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Conservation Laws and Material Momentum in Thermoelasticity

The main goal of this paper is to construct a Lagrangian function such that not only the well-known equations of thermoelasticity, but also material conservation laws can be derived. As action variables, the position $\mathbf{x}$ *of a material particle and a scalar function* η *related to temperature are used. The material momentum for thermoelasticity is derived. Here, by contrast to the purely elastic case, the material momentum depends on a time interval rather than on an instant of time. The balance of material momentum is integrated over time to produce a relation reminiscent of the impulse-momentum equation in classical mechanics.*

1 Introduction

The development of mathematical fracture mechanics focused attention on new material conservation laws because they form a starting point for establishing path-independent integrals. These conservation laws were first derived for a static linear-elastic continuum. Numerous attempts have been made to generalize them for nonlinear, plastic, viscoelastic, and dynamic behavior.

It turns out that these new material conservation laws fit naturally into a physical framework: they are related to the (material) symmetries of the Lagrangian function of the given system which in a static elastic case reduces to the strain energy.

A natural obstacle prevents generalization to thermoelastic systems. The dissipative character of these systems precludes the generation of a Lagrangian for thermoelasticity by simple addition of new terms, involving the fields (which describe the thermal behavior) to the elastic Lagrangian. As long as we are interested in deriving the thermoelastic equations only, it is possible to overcome this obstacle by using adjoint fields as well as original ones in the Lagrangian. Such a formulation, however, is not capable of delivering the new conservation laws because the crucial quantity appearing there involves the Lagrangian itself.

The current paper is an attempt to construct a Lagrangian, from which we can derive all known equations of thermoelasticity, as well as new conservation laws.

2 General Background

The variational approach to many physical theories proved very useful over the years and will be followed here. For a system of particles, for example, we consider an action integral, where the integrand is called the Lagrangian of the system under consideration and the independent variable is the time t. In various field theories, by contrast, we deal with a Lagrangian density because the integration is performed not only over an interval of time, but over a region of space V. In other words, the actions A have the following forms

$$A = \int_{t_0}^{t_1} L(t;\mathbf{x}_1,\mathbf{x}_2 \ldots \mathbf{x}_n,\mathbf{v}_1,\mathbf{v}_2 \ldots \mathbf{v}_n)\, dt \tag{1}$$

for systems of n particles, and

$$A = \int_{t_0}^{t} \int_{V} \mathcal{L}(t,\mathbf{x};\phi,\dot{\phi},\phi_{,i})\, dt\, d^3x \tag{2}$$

for fields. Here $\mathbf{x}_i$ $(i = 1, 2, \ldots n)$ denotes the position of the ith particle, $\mathbf{v}_i$ is its velocity, and all $\mathbf{x}_k$, $\mathbf{v}_k$ are functions of t. In (2) ϕ is a field (or fields), a function of both, and time t and space coordinates $\mathbf{x}$, $\dot{\phi}$, and $\phi_{,i}$ denote partial derivatives with respect to time and x_i, respectively.

According to the stationarity principle, by varying the dependent variables ($\mathbf{x}$ in the first case, ϕ in the second), we derive the equations of motion for the particles or the field equations. In many cases we are also interested in conservation laws for the given system. They can be derived from Noether's theorem [1] and are related to the invariance of the action integral under infinitesimal transformations performed on independent and/or dependent variables.

