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HOMOGENIZATION AND LINEAR THERMOELASTICITY*

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Abstract. We study homogenization of linear dynamic thermoelasticity with rapidly varying coefficients, using a semigroup approach. The resulting homogenized problem exhibits an unusual change in initial temperature.

A formal asymptotic analysis predicts fast time oscillations in the temperature field. These oscillations explain the temperature shift, and show that, for our problem, weak convergence in time is the best convergence that one can obtain.

Introduction. We discuss the problem of "homogenizing" the equations of linear thermoelasticity when the mechanical and thermal properties are periodic and rapidly varying. Following Bensoussan, Lions and Papanicolaou [1] and Sanchez-Palencia [7] and using a semigroup approach, we show rigorously that, as the period of the coefficients goes to zero, the solution of these equations converges to the solution of a related constant coefficient problem, the *homogenized* problem. Then using a formal multiple-scales method, we give what we believe to be a satisfying interpretation of some surprising features of the results.

Thermoelastic behavior is characterized by the coupling of hyperbolic equations of motion and a parabolic heat equation. This leads to several interesting phenomena in the homogenization process.

Fast time oscillations in the temperature field are observed; their phase is completely determined. Thus the solutions can only converge in a weak sense in time to a slowly varying homogenized solution.

Furthermore, the initial data for the homogenized problem are related to the initial data of the inhomogeneous problem by *a linear transformation* which is not a projection. We know of no other examples of such a phenomenon.

In §1, we formulate and prove the existence of a homogenized thermoelastic medium. Section 2 contains the more formal arguments and the fast oscillations results, which are at the root of the observed change in initial data.

1. Homogenization of the thermoelastic problem. To reduce the overwhelmingly cumbersome notations that characterize thermoelasticity, we will place ourselves in a scalar setting, that is, one where the displacement field is taken to be scalar valued. Duvaut and Lions [2] show, using Korn's theorem, that this is no loss of generality.

We consider a domain Ω of $\mathbb{R}^n$. The degree of smoothness of $\delta\Omega$ will depend on the type of boundary conditions adopted. We will always assume that $\delta\Omega$ is smooth enough for one to be in position to apply Rellich's theorem on compact imbeddings of Sobolev spaces (Folland [3, Chap. 6]).

We will refer to $Y=\Pi_{i=1}^{n}]0,y_i^0[$ as the "reference cell"; $|Y|$ is its volume.

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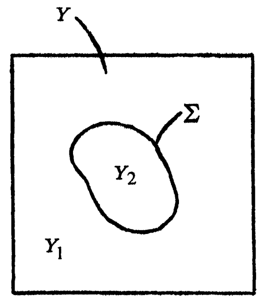

FIG 1.

If Σ is a smooth hypersurface dividing Y into Y_1 and Y_2 (see Fig. 1), we define $a_{ij}(y)$, $\lambda_{ij}(y)$, $\alpha_i(y)$, $\beta(y)$, $\rho(y)$ to be real Y-periodic functions, smooth and bounded on the closure of Y_1 and Y_2 but with Σ as potential surface of discontinuity.

Furthermore, $a_{ij}(y)$, $\lambda_{ij}(y)$ are assumed to be symmetric and strongly elliptic on Y, that is, there exists $\alpha>0$ such that for all ξ's in $\mathbb{R}^n$

$$a_{ij}(y)\ \big(\text{resp.}\lambda_{ij}(y)\big)\ \xi_i\xi_j\geq\alpha\xi_i^2 \quad \text{on } Y, \tag{1.1}$$

$\beta(y)$ and $\rho(y)$ are positive and bounded away from zero. We finally choose α such that α^{-1} is a common upper bound of the L_∞-norms of the coefficients. We extend all coefficients to all of $\mathbb{R}^n$ by periodicity. Our equations are (Kupradze [5])

$$\begin{aligned}\rho\left(\frac{x}{\varepsilon}\right)\frac{\partial^2 u^\varepsilon}{\partial t^2}&=\frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\varepsilon}\right)\left(\frac{\partial u^\varepsilon}{\partial x_j}-\alpha_j\left(\frac{x}{\varepsilon}\right)\tau^\varepsilon\right)\right),\\ \beta\left(\frac{x}{\varepsilon}\right)\frac{\partial \tau^\varepsilon}{\partial t}&=\frac{\partial}{\partial x_i}\left(\lambda_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial \tau^\varepsilon}{\partial x_j}\right)-a_{ij}\left(\frac{x}{\varepsilon}\right)a_j\left(\frac{x}{\varepsilon}\right)\frac{\partial^2 u^\varepsilon}{\partial t\partial x_i}.\end{aligned} \tag{1.2}$$

