is an operator acting on $\mathbb{R}^N_x$, while P acts on $\mathbb{R}^{N+1}_{x,t}$. A completely rigorous proof of (3.72) would involve an argument of the type used in the proof of (3.85) below : the sequence V^ε is replaced by $\Omega(V^\varepsilon)$ where Ω is a pseudo-differential operator of order 0 on $\mathbb{R}^{N+1}_{x,t}$ which makes $P\Omega$ a pseudo-differential operator on $\mathbb{R}^{N+1}_{x,t}$ (for more details see (3.80)-(3.84) below).

Since $\varphi\psi v^\varepsilon$ is supported in a compact subset of $\mathbb{R}^{N+1}_{x,t}$, (3.4) implies (cf. e.g. [13], Corollary 4) that

$$\varphi\psi v^\varepsilon \to 0 \quad \text{in } \mathcal{C}^0_c(\mathbb{R}_t; L^2(\mathbb{R}^N_x)),$$

as ε tends to zero, while $(\mathcal{R}'' \circ \theta)v^\varepsilon$ converges weakly to 0 in $L^2(\mathbb{R}_t; L^2(\mathbb{R}^N_x))$. The term

$$\int_{R^{N+1}} v^\varepsilon \overline{(\mathcal{R}'' \circ \theta)v^\varepsilon} \varphi\psi \, dx\, dt$$

converges to zero as ε tends to zero. We have thus proved that, for any ψ in $\mathcal{C}^\infty_0(\mathbb{R})$,

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}} \mathcal{R}^\varepsilon(t)\psi(t)dt = \lim_{\varepsilon\to 0} \int_{\mathbb{R}} Q^\varepsilon(t)\psi(t)dt. \tag{3.73}$$

We now introduce $\tilde{P}_+$ and $\tilde{P}_-$ to be elements of $\psi^0(\mathbb{R}^N; \mathbb{R})$ with respective principal symbols

$$\begin{aligned} \tilde{p}^0_+(x,\eta)\ (\text{resp. } \tilde{p}^0_-(x,\eta)) &= p^0(\eta, \mp\frac{\lambda(x,\eta)}{\zeta(x)(\rho(x))^{1/2}}) \\ &= p^0\Big(\eta, \mp\Big(\frac{A(x)\eta\cdot\eta}{\rho(x)}\Big)^{1/2}\Big). \end{aligned} \tag{3.74}$$

Note that because of the various properties of $A(x)$ and $\rho(x)$ it is readily verified that $\tilde{P}_\pm$ belongs to $\psi^0(\mathbb{R}^N; \mathbb{R})$.

Remark 3.12. Defining the mappings $S^\pm$ from $\mathbb{R}^N_x \times S^{N-1}_\eta$ into S^N_ξ as

$$\begin{cases} S^\pm_i(x,\eta) = \Big(\dfrac{\rho(x)}{\rho(x) + A(x)\eta\cdot\eta}\Big)^{1/2} \eta_i \quad , \quad 1 \le i \le N, \\ S^\pm_0(x,\eta) = \pm\Big(\dfrac{A(x)\eta\cdot\eta}{\rho(x) + A(x)\eta\cdot\eta}\Big)^{1/2}, \end{cases} \tag{3.75}$$

for any x in $\mathbb{R}_x^N$ and any η in S_η^{N-1}, it is immediately noted, by reasons of homogeneity, that

$$\tilde{p}_\pm^0(x,\eta) = p^0(S^\mp(x,\eta)). \tag{3.76}$$

□

Because $\tilde{P}_+$ (resp. $\tilde{P}_-$) only acts on the spatial directions (*i.e.*, in (x,η)), application of $\tilde{P}_+$ (resp. $\tilde{P}_-$) to the first (respectively second) equation of (3.69) and consideration of estimate (3.67) yields

$$\zeta(x)\sqrt{\rho(x)}\,\frac{\partial}{\partial t}\,(\tilde{P}_\pm v_\pm^\varepsilon) \pm i\Lambda(\tilde{P}_\pm v_\pm^\varepsilon) = \tilde{r}_\pm^\varepsilon, \tag{3.77}$$

where $\tilde{r}_+^\varepsilon$ and $\tilde{r}_-^\varepsilon$ are bounded in $L^\infty(\mathbb{R}_t; L^2(\mathbb{R}_x^N))$ independently of ε.

Set

$$\begin{cases} \tilde{Q}_+^\varepsilon(t) = \dfrac{1}{4}\displaystyle\int_{\mathbb{R}^N} |\tilde{P}_+(\theta(t)v_+^\varepsilon(t))|^2\varphi(x)dx, \\ \tilde{Q}_-^\varepsilon(t) = \dfrac{1}{4}\displaystyle\int_{\mathbb{R}^N} |\tilde{P}_-(\theta(t)v_-^\varepsilon(t))|^2\varphi(x)dx, \\ \tilde{Q}^\varepsilon(t) = \tilde{Q}_+^\varepsilon(t) + \tilde{Q}_-^\varepsilon(t). \end{cases} \tag{3.78}$$

The definition (3.78) of $\tilde{Q}^\varepsilon$ is similar to that of Q^ε in (3.71). Note however that because the operators $\tilde{P}_\pm$ only act in the spatial directions they are applied in (3.78) to the values at any given time t of the fields $\theta\, v_\pm^\varepsilon$, while, in (3.71), the operator P acts in all direction, so that it has to be applied to $\theta\, v_\pm^\varepsilon$ and the resulting fields $P(\theta\, v_\pm^\varepsilon)$ are then evaluated at time t.