The Lagrangian description of the continuum can be derived from (1) when the number n becomes very large and particles are densely distributed. $\mathbf{x}_n(t)$ then becomes a continuous function of a material coordinate $\mathbf{a}$ (related to the original number of a particle n) i.e., $\mathbf{x} = \mathbf{x}(\mathbf{a},t)$. Most often a displacement vector $\mathbf{u} = \mathbf{x} - \mathbf{a}$ is used for the description of a continuum rather than the position $\mathbf{x}$ itself. Then relation (1) transforms into

$$A = \int_{t_0}^{t} \int_{B} L(\mathbf{a},t;\mathbf{u},\mathbf{v},\nabla\mathbf{u})\, d^3a\, dt \tag{3}$$

where the integration is performed over the whole body B. Here $\mathbf{v}$ and $\nabla\mathbf{u}$ denote partial derivatives of $\mathbf{u}$ with respect to t and a_i $(i = 1,2,3)$. The usually accepted equations of an

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elastic continuum are not, however, generally obtained from stationariness of the action integral (3), but rather by extending well-known principles for the discrete systems of particles to a continuum. In other words, the introduction of the new independent variables **a** was not really exploited until 1951 when Eshelby introduced the notion of a force on an elastic singularity [2], related to translations in **a** space. In the absence of defects, the relation obtained by Eshelby becomes a statement of conservation of field (or material) momentum.

New conservation laws, associated with the properties of material space, have been derived by Günther [3] and Knowles and Sternberg [4]. These conservation laws are closely linked to the path-independent integrals of defect (especially fracture) mechanics. A similar situation arises in particle mechanics where the conservation of linear momentum becomes a balance of linear momentum (or equation of motion) if external forces are present. The first of such path-independent quantities for a crack was established in 1968 by Rice [5] and extended by Budiansky and Rice [6]. Eshelby's approach and most others are essentially based on the formulation of the type represented by equation (3), e.g., they use the displacement vector as a basic field quantity. We are now confronted with a very peculiar and unusual (from the field-theroretical point of view) situation. Namely, conservation laws, in general, are related to the transformation on either dependent or independent variables as mentioned before. The equations of continuum mechanics, however, represent balance of linear and angular momentum, which in turn follow from transformations of physical coordinates. But the physical coordinates do not directly enter the formulation either as dependent or as independent variables.

To take full advantage of the Lagrangian field-theoretical formalism, we have to adopt **x** as the dependent variable (instead of **u**) in (3), or alternatively (if we want to achieve formal resemblance to field theory) we might use a formulation based on form (2), where **a** would be playing the role of the field ϕ. This has been done by Golebiewska Herrmann [7]. In this approach two basic quantities were introduced in a natural fashion into the development, namely the physical momentum tensor, i.e., the Piola-Kirchhoff stress tensor and the material momentum tensor related to Eshelby's tensor. The latter was given as

$$b_{ik} = \frac{\partial L}{\partial x_{j,k}} x_{j,i} - L\delta_{ik} \tag{4}$$

in the representation using $\mathbf{a},t$ as independent variables.

As it is seen, the Lagrangian density itself enters the expression for the material momentum tensor b_{ik}. It is therefore of prime importance to be able to produce a suitable expression for the Lagrangian. In this paper we construct material momenta for linear, dynamic, fully coupled thermoelasticity. Thus our first concern is the generation of an appropriate Lagrangian density.

3 A Short Review of Variational Formulation in Thermoelasticity

The equations of linear thermoelasticity are

$$-\sigma_{ij,j}(\mathbf{a},t) - f_i(\mathbf{a},t) + \rho(\mathbf{a})\ddot{u}_i(\mathbf{a},t) = 0 \tag{5a}$$

$$-q_{i,i}(\mathbf{a},t) - r(\mathbf{a},t) + \dot{s}(\mathbf{a},t) = 0 \tag{5b}$$

where the u_i's are the components of the displacement vector, usually defined as,

$$u_i(\mathbf{a},t) = x_i(\mathbf{a},t) - a_i$$

where $\rho(\mathbf{a})$ is the density of the medium, f_i represents body forces, σ_{ij} is the Cauchy stress tensor, and r and q_i are the heat source and the heat flux, respectively. Finally, s denotes the entropy of the system. The aforementioned quantities satisfy certain relations, namely