In (1.2), u^ε represents the displacement field and τ^ε the temperature increment field. The first equation is the scalar version of the equations of motion and the second is the heat conduction equation. The coupling between the equations results from consideration of the interaction between deformation and temperature: a temperature change induces strain and conversely. Finally, the rapid spatial oscillations in the coefficients translate the periodic structure of the body which comes from the assembling of ε-scaled versions of the reference cell Y. This body has to be thought of as made of a composite material where both constituents behave thermoelastically.

For the sake of simplicity we will only consider Dirichlet boundary conditions throughout:

$$u^\varepsilon=0, \qquad \tau^\varepsilon=0 \quad \text{on } \partial\Omega. \tag{1.3}$$

And for initial conditions, we will have:

$$u^\varepsilon(x,0)=f(x), \quad \frac{\partial u^\varepsilon}{\partial t}(x,0)=g(x), \quad \tau^\varepsilon(x,0)=k(x). \tag{1.4}$$

Our goal is to study the behavior of u^ε and τ^ε as ε, the period, goes to zero.

We define H to be:

$$H=H_0^1(\Omega)\times L_2(\Omega)\times L_2(\Omega). \tag{1.5}$$

On H, we define the operator A_ε:

(1.6)

$$A_\varepsilon = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{\rho\left(\frac{x}{\varepsilon}\right)} \frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_j}\right) & 0 & -\frac{1}{\rho\left(\frac{x}{\varepsilon}\right)} \frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\varepsilon}\right)\alpha_j\left(\frac{x}{\varepsilon}\right)\cdot\right) \\ 0 & -\frac{1}{\beta\left(\frac{x}{\varepsilon}\right)} a_{ij}\left(\frac{x}{\varepsilon}\right)\alpha_j\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_i} & \frac{1}{\beta\left(\frac{x}{\varepsilon}\right)}\frac{\partial}{\partial x_i}\left(\lambda_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_j}\right) \end{bmatrix}$$

with domain

(1.7) $D(A_\varepsilon) = \{U = (u, u_t, \tau) \in H_0^1(\Omega) \times L_2(\Omega) \times H_0^1(\Omega)$ such that $A_\varepsilon U$ (taken in a distributional sense) belongs to $H\}$.

Then the following proposition holds:

PROPOSITION 1.1. *A_ε generates in H a strongly continuous semigroup of operators $S_\varepsilon(t)$ such that*:

(1.8) $$\|S_\varepsilon(t)\| \le \alpha^{-1} \quad (\forall t > 0).$$

Proof. We first consider for a fixed ε the norm

(1.9) $$|U|_\varepsilon^2 = \int_\Omega \left[a_{ij}\left(\frac{x}{\varepsilon}\right) \frac{\partial u}{\partial x_j}\frac{\partial \tilde{u}}{\partial x_i} + \rho\left(\frac{x}{\varepsilon}\right) u_t \tilde{u}_t + \beta\left(\frac{x}{\varepsilon}\right)\tau\tilde{\tau} \right] dx$$

where $\tilde{}$ denotes complex conjugation.

In view of the properties of the coefficients, $|\cdot|_\varepsilon$ is a norm on H, equivalent to the natural Sobolev norm on H, noted $\|\cdot\|$, that is, if U is in H,

(1.10) $$\alpha\|U\|^2 \le |U|_\varepsilon^2 \le \alpha^{-1}\|U\|^2.$$

In the norm $|\cdot|_\varepsilon$, A_ε generates a semigroup of contractions. Indeed, the domain $D(A_\varepsilon)$ is dense, since, though $\mathcal{C}_0^\infty(\Omega)$ functions do not belong to it, $\mathcal{C}_0^\infty(\Omega)$ functions whose conormal derivatives are 0 together with their third component on the only possible surfaces of discontinuity for the coefficients (i.e., the ε-scaled versions of Σ in each of the cells making up Ω) do belong to the domain $D(A_\varepsilon)$. Checking that A_ε is closed, that the range of $(1 - A_\varepsilon)$ is H itself and that A_ε is dissipative offers no special difficulties (see Francfort [4] for full details). Note that the measure of the dissipation,