Remark 3.13. In the spirit of Remark 3.11, it can be checked that, at the possible expense of the extraction of a subsequence,

$$\tilde{Q}^\varepsilon(t) \to \tilde{Q}(t) \quad \text{uniformly on any compact interval of } \mathbb{R}.$$

Indeed, in view of (3.67),

$$\tilde{P}_\pm(\theta(t)v_\pm^\varepsilon(t)) \quad \text{is bounded in } L^\infty(\mathbb{R}_t; L^2(\mathbb{R}_x^N)),$$

and thus multiplication of (3.77) by $\zeta^2\,\varphi\theta^2\tilde{P}_{\pm}v^\varepsilon_{\pm}$ will yield an equation of the type (3.55) for $\tilde{Q}^\varepsilon_{\pm}(t)$ from which the equicontinuity will be deduced. □

We now prove that

$$
(3.79)\qquad \begin{cases} \lim\limits_{\varepsilon\to 0}\displaystyle\int_{\mathbb{R}} \tilde{Q}^\varepsilon_{+}(t)\psi(t)dt = \lim\limits_{\varepsilon\to 0}\int_{\mathbb{R}} Q^\varepsilon_{+}(t)\psi(t)dt, \\ \lim\limits_{\varepsilon\to 0}\displaystyle\int_{\mathbb{R}} \tilde{Q}^\varepsilon_{-}(t)\psi(t)dt = \lim\limits_{\varepsilon\to 0}\int_{\mathbb{R}} Q^\varepsilon_{-}(t)\psi(t)dt. \end{cases}
$$

We only prove the first relation in (3.79). The proof of the other equality is identical. To this effect notation (3.13) and definition (3.66) of v^ε_{+} are recalled and the following expression is obtained

$$
\theta v^\varepsilon_{+} = \zeta(x)\sqrt{\rho(x)}\,V^\varepsilon_0 - i\theta'\Lambda(\theta v^\varepsilon),
$$

where θ' is an element of $C_0^\infty(\mathbb{R})$ with $\theta'\equiv 1$ on supp θ.

Since θv^ε is compactly supported in $\mathbb{R}^{N+1}_y$ (cf. (3.65)) it can be expressed as

$$
\theta v^\varepsilon = \Delta_x^{-1}\Big(\sum_{j=1}^{N}\frac{\partial}{\partial x_j}(\zeta\,V^\varepsilon_j)\Big),
$$

where Δ_x^{-1} is to be seen as the inverse of the Laplacian with Dirichlet boundary conditions on a domain large enough to contain the x-projection K' of the common support of the V^ε_j's (and of θv^ε) and ζ has been defined to be identically 1 on K'. Thus θv^ε_{+} reads as

$$
(3.80)\qquad \theta v^\varepsilon_{+} = \zeta(x)\sqrt{\rho(x)}\,V^\varepsilon_0 - i\theta'\sum_{j=1}^{N}\Lambda_j\,V^\varepsilon_j,
$$

where

$$
(3.81)\qquad \Lambda_j \equiv \Lambda\circ\,\Delta_x^{-1}\circ\frac{\partial}{\partial x_j}\circ\zeta.
$$

We now introduce the pseudo-differential operator Ω in $\psi^0(\mathbb{R}^{N+1}_y;\mathbb{R})$ with principal symbol (independent of y)

$$
(3.82)\qquad \omega^0(\xi) = \Big(1-\chi\Big(\frac{|\eta|}{\sqrt{\tau^2+|\eta|^2}}\Big)\Big),
$$

where χ has been defined in (3.50).

Consider a cone $\mathcal{C}_\alpha$ of angle α and direction $\tau = \pm 1$, $\eta = 0$ in $\mathbb{R}^{N+1}$. Then, if $(\eta, \tau) \in \mathbb{R}^{N+1}$ lies outside $\mathcal{C}_\alpha$,

$$\frac{|\eta|}{|\tau|} \geq tg\alpha,$$

in which case

$$\frac{|\eta|}{\sqrt{\tau^2 + |\eta|^2}} \geq \frac{1}{\sqrt{1 + \frac{1}{tg^2\alpha}}} = \sin\alpha.$$

For any (small) α we are at liberty to choose χ in (3.50) such that

$$\chi(s) = \begin{cases} 1 & \text{if } |s| \leq \sin\alpha, \\ 0 & \text{if } |s| \geq \sin 2\alpha, \end{cases}$$

in which case

$$\chi\Big(\frac{|\eta|}{\sqrt{\tau^2 + |\eta|^2}}\Big) = \begin{cases} 1 & \text{if } (\eta, \tau) \in \mathcal{C}_\alpha, \\ 0 & \text{if } (\eta, \tau) \notin \mathcal{C}_{2\alpha}, \end{cases}$$

so that

$$\omega^0(y, \xi) = \begin{cases} 0 & \text{if } (\eta, \tau) \in \mathcal{C}_\alpha, \\ 1 & \text{if } (\eta, \tau) \notin \mathcal{C}_{2\alpha}. \end{cases}$$

By virtue of Remark 3.4 we are in a position to choose α such that $\mathcal{C}_{2\alpha} \cap \operatorname{supp} \nu = \emptyset$ and direct application of Theorem 3.1 and Lemma 3.1 yields, for any φ in $C_0^\infty(\mathbb{R}_y^{N+1})$ and any j with $1 \leq j \leq N$,

$$\|\varphi(1 - \Omega)(V_j^\varepsilon)\|^2_{L^2(\mathbb{R}_y^{N+1})} \overset{\varepsilon \to 0}{\longrightarrow} < \nu, \varphi^2 \chi^2(|\eta|) >= 0,$$

since $\chi \equiv 0$ outside $\mathcal{C}_{2\alpha}$ and $\tau^2 + |\eta|^2 = 1$ on $\operatorname{supp} \nu$. Thus

$$(1 - \Omega)V_j^\varepsilon \to 0 \quad \text{strongly in } L^2_{\text{loc}}(\mathbb{R}_y^{N+1}), \tag{3.83}$$

as ε tends to zero.