$$\begin{aligned} \sigma_{ij}(\mathbf{a},t) &= c_{ijkp}(\mathbf{a})\, u_{k,p}(\mathbf{a},t) - \beta_{ij}(\mathbf{a})\tau(\mathbf{a},t) \\ q_i(\mathbf{a},t) &= \lambda_{ij}(\mathbf{a})\, \tau_{,j}(\mathbf{a},t) \\ s(\mathbf{a},t) &= \beta(\mathbf{a})\, \tau(\mathbf{a},t) + \beta_{ij}(\mathbf{a})\, u_{i,j}(\mathbf{a},t) \end{aligned} \tag{6}$$

where we introduced the temperature increment $\tau(\mathbf{a},t)$ and where c_{ijkp}, β_{ij}, λ_{ij}, β are the elastic and thermoelastic coefficients. In most cases these coefficients are constant.

A characteristic feature of these equations resides in the coupling of the equations of motion (5*a*) with a differentiated form of the second law of thermodynamics (5*b*). The latter equation induces dissipation and the whole system becomes nonconservative.

The basic purpose of a variational formulation of the problem will always be the generation of the complete set of equations of Euler's equations associated with a given Lagrangian density. The dissipative character of the thermoelastic system is of essential importance and precludes the possibility of constructing the Lagrangian density by simple modification of its elastic analog (e.g., by adding additional terms).

To the authors' knowledge two methods have been suggested to overcome this difficulty. The first one is based on the introduction of adjoint fields that render the system conservative. As such, it applies to any type of dissipative system. Implementations can be found in Morse and Feshback [8] for the heat equation and in Herrmann and Tasi [9] for thermoelasticity. It is essential to note that this method generates a set of equations for the adjoint fields as well. In the case of the heat equation, the adjoint field is a solution of the backward heat equation. As such, it blows up exponentially fast after a finite time for any kind of nonzero initial data. Of course, this last feature is irrelevant as long as one's goal is simply to generate the heat equation from a Lagrangian, but it forsakes any hope to obtain a meaningful material momentum, since such a momentum always involves $\mathcal{L}$, thereby coupling field and adjoint fields.

The second method has been specifically designed for thermoelasticity by Biot [10]. A field **s**, called the entropy displacement field, is defined through

$$div\, \mathbf{s} = -s \tag{7}$$

where s is the entropy of the system (as mentioned before).

Recently Rafalski [11] has suggested a Lagrangian density based on convolution in time. This feature alleviates the obstacle created by the dissipation and avoids the trap of adjoint fields. Roughly speaking, the loss of energy of the system is compensated, through the convolution in time, by the gain of energy of that same system run backward. Formally, this is done by introducing symmetrized products of two quantities at two different times, as shown in the next section. By contrast to the method of adjoint fields, the system is run backward only as long as it is meaningful. Note, however, that Rafalski's formulation still uses entropy displacements.

The few papers mentioned in this section offer, in our opinion, a representative sample of the variational techniques currently used in thermoelasticity in order to derive the thermoelastic equations. In the next section we will construct a Lagrangian density where neither adjoint fields nor entropy displacements are considered.

Before we proceed we should mention some contributions related to new conservation laws for thermoelasticity. Gurtin [12] derived a path-independent integral for the static case. Herrmann [13] found a generalization of Eshelby's tensor for thermoelasticity. However, great difficulties arise in handling the time dependence. The material momenta produced in [13]

are associated with the Laplace transform in time of the original equations, and their connection with the original material momenta has not been established. The author suggested that whenever the time operator occurred, the multiplication had to be carried out as a convolution. In the work mentioned in the foregoing, the entropy displacement s has been used. (The practicality of the notion of s is questionable since it involves the solving of equation (7) explicitly.)

4 A New Lagrangian for Thermoelasticity

To our knowledge, Gurtin [14] is the first to have considered nonlocal Lagrangians in the case of elasticity. As it was previously mentioned, Rafalski then proceeded to apply this concept to thermoelasticity in [11]. In our formulation we keep this essential feature, but modify the fields previously introduced.