(1.11) $$\operatorname{Re}(A_\varepsilon U, U) = -\operatorname{Re}\left(\int_\Omega \lambda_{ij}\left(\frac{x}{\varepsilon}\right) \frac{\partial \tau}{\partial x_j}\frac{\partial \tilde{\tau}}{\partial x_i} dx \right) \le -\alpha|\nabla\tau|_{L_2(\Omega)}^2$$

(in view of the properties of the λ_{ij}'s), is precisely the physical dissipation due to the heat fluxes through the domain.

The result then follows from the application of Lumer–Phillips's theorem (Yosida [8, Chap. 9]). Therefore,

(1.12) $$|S_\varepsilon(t)U|_\varepsilon \le |U|_\varepsilon \quad \text{for any } U \text{ in } H,$$

and thus, using (1.10),

(1.13) $$\|S_\varepsilon(t)U\| \le \alpha^{-1}\|U\|,$$

which completes the proof. □

We now leave the time dependent formulation and examine the behavior of the resolvent of A_ε, $R_\lambda(A_\varepsilon)$ as ε goes to 0. At the end of this section we will reintroduce the time dependence by using some basic properties of semigroups.

It is a direct consequence of (1.8) (Yosida [8, Chap. 9]) that the right half complex plane belongs to the resolvent set of A_ε, for every ε. Let us consider $F=(f,g,k)$ to be an element of H. We take λ to be real strictly positive. The following string of equivalences holds:

(1.14)

$$R_\lambda(A_\varepsilon)F=U_\varepsilon, \quad (U_\varepsilon=(u^\varepsilon,u_t^\varepsilon,\tau^\varepsilon))$$

$$\Leftrightarrow \quad \lambda u^\varepsilon - u_t^\varepsilon = f,$$

$$\rho\left(\frac{x}{\varepsilon}\right)\lambda u_t^\varepsilon - \frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\varepsilon}\right)\left(\frac{\partial u^\varepsilon}{\partial x_j} - \alpha_j\left(\frac{x}{\varepsilon}\right)\tau^\varepsilon\right)\right) = \rho\left(\frac{x}{\varepsilon}\right)g,$$

$$\beta\left(\frac{x}{\varepsilon}\right)\lambda\tau^\varepsilon - \frac{\partial}{\partial x_i}\left(\lambda_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial \tau^\varepsilon}{\partial x_j}\right) + a_{ij}\left(\frac{x}{\varepsilon}\right)\alpha_j\left(\frac{x}{\varepsilon}\right)\frac{\partial u_t^\varepsilon}{\partial x_i} = \beta\left(\frac{x}{\varepsilon}\right)k,$$

(1.15)

$$\Leftrightarrow \quad \lambda u^\varepsilon - u_t^\varepsilon = f,$$

$$\lambda^2\rho\left(\frac{x}{\varepsilon}\right)u^\varepsilon - \frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\varepsilon}\right)\left(\frac{\partial u^\varepsilon}{\partial x_j} - \alpha_j\left(\frac{x}{\varepsilon}\right)\tau^\varepsilon\right)\right) = \rho\left(\frac{x}{\varepsilon}\right)(\lambda f + g),$$

$$\lambda\beta\left(\frac{x}{\varepsilon}\right)\tau^\varepsilon - \frac{\partial}{\partial x_i}\left(\lambda_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial \tau^\varepsilon}{\partial x_j}\right) + \lambda a_{ij}\left(\frac{x}{\varepsilon}\right)\alpha_j\left(\frac{x}{\varepsilon}\right)\frac{\partial u^\varepsilon}{\partial x_i}$$

$$= \beta\left(\frac{x}{\varepsilon}\right)k + a_{ij}\left(\frac{x}{\varepsilon}\right)\alpha_j\left(\frac{x}{\varepsilon}\right)\frac{\partial f}{\partial x_i}.$$