Expression (3.80) for θv_+^ε is then rewritten as

$$\theta v_+^\varepsilon = \{\zeta(x)\sqrt{\rho(x)}\, V_0^\varepsilon - i\theta' \sum_{j=1}^{N} \Lambda_j(\Omega V_j^\varepsilon)\} - i\theta' \sum_{j=1}^{N} \Lambda_j((1 - \Omega)V_j^\varepsilon).$$

It is then remarked that $\theta'\Lambda_j \circ \Omega$ may be identified with the pseudo-differential operator of order 0 (an element of $\psi_c^0(\mathbb{R}_y^{N+1};\mathbb{R})$) whose principal symbol is

$$\theta'(t)\,\lambda(x,\eta)(-\frac{i\eta}{|\eta|^2})\zeta(x)\Big(1-\chi\Big(\frac{|\eta|}{\sqrt{\tau^2+|\eta|^2}}\Big)\Big). \tag{3.84}$$

(The above expression is easily verified to be an admissible principal symbol in view of the properties of the support of χ.)

Note however that $\Lambda_j \circ (1-\Omega)$ is *not* a pseudo-differential operator on $\mathbb{R}_y^{N+1}$, but that, because of (3.83) together with the definition (3.81) of Λ_j,

$$\theta'\Lambda_j \circ (1-\Omega)V_j^\varepsilon \to 0 \quad \text{strongly in } L^2(\mathbb{R}_y^{N+1}).$$

(Λ_j, which acts only in the spatial directions, sends boundedly $L^2(\mathbb{R}_t; L^2(\mathbb{R}_x^N))$ into itself.)

Thus the H-measure κ_+ associated to (a subsequence of) θv_+^ε identifies with that of (a subsequence of) $\{\zeta(x)\sqrt{\rho(x)}V_0^\varepsilon - i\theta'\sum_{j=1}^N \Lambda_j(\Omega\, V_j^\varepsilon)\}$ which can in turn be computed from ν through direct application of Theorem 3.1 and Lemma 3.1. We obtain

$$\kappa_+ = |q(y,\xi)|^2\nu,$$

where, in view of (3.84) and the definition of the symbol $\lambda(x,\eta)$,

$$q(y,\xi) = \zeta(x)\sqrt{\rho(x)}\tau - \theta'\zeta^2(x)\sqrt{A(x)\eta\eta}\,(1-\chi(|\eta|)).$$

The sequence V_i^ε $(0 \le i \le N)$ has its support in K (cf. (3.65)) on which ζ and θ' are identically 1; further, according to the localization property (3.17), the support of ν lies in the null set of $Q(x,\xi)$; finally it does not intersect the support $C_{2\alpha}$ of χ. Thus κ_+ also reads as

$$\kappa_+ = |\sqrt{\rho(x)}\tau - \sqrt{A(x)\eta\eta}|^2\nu,$$

and

$$\operatorname{supp}\kappa_+ \subset \{(y,\xi)\in\mathbb{R}_y^{N+1}\times S_\xi^N \mid \sqrt{\rho(x)}\tau = -\sqrt{A(x)\eta\eta}\}, \tag{3.85}$$

where we recall that κ_+ is the H-measure associated to (a subsequence of) θv_+^ε.

The sought equality (cf. (3.79)) is then an immediate corollary of the following

Lemma 3.3. *Let P be an element of $\psi^0(\mathbb{R}^{N+1};\mathbb{R})$ whose principal symbol p^0 is of the form $p^0(\eta,\tau)$ (with $p^0 \in C^\infty(S^N)$). Let Π be an element of $\psi^1(\mathbb{R}^N;\mathbb{R})$ (which thus maps $L^2(\mathbb{R}^N)$ into $H^{-1}(\mathbb{R}^N)$) with principal symbol $\pi(x,\eta)$. Define $\tilde{P}$ to be an element of $\psi^0(\mathbb{R}^N;\mathbb{R})$ with principal symbol $\tilde{p}^0$ defined as*

$$\tilde{p}^0(x,\eta)(=p^0(\eta,\pi(x,\eta)))=p^0\Big(\frac{\eta}{\sqrt{|\eta|^2+\pi^2(x,\eta)}}\,,\,\frac{\pi(x,\eta)}{\sqrt{|\eta|^2+\pi^2(x,\eta)}}\Big).$$

Assume that w^ε is a compactly supported sequence of elements of $L^\infty(\mathbb{R}_t;L^2(\mathbb{R}^N_x))$ with the following properties:

$$\text{(3.86)}\qquad \begin{aligned}&w^\varepsilon \rightharpoonup 0 \text{ weakly in } L^2(\mathbb{R}^{N+1}_{x,t})\\ &\text{and strongly in } L^2(\mathbb{R}_t;H^{-1}(\mathbb{R}^N_x)),\end{aligned}$$

as ε tends to zero,

$$\text{(3.87)}\qquad \operatorname{supp}\omega\subset\{(y,\xi)\in\mathbb{R}^{N+1}_{x,t}\times S^N_{\eta,\tau}\mid \tau=\pi(x,\eta),\eta\neq 0,\tau\neq\pm 1\},$$

where ω is the H-measure associated to (a subsequence of) w^ε.
Then (for a subsequence)

$$[Pw^\varepsilon](t)-\tilde{P}(w^\varepsilon(t))\to 0 \quad \text{strongly in } L^2_{\rm loc}(\mathbb{R}^{N+1}_{x,t}),$$

as ε tends to zero. □

This lemma, once proved, is applied to $w^\varepsilon=\theta v^\varepsilon_\pm$, $\omega=\kappa_\pm$ (the H-measure associated to (a subsequence of) $\theta v^\varepsilon_\pm$) and to $\Pi=\mp 1/(\zeta\sqrt{\rho})\Lambda$ (any element of $\psi^1(\mathbb{R}^N_x;\mathbb{R})$ with principal symbol $\mp(A(x)\eta\cdot\eta/\rho(x))^{1/2}$). Upon invoking the spatial compactness of the support of v^ε_+, (3.67), (3.69), Aubin's compactness lemma (cf. e.g. [13], Corollary 4) and (3.85) Lemma 3.3 applies, which immediately implies that the sought equality (3.79) holds true.