If f and g are two functions of time, we define

$$[f,g]_{t_0}^{+} = f(t)\, g(t_0 - t) + f(t_0 - t)\, g(t)$$

$$[f,g]_{t_0}^{-} = f(t)\, g(t_0 - t) - f(t_0 - t)\, g(t)$$

Let t_0 be a strictly positive real number. If $\{x_i\}$ $(i = 1,2,3)$ and η are functions of $\mathbf{a}$ and t, we define $\mathfrak{L}(\mathbf{a},t,t_0; x_i,\dot{x}_i,x_{i,j},\eta,\dot{\eta},\eta_{,i},\dot{\eta}_{,i})$ to be

$$-\tfrac{1}{4}\, c_{ijkp}(\mathbf{a})[x_{k,p} - \delta_{kp}, x_{i,j} - \delta_{ij}]_{t_0}^{+} + \tfrac{1}{2}\, \beta_{ij}(\mathbf{a})[\dot{\eta}, x_{i,j} - \delta_{ij}]_{t_0}^{+} + \tfrac{1}{4}\, \beta(\mathbf{a})[\dot{\eta},\dot{\eta}]_{t_0}^{+} - \tfrac{1}{4}\, \rho(\mathbf{a})[\dot{x}_i,\dot{x}_i]_{t_0}^{+} + \tfrac{1}{4}\, \lambda_{ij}(\mathbf{a})[\dot{\eta}_{,j},\dot{\eta}_{,i}]_{t_0}^{+} + \tfrac{1}{2}\, [f_i,x_i]_{t_0}^{+} - \tfrac{1}{2}\, [r,\eta]_{t_0}^{+} \tag{8}$$

where f_i and r are given functions of $\mathbf{a}$ and t and all coefficients entering (8) are assumed to be smooth, real, and symmetric and satisfy the usual hypotheses of strong ellipticity which guarantee that the equations of thermoelasticity are well posed (see equation (5)). The newly introduced field η is related to the temperature τ by the relation $\tau = \dot{\eta}$.

The reader has undoubtedly noticed the nonlocality in time of $\mathfrak{L}(\mathbf{a},t,t_0;x_i,\dot{x}_i,x_{i,j},\eta,\dot{\eta},\eta_{,i},\dot{\eta}_{,i})$. Time t_0 appears explicitly in $\mathfrak{L}$, not only in the limit of the integral. To underline this fact we will use from now on the notation $\mathfrak{L}_{t_0}$ instead of $\mathfrak{L}$. This time nonlocality will turn out to be essential for our purpose.

As a first test of the adequacy of our Lagrangian density, we should be able to easily recover the classical Lagrangian density of linear elasticity. This is achieved by canceling all terms where η is present and by collapsing t with $t_0 - t$. Then $[\dot{x}_i,\dot{x}_i]_{t_0}^{+}$ becomes $-2\dot{x}_i(\mathbf{a},t)\, \dot{x}_i(\mathbf{a},t)$ since $\partial/\partial t = -\partial/\partial(t_0 - t)$ and $\mathfrak{L}t_0$ becomes

$$-\tfrac{1}{2}\, c_{ijkp}(\mathbf{a})(x_{k,p} - \delta_{kp})(\mathbf{a},t)\,(x_{i,j} - \delta_{ij})(\mathbf{a},t) + \tfrac{1}{2}\, \rho(\mathbf{a})\, \dot{x}_i(\mathbf{a},t)\, \dot{x}_i(\mathbf{a},t) + f_i(\mathbf{a})\, x_i(\mathbf{a},t) \tag{9}$$

as announced.