The last two equations (1.15) have a unique solution $v^\varepsilon=\lambda u^\varepsilon$, τ^ε in $(H_0^1(\Omega))^2$, since the Dirichlet form d_ε defined as

$$d_\varepsilon((v^\varepsilon,\tau^\varepsilon),(\zeta,\eta)) = \frac{1}{\lambda}\int_\Omega a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial v^\varepsilon}{\partial x_j}\frac{\partial \bar{\zeta}}{\partial x_i}dx + \lambda\int_\Omega \rho\left(\frac{x}{\varepsilon}\right)v^\varepsilon\bar{\zeta}\,dx$$

(1.16)
$$-\int_\Omega a_{ij}\left(\frac{x}{\varepsilon}\right)\alpha_j\left(\frac{x}{\varepsilon}\right)\tau^\varepsilon\frac{\partial \bar{\zeta}}{\partial x_i}dx + \int_\Omega \lambda_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial \tau^\varepsilon}{\partial x_j}\frac{\partial \bar{\eta}}{\partial x_i}dx$$

$$+\lambda\int_\Omega \beta\left(\frac{x}{\varepsilon}\right)\tau^\varepsilon\bar{\eta}\,dx + \int_\Omega a_{ij}\left(\frac{x}{\varepsilon}\right)\alpha_j\left(\frac{x}{\varepsilon}\right)\frac{\partial v^\varepsilon}{\partial x_i}\bar{\eta}\,dx$$

is strictly coercive on $(H_0^1(\Omega))^2$, in view of the properties of the coefficients.

If we manage to find a limit for u^ε, τ^ε as ε goes to zero, then going back up through the string (1.14) will enable us to obtain the limit of $R_\lambda(A_\varepsilon)F$.

Performing the limiting process in (1.15) is the task of the homogenization method. Rather than exposing all the details of the argument, we merely mention the different steps that were performed, underlining only the ones that are not standard. For further details the reader is to refer to Bensoussan, Lions and Papanicolaou [1, Chap. 1, esp. §§3, 9 and 13], or, for our problem, to Francfort [4].

Firstly, one shows that u_ε and τ_ε are bounded in $(H_0^1(\Omega))^2$, which immediately implies the existence of a weakly convergent subsequence in $(H_0^1(\Omega))^2$ converging to

(u, τ). Since we ultimately show that any convergent subsequence converges to the same limit, we do not distinguish between the sequence and subsequences of this sequence.

Then, defining

$$\begin{aligned}
&\sigma_i^\varepsilon = a_{ij}\left(\frac{x}{\varepsilon}\right)\left(\frac{\partial u^\varepsilon}{\partial x_j} - \alpha_j\left(\frac{x}{\varepsilon}\right)\tau^\varepsilon\right) && \text{the stress,}\\
(1.17)\qquad &\kappa_i^\varepsilon = \lambda_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial \tau^\varepsilon}{\partial x_j} && \text{the heat flux,}\\
&\nu^\varepsilon = a_{ij}\left(\frac{x}{\varepsilon}\right)\alpha_j\left(\frac{x}{\varepsilon}\right)\frac{\partial u^\varepsilon}{\partial x_i},
\end{aligned}$$

it is easy to conclude that these quantities converge weakly in $L_2(\Omega)$ to σ_i, κ_i, ν, which in turn satisfy:

$$(1.18)\qquad \bar{\rho}\lambda^2 u - \frac{\partial \sigma_i}{\partial x_i} = \bar{\rho}(\lambda f + g), \qquad \bar{\beta}\lambda\tau - \frac{\partial \kappa_i}{\partial x_i} + \lambda\nu = \bar{\beta}k + \overline{a_{ij}\alpha_j}\frac{\partial f}{\partial x_i},$$

where, from now on, $\overline{}$ will denote the Y-average $\frac{1}{|Y|}\int_Y dy$.