Proof of Lemma 3.3. Since P and $\tilde{P}$ are uniquely determined by p^0 and $\tilde{p}^0$, up to pseudo-differentials operators of order -1 which are bounded mappings from $H^{-1}(\mathbb{R}^{N+1}_y)$ into $L^2(\mathbb{R}^{N+1}_y)$ and $H^{-1}(\mathbb{R}^N_x)$ into $L^2(\mathbb{R}^N_x)$ respectively, the second convergence in (3.86) enables us to focus on the principal part of P and $\tilde{P}$ and to choose for symbol for $\tilde{P}$, $\tilde{p}(x,\eta)$ with

$$\tilde{p}(x,\eta) = \tilde{p}^0(x,\eta/|\eta|)(1-\chi(|\eta|)),$$

where $\chi(t)$ has been defined in (3.50).

We reintroduce in this proof the pseudo-differential operator Ω in $\psi^0(\mathbb{R}_y^{N+1};\mathbb{R})$ defined through its principal symbol in (3.82). Note that Ω depends on χ.

Let $\zeta(x)$ (resp. $\theta(t)$) be an element of $C_0^\infty(\mathbb{R}_x^N)$ (resp. $C_0^\infty(\mathbb{R}_t)$) with $\zeta(x)$ (resp. $\theta(t)$) $\equiv 1$ on the $\mathbb{R}_x^N$ (resp. $\mathbb{R}_t$)-projection of the common compact support K of w^ε and remark that, through direct application of Theorem 3.1

$$\text{(3.88)} \qquad \operatorname{supp}\omega \subset \{(y,\xi) \in \mathbb{R}_y^{N+1} \times S_\xi^N \mid y=(x,t) \text{ with } \zeta(x)=1 \text{ and } \theta(t)=1\}.$$

Then, as ε tends to zero,

$$\text{(3.89)} \qquad \|\zeta\theta(1-\Omega)w^\varepsilon\|^2_{L^2(\mathbb{R}_y^{N+1})} \to <\omega, \chi^2(|\eta|)> .$$

Furthermore it is remarked that $\theta\tilde{P}\text{o}\zeta\text{o}\Omega$ is a pseudo-differential operator of order 0 (an element of $\psi^0(\mathbb{R}_y^{N+1};\mathbb{R})$) with principal symbol

$$\theta(t)\tilde{p}^0(x,\frac{\eta}{|\eta|})\zeta(x)\Big(1-\chi(\Big(\frac{|\eta|}{\sqrt{|\eta|^2+\tau^2}}\Big)\Big).$$

The estimate of the term $[Pw^\varepsilon](t)-\tilde{P}(w^\varepsilon(t))$ is performed with the help of the following decomposition:

$$[Pw^\varepsilon](t)-\tilde{P}(w^\varepsilon(t)) = [\{P-\theta\tilde{P}\circ\zeta\circ\Omega\}w^\varepsilon](t)-\tilde{P}([\zeta\theta(1-\Omega)w^\varepsilon](t)).$$

Since $\tilde{P}$ acts boundedly on $L^2(\mathbb{R}_x^N)$,

$$\overline{\lim_{\varepsilon\to 0}}\|\tilde{P}([\zeta\theta(1-\Omega)w^\varepsilon](t))\|^2_{L^2(\mathbb{R}_y^{N+1})} \le C\overline{\lim_{\varepsilon\to 0}}\|[\zeta\theta(1-\Omega)w^\varepsilon]\|^2_{L^2(\mathbb{R}_y^{N+1})},$$

where C is a constant that does not depend upon ε. By virtue of (3.89) we obtain

$$\text{(3.90)} \qquad \overline{\lim_{\varepsilon\to 0}}\|\tilde{P}([\zeta\theta(1-\Omega)w^\varepsilon](t)\|^2_{L^2(\mathbb{R}_y^{N+1})} \le C<\omega,\chi^2(|\eta|)> .$$

On the other hand the limit of the term $\psi[\{P - \theta\tilde{P} \circ \zeta \circ \Omega\}w^\varepsilon]$ ($\psi \in C_0^\infty(\mathbb{R}_y^{N+1})$) can be immediately computed through direct application of Theorem 3.1 and Lemma 3.1, which, in view of (3.88) leads to

$$\|\psi\{P-\theta\tilde{P} \circ \zeta \circ \Omega\}w^\varepsilon\|^2_{L^2(\mathbb{R}_y^{N+1})} \to < \omega, \psi^2 |p^0(\xi) - \tilde{p}^0(x, \frac{\eta}{|\eta|})(1 - \chi(|\eta|))|^2 >,$$
$$= < \omega, \psi^2 |p^0(\eta, \tau) - p^0(\eta, \pi(x, \eta))(1 - \chi(|\eta|))|^2 > .$$

In view of (3.87), the above convergence also reads as

$$\|\psi\{P - \theta\tilde{P} \circ \zeta \circ \Omega\}w^\varepsilon\|^2_{L^2(\mathbb{R}_y^{N+1})} \to < \omega, \psi^2 |p^0(\eta, \tau)|^2\, \chi^2(|\eta|) >,$$

as ε tends to zero.

The above convergence, together with (3.90), implies that, for any ψ in $C_0^\infty(\mathbb{R}_y^{N+1})$,

$$\text{(3.91)} \qquad \overline{\lim_{\varepsilon \to 0}} \|\psi([Pw^\varepsilon](t) - \tilde{P}(w^\varepsilon(t)))\|^2_{L^2(\mathbb{R}_y^{N+1})} \leq C_\psi < \omega, \chi^2(|\eta|) >,$$

where C_ψ is a constant that only depends on ψ.

Thus far the support of χ has only been restricted to be such that

$$\chi(s) = 1, \quad \text{if } |s| \leq \sin\, \alpha;$$

indeed we have not used the fact that

$$\chi(s) = 0, \quad \text{if } |s| \geq \sin\, \alpha.$$

Consideration of a sequence χ_n with

$$\begin{cases} 0 \leq \chi_n \leq 1, \\ \chi_n(x) \searrow^{n \to +\infty} 0 \quad \text{if } s \neq 0 \quad \text{and } 1 \quad \text{if } s = 0, \end{cases}$$

permits to conclude through application of dominated convergence that

$$< \omega, \chi_n^2(|\eta|) > \to < \omega, 1_{\{\eta = 0\}} >,$$

as n tends to $+\infty$, where $1_{\{\eta=0\}}$ denotes the characteristic function on S_ξ^N of the set $\{\eta = 0, \tau = \pm 1\}$.