We now postulate the *stationarity of A* at the field point (x_i,η). We perform the classical steps leading to Euler's equations, taking full account of the symmetry of the coefficients and of the convolution in time. We finally obtain the following set of equations for $0 \le t \le t_0$:

$$-\frac{\partial}{\partial a_j}\{c_{ijkp}(\mathbf{a})[x_{k,p}(a,t) - \delta_{kp}] - \beta_{ij}(\mathbf{a})\, \dot{\eta}(\mathbf{a},t)\} + \rho(\mathbf{a})\, \ddot{x}_i(\mathbf{a},t) - f_i(\mathbf{a},t) = 0$$

$$-\beta(\mathbf{a})\, \ddot{\eta}(\mathbf{a},t) - \beta_{ij}(\mathbf{a})\, \dot{x}_{i,j}(\mathbf{a},t) + \frac{\partial}{\partial a_j}[\lambda_{ij}(\mathbf{a},t)\, \dot{\eta}_{,j}(\mathbf{a},t)] + r(\mathbf{a},t) = 0 \tag{10}$$

Equation (10) is exactly the equation governing the behavior of a thermoelastic body for $t \le t_0$, as given by (5).

Following the terminology of the calculus of variations, we conclude that $\mathfrak{L}_{t_0}$ *is a Lagrangian for thermoelasticity* (for $t \le t_0$). Note that in contrast with the method of adjoint fields, the Euler's equations of motion consist only of the thermoelastic equations of motion: there are no additional equations. This remarkable feature is entirely due to the symmetry of the convolution in time since if $E(t)$ represents the Euler-Lagrange expression at time t, and $\delta B(t)$ the variation of the appropriate fields at t, we have:

$$\int_0^{t_0} \{A(t)\, \delta B(t_0 - t) + A(t_0 - t)\, \delta B(t)\}\, dt = 2\int_0^{t_0} A(t)\, \delta B(t_0 - t)\, dt \tag{11}$$

Another consequence of (11) is that even though t_0 appears in a crucial way in (8), it does not enter in any way the equations (10), which was to be expected in as much as Euler's equations should be a statement about the time evolution of a given system, a statement that cannot possibly involve the future of this system.

It is actually possible to produce a set of "natural" boundary and initial conditions from the variation of A. Demanding (u_i,η) to vanish at time $t = 0$ and A to be stationary over all elements of $C^{\infty}(\bar{B} \times [0,t_0], \mathbb{R}^4)$ that vanish at $t = 0$, we obtain as boundary and initial conditions:

Boundary Conditions

$$\sigma_{ij}(\mathbf{a},t)\, n_i = 0, \quad \text{(traction free boundary)}$$

$$q_i(\mathbf{a},t)\, n_i = 0, \quad \text{(flux free boundary)} \tag{12}$$

n_i being the ith component of the outwardly directed normal to ∂B, the boundary of the body.

Initial Conditions

$$x_i(\mathbf{a},0) = a_i, \quad \text{(no initial displacement)}$$

$$\dot{x}_i(\mathbf{a},0) = 0, \quad \text{(no initial velocity)} \tag{13}$$

$$\eta(\mathbf{a},0) = 0$$

$$s(\mathbf{a},0) = 0, \quad \text{(no initial entropy)} \tag{14}$$

Note that the first equation of (14) is satisfied by choosing

$$\eta(\mathbf{a},t) = \int_0^t \tau(\mathbf{a},t')\, dt' \tag{15}$$

which determines η completely.

The system of equations (10), (13)-(15) is a well-posed, boundary-initial-value problem in x_i and η, provided that some regularity conditions are met by f_i and r, the external "loadings" (refer to Dafermos [15] for example). Once more the convolution is essential to obtain a meaningful set of boundary and initial conditions. This has to be contrasted with the classical Lagrangian for elasticity given by (9) which does not give rise to a well-posed initial boundary-value problem.

In the next section we generate material momenta from our Lagrangian density and we briefly examine the connection between these material momenta and the existence of path-independent integrals in thermoelasticity.

5 Conservation of the Material Momentum

The conservation laws can be derived in various ways. A familiar one involving use of the Lagrangian for the system under consideration consists of allowing a larger class of variations (with varying boundaries) and to impose the invariance conditions on the corresponding action integral (refer e.g., to Logan [16]). The alternative simpler procedure was suggested for an elastic continuum by Golebiewska Herrmann [7]. It involves the Lagrangian density only and consists of differentiating it with respect to independent

variables. The terms are grouped in such a way that the use of the equations of motion can be made. We follow that approach. In view of (8), this is a straightforward, though lengthy computation. After numerous integrations by parts, we obtain:

$$2\frac{d\mathcal{L}_{t_0}}{da_m} = 2\frac{\partial \mathcal{L}_{t_0}}{\partial a_m}\Big|_{\exp} - [(A),x_{i,m}]_{t_0}^{+} + [(B),\eta_{,m}]_{t_0}^{+}$$
$$- \partial_j\{[\sigma_{ij},x_{i,m}]_{t_0}^{+} - [q_j,\eta_{,m}]_{t_0}^{+}\} + \partial_t\{\rho(\mathbf{a})[\dot{x}_i,x_{i,m}]_{t_0}^{-}$$
$$- [s,\eta_m]_{t_0}^{-} - \tfrac{1}{2}\lambda_{ij}(\mathbf{a})[\eta_{,j},\eta_{,im}]_{t_0}^{-}\} \tag{16}$$

where (A) and (B) denote the left-hand sides of equations $(5a)$ and $(5b)$ and

$$\frac{\partial \mathcal{L}}{\partial_{a_m}}\Big|_{\exp}$$

refers to explicit differentiation of $\mathcal{L}_{t_0}$ with respect to a_i.

We define $B_{mj}(\mathbf{a},t,t_0)$ and $B_m(\mathbf{a},t,t_0)$ to be

$$B_{mj}(\mathbf{a},t,t_0) = -\tfrac{1}{2}[\sigma_{ij},x_{i,m}]_{t_0}^{+} + \tfrac{1}{2}[q_j,\eta_{,m}]_{t_0}^{-} - \mathcal{L}_{t_0}\delta_{jm}$$
$$B_m(\mathbf{a},t,t_0) = +\tfrac{1}{2}\rho(\mathbf{a})[\dot{x}_i,x_{i,m}]_{t_0}^{-} - \tfrac{1}{2}[s,\eta_{,m}]_{t_0}^{-}$$
$$- \tfrac{1}{4}\lambda_{ij}(\mathbf{a})[\eta_{,j},\eta_{,im}]_{t_0}^{-} \tag{17}$$

Then taking $(5a)$ and $(5b)$ into account, (16) reads as

$$\frac{\partial B_{mj}}{\partial a_j} + \frac{\partial B_m}{\partial t} = -\frac{\partial \mathcal{L}_{t_0}}{\partial a_m}\Big|_{\exp} \tag{18}$$

If we assume *homogeneity of the body and space uniformity of the mechanical and thermal loadings*, the explicit derivatives vanish and we are left with space-time, divergence-free quantities composed of B_{mj} and B_m. Following the terminology of Golebiewska Herrmann [7], we will refer to (18) with vanishing right-hand side as to a *material conservation law*. Introducing the symbolic notation $\mathfrak{B} = (B_{mj},B_m)$, the latter can be written shortly as

$$div_4\,\mathfrak{B} = 0 \tag{19}$$

The relation (18), however, has an even more general meaning, since

$$\frac{\partial \mathcal{L}_{t_0}}{\partial a_m}\Big|_{\exp}$$

will be different from zero in any part of the body where a defect or an inhomogeneity is present. In this respect (18) can be viewed as a balance of material momentum.