Since the H-measure ω does not charge the points $\{\eta = 0, \tau = \pm 1\}$, (cf. (3.87)),

$$< \omega, 1_{\{\eta=0\}} > = 0,$$

which in view of (3.91) proves the desired result. □

In view of (3.73), (3.79), decomposition (3.63) and convergences (3.64) hold true. By virtue of (3.56) and Remark 3.13 we finally obtain

$$\mathcal{R}(0) = \tilde{Q}(0), \tag{3.92}$$

where

$$\tilde{Q}(0) = \frac{1}{4} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} (|\tilde{P}_+ v^\varepsilon_{+0}|^2 + |\tilde{P}_- v^\varepsilon_{-0}|^2)\varphi(x)dx, \tag{3.93}$$

and $v^\varepsilon_{+0}, v^\varepsilon_{-0}$ are given in (3.70). Note that $\theta(0)$ does not appear in (3.93) since $\theta(0) = 1$.

Denote by $\tilde{\nu}_+$ and $\tilde{\nu}_-$ the H-measures (defined on $\mathbb{R}^N_x \times S^{N-1}_\eta$) associated to a subsequence of v^ε_{+0} and v^ε_{-0}. Then

$$\tilde{Q}(0) = \frac{1}{4}\{< \tilde{\nu}_+, |\tilde{p}^0_+|^2\varphi > + < \tilde{\nu}_-, |\tilde{p}^0_-|^2\varphi >\}. \tag{3.94}$$

Appealing to (3.76) in Remark 3.12 permits to rewrite (3.94) as

$$\tilde{Q}(0) = \frac{1}{4}\{< \tilde{\nu}_+, \varphi|p^0(S^-)|^2 > + < \tilde{\nu}_-, \varphi|p^0(S^+)|^2 >\}. \tag{3.95}$$

Collecting (3.61)-(3.62), (3.92), (3.95) we finally obtain

$$\ll \tau\nu, \{\phi, Q\} \gg = \frac{1}{4}\Big\{< \tilde{\nu}_+, \phi_{(t=0,S^-)} > + < \tilde{\nu}_-, \phi_{(t=0,S^+)} >\Big\}, \tag{3.96}$$

for all ϕ's of the form

$$\phi = |p^0(\eta, \tau)|^2 \psi(t)\varphi(x),$$

with p^0 in $\mathcal{C}^\infty(S^N_\xi), \varphi$ in $\mathcal{C}^\infty_0(\mathbb{R}^N_x), \psi$ in $\mathcal{C}^\infty_0([0,T))$.

Invoking an immediate density argument permits to extend ϕ to be an arbitrary element of $C_0^\infty([0,T) \times \mathbb{R}_x^N \times S_\xi^N)$ in (3.96).

We have thus proved the following

Theorem 3.4. *The H-measure ν defined in Theorem 3.3 satisfies, for any ϕ in $C_0^\infty(\mathbb{R}_x^N \times [0,T) \times S_\xi^N)$,*

$$\ll \xi_0 \nu, \{\phi, Q\} \gg = \frac{1}{4}\Big\{ < \tilde{\nu}_+, \phi_{(t=0,S^-)} > + < \tilde{\nu}_-, \phi_{(t=0,S^+)} > \Big\}, \tag{3.97}$$

where S^+ and S^- have been defined in (3.75) and $\tilde{\nu}_+, \tilde{\nu}_-$ are the H-measures associated to $v^\varepsilon_{+0}, v^\varepsilon_{-0}$ defined as

$$v^\varepsilon_{\pm 0} = \zeta\sqrt{\rho}\beta^\varepsilon \mp i\Lambda\gamma^\varepsilon. \tag{3.98}$$

In (3.98), $\zeta(x)$ is any element of $C_0^\infty(\mathbb{R}_x^N)$ with $\zeta(x) \equiv 1$ on the support of β^ε and γ^ε, and Λ is any element of $\psi_c^1(\mathbb{R}_x^N;\mathbb{R})$ with principal symbol $\zeta(x)(A(x)\eta\cdot\eta)^{1/2}$. □

Remark 3.14. Since β^ε and γ^ε are compactly supported in a compact set K^0 on which $\zeta(x) \equiv 1$ (recall that $\zeta(x) \equiv 1$ on K' and that K' - the $\mathbb{R}_x^N$-projection of the support K of θv^ε (cf. (3.65))- contains K^0), the choice of the specific function ζ (in $C_0^\infty(\mathbb{R}_x^N)$) in the definition (3.66) of $v^\varepsilon_\pm$ will not affect the H-measures $\tilde{\nu}_\pm$. Indeed direct application of Theorem 3.1 will restrict the supports of $\tilde{\nu}_\pm$ to K^0 on which $\zeta(x) \equiv 1$. This will be illustrated in more details in Subsection 3.3. □

By virtue of Remark 3.5, $\ll \tau\nu, \{\phi, Q\} \gg$ defines the trace of $\partial Q/\partial\tau\ \tau\nu$ at time $t = 0$. Equation (3.97) enables us to recover the initial value of the measure $\partial Q/\partial\tau\ \tau\nu$ and thus (cf. Remark 3.6) of ν.

Remark 3.15. As anounced in Remark 3.2, (3.18) is immediately recovered from (3.97) by taking ϕ to be compactly supported in $(0,T)\times\mathbb{R}_x^N \times S_\xi^N$ since in such a case $\ll\ ,\ \gg = <\ ,\ >$.

Note also that the derived equation is indeed an equation for $\xi_0\nu(=\tau\nu)$ and not for ν, as previously emphasized in Remark 3.3. □

Define at this point the measure $\tilde{\pi}_\pm$ on $\mathbb{R}_x^N \times S_\xi^N$ as follows

$$(3.99) \qquad < \tilde{\pi}_\pm, \tilde{\phi} >= \int_{\mathbb{R}^N_x \times S^N_\xi} \tilde{\phi}(x, S^\mp(x,\eta))\tilde{\nu}_\pm(dx\, d\eta),$$

for any $\tilde{\phi}$ in $C_0^\infty(\mathbb{R}^N_x \times S^N_\xi)$.