An essential difference with the elastic case is evidenced in (17). The material momentum $\mathfrak{B}$ is nonlocal in time. This is a direct consequence of the dissipation and it has profound implications on the possibility of generating path-independent integrals. Such integrals are usually obtained by application of the divergence theorem to a material balance law. A direct inspection of (19) shows that if we consider a domain of the form $Gx[0,t_0]$, where G is a subdomain of B, a significant simplification occurs due to the symmetry property of the convolution. Defining $I_{mj}(\mathbf{a},t_0)$ to be

$$\int_0^{t_0} B_{mj}(\mathbf{a},t,t_0)\,dt = \int_0^{t_0}\{\mathcal{L}_{t_0}\delta_{jm} - \sigma_{ij}(\mathbf{a},t)\,x_{i,m}(\mathbf{a},t_0-t)$$
$$+ q_j(\mathbf{a},t)\,\eta_{,m}(\mathbf{a},t_0-t)\}\,dt \tag{20}$$

we obtain from (19)

$$\int_{\partial G} I_{mj}(\mathbf{a},t_0)\,n_j d\sigma = -\Big\{\int_G (\rho\dot{x}_i(\mathbf{a},t)\,x_{i,m}(\mathbf{a},t_0-t)$$
$$-s(\mathbf{a},t)\,\eta_{,m}(\mathbf{a},t_0-t)$$
$$- \tfrac{1}{2}\lambda_{ij}\eta_{,j}(\mathbf{a},t)\,\eta_{,im}(\mathbf{a},t_0-t)\,d^3a\Big\}\Big|_{t=0}^{t=t_0} \tag{21}$$

where ∂G denotes the boundary of G, and n_j is the jth component of the outer normal to ∂G. We assume that this boundary is smooth enough for application of the divergence theorem.

Making use of relation (15), equation (21) becomes

$$\int_{\partial G} I_{mj}(\mathbf{a},t_0)\,n_j d\sigma = -\Big\{\int_G \rho\dot{x}_i(\mathbf{a},t)\,x_{i,m}(\mathbf{a},t_0 - t)\,d^3a\Big\}\Big|_{t=0}^{t=t_0}$$
$$-\int_G s(\mathbf{a},0)\,\eta_{,m}(\mathbf{a},t_0)\,d^3a \tag{22}$$

This last equation is the closest we can get to path independence: observe that the right-hand side depends only on $t = 0$ and $t = t_0$; it is the material analog of an impulse-momentum type relation.

In contrast with the elastic case, there is no true path independence even in a quasi-static case unless the initial entropy of the system happens to be equal to zero.

But the most striking difference with the classical path-independent integrals of fracture mechanics lies in the nonlocal character of all the quantities defined. It is only after specifying over which time length the system is observed that it becomes possible to devise for that time interval quantities such as $I_{mj}(\mathbf{a},t_0)$.

In a later paper we will show that, in analogy with the elastic case, the quantity

$$\int_S I_{mj}(\mathbf{a},t_0)\,n_j d\sigma$$

where S is the surface around a crack in a given body, and B is related to the total energy change of the system due to an infinitesimal translation in the mth direction of this crack, over the interval $[0,t_0]$. What this suggests is that it is not possible, as it is in elasticity, to determine the instantaneous force acting on the crack and to decide whether the crack will start propagating at a given instant in time, but that we should be able to determine if a crack will start propagating during any given time interval.

6 Concluding Remarks

The proposed Lagrangian differs from usual Lagrangians because of its nonlocality in time. In many physical theories there exists a strong relation between energy and time: in classical mechanics or field theory, invariance with respect to translation in time leads to conservation of energy; in quantum mechanics, Heisenberg's principle couples time and energy. Because the basic feature distinguishing a thermoelastic system for an elastic one lies in its dissipative character, it seems justified to adopt an expression where the time dependence differs from the one most often used in elasticity.

Let us emphasize once more that only one set of equations is derived (from the variational principle) in contradistinction to the formulation using adjoint fields.

In a following paper we will concentrate on quasi-static behavior and on the possible relation of material momentum to energy release rates.

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