Definition (3.99) is easily checked to define $\tilde{\pi}_\pm$ as Radon measures and (3.97) reads as

$$(3.100) \qquad \ll \tau\nu, \{\phi, Q\} \gg = \frac{1}{4}\{< \tilde{\pi}_+, \phi(t=0) > + < \tilde{\pi}_-, \phi(t=0) >\},$$

for any ϕ in $C_0^\infty([0,T) \times \mathbb{R}^N_x \times S^N_\xi)$.

Remark 3.16. In the case where $\tilde{\nu}_\pm$ have smooth enough densities, $\tilde{\pi}_\pm$ identify with measures concentrated on the images of S^{N-1}_η by the mappings $S^\pm$. □

In the light of (3.100), Lemma 3.2 and Remarks 3.6, 3.8, 3.10 we can finally restate Theorems 3.3 and 3.4 as

Corollary 3.1. *The measure $\xi_0\nu$ is the transported value along the integral curves of (3.34)-(3.35) -that live on $\mathbb{R}^{N+1}_y \times S^N_\xi$ and on the null set of Q- of its initial value*

$$(3.101) \qquad \tau\nu_{|t=0} = \xi_0\nu_{|t=0} = \frac{1}{4\rho(x)\xi_0}(\tilde{\pi}_+ + \tilde{\pi}_-),$$

where $\tilde{\pi}_\pm$ have been defined in (3.99). □

Remark 3.17. Because of the specific form of $S^\pm$ defined in (3.75) the measures $\tilde{\pi}_\pm$ have their support on the null set of Q, which agrees with the statements that the support of ν is included in the null set of Q and that the curves (3.34)-(3.35) conserve the null set of Q (cf. Remark 3.8). Thus in particular $\xi_0 = \tau$ is not zero in formula (3.101). □

3.2.3. The limit energy density.

We remind the reader of our initial task, namely that of computing the limit energy density d of d^ε defined as

$$d^\varepsilon(x,t) = \frac{1}{2}[\rho(\frac{\partial v^\varepsilon}{\partial t})^2 + A \operatorname{grad} v^\varepsilon \operatorname{grad} v^\varepsilon](x,t).$$

Recalling (3.15), (3.16) we obtain, for any φ in $C_0^\infty((0,T)\times \mathbb{R}^N)$,

$$< d, \varphi > = \frac{1}{2}\int_{\mathbb{R}_y^{N+1}} \varphi(t,x)\int_{S_\xi^N}(\rho(x)\xi_0^2 + \sum_{i,j=1}^N a_{ij}(x)\xi_i\,\xi_j)\nu(dy\,d\xi).$$

The support of ν lies in the null set of $Q(x,\xi)$ on which $\frac{1}{2}(\rho(x)\xi_0^2 + \sum_{i,j=1}^N a_{ij}(x)\xi_i\,\xi_j)$ may be replaced by $\rho(x)\xi_0^2$ thereby yielding

$$< d, \varphi > = \int_{\mathbb{R}_y^{N+1}} \rho(x)\varphi(t,x)\int_{S_\xi^N} \xi_0^2\nu(dx\,d\xi). \tag{3.102}$$

We have thus obtained the following

Theorem 3.5. *The measure limit d of the energy density $d^\varepsilon(x,t)$ defined as*

$$d^\varepsilon(x,t) = \frac{1}{2}\{\rho(x)(\frac{\partial v^\varepsilon}{\partial t})^2 + A(x)\operatorname{grad} v^\varepsilon \operatorname{grad} v^\varepsilon\},$$

for v^ε solution to (3.1), (3.3), is given by

$$d = \rho(x)\int_{S_\xi^N} \xi_0^2\nu(dx\,d\xi) \tag{3.103}$$

where ν is defined through Theorem 3.4 or equivalently Corollary 3.1. □

3.3. Slowly modulated periodic initial conditions.

This final subsection specializes the initial conditions β^ε and γ^ε to be of the form anounced in (3.5), (3.6) of Remark 3.1.

Our first goal is the computation of the measures $\tilde{\nu}_+$ and $\tilde{\nu}_-$ associated to $v_{\pm 0}^\varepsilon = \zeta\sqrt{\rho}\beta^\varepsilon \mp i\Lambda\gamma^\varepsilon$. To this effect we note that, since γ^ε has a compact support independent of ε,

$$\gamma^\varepsilon = \Delta^{-1}(\operatorname{div}(\operatorname{grad}\gamma^\varepsilon)), \tag{3.104}$$

where Δ^{-1} is to be seen as the inverse of the Laplacian with Dirichlet boundary conditions on a domain large enough to contain the common support K^0 of the γ^ε's. In the language of pseudo-differential operators (3.104) reads as

$$\gamma^\varepsilon = \sum_{i=1}^N \Delta_i\Big(\frac{\partial\gamma^\varepsilon}{\partial x_i}\Big),$$

where Δ_i is the element of $\psi^{-1}(\mathbb{R}^N;\mathbb{R})$ with principal symbol $-i\eta_i/|\eta|^2$. Thus

$$v^{\varepsilon}_{\pm 0} = \sqrt{\rho}\beta^{\varepsilon} \mp i \sum_{i=1}^{N} (\Lambda\Delta_i)\Big(\frac{\partial\gamma^{\varepsilon}}{\partial x_i}\Big). \tag{3.105}$$

But, in view of (3.5),

$$\frac{\partial\gamma^{\varepsilon}}{\partial x_i} = \frac{\partial\alpha}{\partial y_i}(x,\frac{x}{\varepsilon}) + \varepsilon\,\frac{\partial\alpha}{\partial x_i}(x,\frac{x}{\varepsilon}) = \Big(\frac{\partial\alpha}{\partial y_i}\Big)^{\varepsilon}(x) + \varepsilon\Big(\frac{\partial\alpha}{\partial x_i}\Big)^{\varepsilon}(x),$$

thus (3.105) reads as

$$v^{\varepsilon}_{\pm 0} = \zeta\sqrt{\rho}\,\beta^{\varepsilon} \mp i \sum_{i=1}^{N} \Lambda\Delta_i\Big[\Big(\frac{\partial\alpha}{\partial y_i}\Big)^{\varepsilon}\Big] \pm r^{\varepsilon}_0,$$

where r^{ε}_0 converges to 0 strongly in $L^2(\mathbb{R}^N)$ as ε tends to zero. The computation of $\tilde{\nu}_{\pm}$ is now reduced to that of the H-measure associated to (a subsequence of)

$$w^{\varepsilon}_{\pm 0} = \zeta\sqrt{\rho}\beta^{\varepsilon} \mp i \sum_{i=1}^{N} \Lambda\Delta_i\Big[\Big(\frac{\partial\alpha}{\partial y_i}\Big)^{\varepsilon}\Big].$$

Denote by $\tilde{\mu}$ the $\mathbb{R}^{N+1}\times\mathbb{R}^{N+1}$-valued H-measure associated to (a subsequence of) $(\beta^{\varepsilon}, (\operatorname{grad}_y \alpha)^{\varepsilon})$; Theorem 3.1 implies that the support of $\tilde{\mu}$ lies on the set of points x where $\zeta(x) \equiv 1$ (cf. Remark 3.14).

The H-measures $\tilde{\nu}_{+}$ and $\tilde{\nu}_{-}$ are immediately computable from the knowledge of $\tilde{\mu}$ and through application of Lemma 3.1 (especially (3.11)). We obtain,

$$\begin{aligned}\tilde{\nu}_{\pm}(dx\,d\eta) =& \rho(x)\,\tilde{\mu}_{00}(dx\,d\eta) + (A(x)\eta\cdot\eta)\Big(\sum_{i,j=1}^{N} \eta_i\eta_j\tilde{\mu}_{ij}(dx\,d\eta)\Big)\\ &\mp 2(\rho(x)(A(x)\eta\cdot\eta))^{1/2}\sum_{j=1}^{N}\eta_j\,\mathrm{Re}[\tilde{\mu}_{0j}(dx\,d\eta)].\end{aligned} \tag{3.106}$$

We are left with the task of computing $\tilde{\mu}$ explicitly. To this effect, we appeal to the vector-valued analogue of a result obtained (in the scalar case) by L. TARTAR ([15], Section 2) and P. GERARD ([5], Prop. 1.5 or [7], Example 2.4).

Lemma 3.4. *Let $\gamma(x,y)$ be an element of $[\mathcal{C}_0^\infty(\mathbb{R}^N \times \mathcal{T})]^P$ such that*

$$\int_{\mathcal{T}} \gamma_i(x,y)dy = 0, 1 \leq i \leq P.$$

Then there exists a H-measure μ associated to the whole sequence γ^ε with

$$\gamma^\varepsilon(x) = \gamma(x, \frac{x}{\varepsilon}).$$

It is given by

$$\mu_{ij}(dx\, d\eta) = \sum_{\substack{k \in Z^N \\ k \neq 0}} \gamma_{ik}(x)\overline{\gamma}_{jk}(x)dx \otimes \delta_{k/|k|}(\eta), 1 \leq i,j \leq P,$$

where the γ_{ik}'s are the Fourier coefficients of $\gamma_i(x,y)$ $(1 \leq i \leq P)$. □

Remark 3.18. It should be pointed out that this lemma remains valid when γ is less regular (cf. [15], Section 2). □

The proof of Lemma 3.4 will not be presented here because it is identical to that of the scalar-valued case.

Direct application of the above lemma yields

$$(3.107) \quad \begin{cases} \tilde{\mu}_{00} = \sum_{\substack{k \in Z^N \\ k \neq 0}} |\beta_k(x)|^2 dx \otimes \delta_{k/|k|}(\eta), \\ \tilde{\mu}_{ij} = 4\pi^2 \sum_{\substack{k \in Z^N \\ k \neq 0}} k_i k_j |\alpha_k(x)|^2 dx \otimes \delta_{k/|k|}(\eta), 1 \leq i,j \leq N, \\ \tilde{\mu}_{oj} = -i2\pi \sum_{\substack{k \in Z^N \\ k \neq 0}} k_j \beta_k(x)(x)\overline{\alpha}_k(x) dx \otimes \delta_{k/|k|}(\eta), 1 \leq j \leq N. \end{cases}$$

Collecting (3.106), (3.107) finally leads to

$$(3.108) \quad \begin{aligned} \tilde{\nu}_\pm(dx\, d\eta) = \sum_{\substack{k \in Z^N \\ k \neq 0}} &\{\rho(x)\, |\beta_k(x)|^2 + 4\pi^2(A(x)k \cdot k)|\alpha_k(x)|^2 \\ &\mp 4\pi(\rho(x)A(x)k \cdot k)^{1/2} \mathrm{Im}[\beta_k(x)\overline{\alpha}_k(x)]\} dx \otimes \delta_{k/|k|}(\eta). \end{aligned}$$

The next step in our analysis is dictated by (3.101) in Corollary 3.1; we should thus compute $\tilde{\pi}_\pm$. These are given in terms of $\tilde{\nu}_\pm$ by (3.99). We obtain, with the help of (3.108),

$$< \tilde{\pi}_\pm, \tilde{\phi} > = \sum_{\substack{k\in Z^N \\ k\neq 0}} \int_{\mathbb{R}^N} \{\rho(x)\,|\beta_k(x)|^2 + 4\pi^2(A(x)k\cdot k)|\alpha_k(x)|^2$$

$$\mp 4\pi(\rho(x)A(x)k\cdot k)^{1/2}\mathrm{Im}[\beta_k(x)\overline{\alpha}_k(x)]\}\tilde{\phi}(x, S^{\mp}(x, \frac{k}{|k|}))dx,$$

for any $\tilde{\phi}$ in $C_0^\infty(\mathbb{R}_x^N \times S_\xi^N)$.

Thus

$$< \tilde{\pi}_\pm, \tilde{\phi} > = \sum_{\substack{k\in Z^N \\ k\neq 0}} \int_{\mathbb{R}^N} \{\rho(x)\,|\beta_k(x)|^2 + 4\pi^2(A(x)k\cdot k)|\alpha_k(x)|^2$$

$$\mp 4\pi(\rho(x)A(x)k\cdot k)^{1/2}\mathrm{Im}[\beta_k(x)\overline{\alpha}_k(x)]\}dx \otimes \delta_{S^{\mp}(x,k/|k|)}(\xi),$$

and we can specialize Corollary 3.1 to the present setting as

Corollary 3.2. *If the initial conditions for (3.1) are of the form*

$$\begin{cases} \gamma^\varepsilon(x) = \varepsilon\ \alpha(x, \dfrac{x}{\varepsilon}), \\ \beta^\varepsilon(x) = \beta(x, \dfrac{x}{\varepsilon}), \end{cases}$$

with α and β elements of $C_0^\infty(\mathbb{R}^N \times \mathcal{T})$ satisfying

$$\int_{\mathcal{T}} \beta(x,y)dy = 0,$$

then

(3.109)

$$\xi_0\nu_{|t=0} = \frac{1}{4\rho(x)\xi_0} \sum_{\pm} \sum_{\substack{k\in Z^N \\ k\neq 0}} \{\rho(x)\,|\beta_k(x)|^2 + 4\pi^2(A(x)k\cdot k)|\alpha_k(x)|^2$$

$$\mp 4\pi(\rho(x)A(x)k\cdot k)^{1/2}\ \mathrm{Im}[\beta_k(x)\overline{\alpha}_k(x)]\}dx \otimes \delta_{S^{\mp}(x,k/|k|)}(\xi),$$

where $S^\pm$ have been defined in (3.75) and $\alpha_k(x), \beta_k(x)$ denote the Fourier coefficients (in y) of $\alpha(x,y)$ and $\beta(x,y)$ respectively. The measure $\xi_0\nu$ is the transported valued of its initial value given by (3.109) along the integral curves of (3.34)-(3.35) that live on $\mathbb{R}_y^{N+1} \times S_\xi^N$ and on the null set of Q. □

Corollary 3.2 delivers as explicit an expression for ν as possible in the case where ρ and A exhibit an x-dependence. In the constant coefficient case however the H-measure ν can be expressed solely in terms of its initial value. Indeed the integral curves of (3.34)-(3.35) that live on $\mathbb{R}_y^{N+1} \times S_\xi^N$ and the null set of Q are

$$\begin{cases} \overline{y}_0(x) = \rho\xi_0 s + y_0 \quad , \quad \overline{\xi}_0(s) = \xi_0, \\ \overline{y}_i(s) = -(\sum_{j=1}^{N} a_{ij}\,\xi_j)s + y_i \quad , \quad \overline{\xi}_i(s) = \xi_i \quad , \quad 1 \le i \le N. \end{cases}$$

We are interested in curves that originate on the hyperplane ($y_0 = 0$, $y_i = x_i, 1 \le i \le N$) and we conclude that

$$(\xi_0\nu)(x,t,\xi) = (\xi_0\nu)(x + \sum_{j=1}^{N} \frac{a_{ij}\xi_j}{\rho\xi_0} t, 0, \xi).$$

Upon recalling the expressions (3.75) for $S^\pm$ -which become independent of x- we obtain the following

Corollary 3.3. *If, in the context of Corollary 3.2, the coefficients that enter (3.1) are constant (i.e., if ρ and $a_{ij}, 1 \le i,j \le N$ do not depend on x) then*

$$\begin{aligned} \nu(x,t,\xi) = & \frac{1}{4\rho} \sum_{\pm} \sum_{\substack{k \in Z^N \\ k \neq 0}} \frac{1}{(S_0^\mp(k/|k|))^2} \Big\{ \rho |\beta_k(x \mp \frac{Ak}{(\rho Ak\cdot k)^{1/2}} t)|^2 \\ & + 4\pi^2 (Ak\cdot k) |\alpha_k(x \mp \frac{Ak}{(\rho\, Ak\cdot k)^{1/2}} t|^2 \\ & \mp 4\pi(\rho\, Ak\cdot k)^{1/2} \,\mathrm{Im}[\beta_k(x \mp \frac{Ak}{(\rho\, Ak\cdot k)^{1/2}} t) \\ & \overline{\alpha}_k(x \mp \frac{Ak}{(\rho\, Ak\cdot k)^{1/2}} t)] \Big\} dx \otimes \delta_{S^\mp(k/|k|)}(\xi). \end{aligned} \tag{3.110}$$

□

Remark 3.19. We conclude this study by returning to our initial problem, namely the computation of the measure limit d of the energy density d^ϵ. In the constant coefficient case d was computed in Theorem 2.4. We now apply

Theorem 3.4 with ν given by (3.110) and obtain, in view of (3.103),

(3.111)

$$d = \frac{1}{4}\sum_{\pm}\sum_{\substack{k\in Z^N\\ k\neq 0}}\Big\{\rho|\beta_k(x \pm \frac{Ak}{(\rho Ak\cdot k)^{1/2}}t)|^2 + 4\pi^2(Ak\cdot k)|\alpha_k(x \pm \frac{Ak}{(\rho Ak\cdot k)^{1/2}}t)|^2 \pm 4\pi(\rho Ak\cdot k)^{1/2}\,\mathrm{Im}[\beta_k(x \pm \frac{Ak}{(\rho Ak\cdot k)^{1/2}}t)\overline{\alpha}_k(x \pm \frac{Ak}{(\rho Ak\cdot k)^{1/2}}t)]\Big\}.$$

Expression (3.111) is identical to expression (2.64) obtained in Theorem 2.4 as it should be. □

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