It remains to determine σ_i, κ_i, and ν. This is the core of homogenization. To this effect we define $\chi_k(y)$, $\Theta_k(y)$, $\Psi(y)$ to be the *unique periodic solutions*, up to a constant, in $H^1(Y)$ of:

$$\begin{aligned}
&-\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial \chi_k}{\partial y_j}\right) = -\frac{\partial a_{ik}}{\partial y_i}(y),\\
(1.19)\qquad &-\frac{\partial}{\partial y_i}\left(\lambda_{ij}(y)\frac{\partial \Theta_k}{\partial y_j}\right) = -\frac{\partial \lambda_{ik}}{\partial y_i}(y),\\
&-\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial \Psi}{\partial y_j}\right) = -\frac{\partial}{\partial y_i}(a_{ij}(y)\alpha_j(y)).
\end{aligned}$$

Ψ can be considered as nonstandard with respect to the "classical" case. The functions:

$$(1.20)\qquad w_k^\varepsilon = x_k - \varepsilon\chi_k\left(\frac{x}{\varepsilon}\right), \qquad z_k^\varepsilon = x_k - \varepsilon\Theta_k\left(\frac{x}{\varepsilon}\right)$$

satisfy:

$$\begin{aligned}
&\int_\Omega a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial w_k^\varepsilon}{\partial x_j}\frac{\partial \tilde{\omega}}{\partial x_i}dx = 0,\\
(1.21)\qquad &\int_\Omega \lambda_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial z_k^\varepsilon}{\partial x_j}\frac{\partial \tilde{\mu}}{\partial x_i}dx = 0 \qquad \text{for any } \omega, \mu \text{ in } H_0^1(\Omega).
\end{aligned}$$

Taking ω and μ to be $\mathcal{C}_0^\infty(\Omega)$ functions and making use of (1.16), (1.21), we have:

(1.22)

$$\begin{aligned}
d_\varepsilon((\lambda u^\varepsilon, \tau^\varepsilon), (\omega w_k^\varepsilon, \mu z_k^\varepsilon)) &- \int_\Omega a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial w_k^\varepsilon}{\partial x_j}\frac{\partial(\tilde{\omega}u^\varepsilon)}{\partial x_i}dx - \int_\Omega \lambda_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial z_k^\varepsilon}{\partial x_j}\frac{\partial(\tilde{\mu}\tau^\varepsilon)}{\partial x_i}dx\\
&= \int_\Omega \rho\left(\frac{x}{\varepsilon}\right)(\lambda f + g)\tilde{\omega}w_k^\varepsilon dx + \int_\Omega\left(\beta\left(\frac{x}{\varepsilon}\right)k + a_{ij}\left(\frac{x}{\varepsilon}\right)\alpha_j\left(\frac{x}{\varepsilon}\right)\frac{\partial f}{\partial x_i}\right)\tilde{\mu}z_k^\varepsilon dx.
\end{aligned}$$

In (1.22), we have in essence subtracted from the variational formulation of (1.15) appropriate expressions equal to 0 in order to eliminate products of weak convergences.

It is then possible to go to the limit in (1.22) in a way identical to Bensoussan, Lions and Papanicolaou [1, Chap. 1, §3]. Upon our performing this limiting process, σ_i and κ_i come out to be:

$$\tag{1.23} \begin{aligned} \sigma_i &= \overline{\left(a_{ij}-a_{kj}\frac{\partial\chi_i}{\partial y_k}\right)}\frac{\partial u}{\partial x_j}-\overline{\left(a_{ij}\alpha_j-a_{kj}\alpha_j\frac{\partial\chi_i}{\partial y_k}\right)}\tau,\\ \kappa_i &= \overline{\left(\lambda_{ij}-\lambda_{kj}\frac{\partial\Theta_i}{\partial y_k}\right)}\frac{\partial\tau}{\partial x_j}. \end{aligned}$$

Determining ν requires some additional effort and the use of Ψ. One defines g^ε to be:

$$\tag{1.24} g^\varepsilon = 1+\varepsilon\Psi\left(\frac{x}{\varepsilon}\right);$$

then it satisfies, for any ω in $H_0^1(\Omega)$:

$$\tag{1.25} \int_\Omega a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial g^\varepsilon}{\partial x_j}\frac{\partial\tilde{\omega}}{\partial x_i}dx=\int_\Omega a_{ij}\left(\frac{x}{\varepsilon}\right)\alpha_j\left(\frac{x}{\varepsilon}\right)\frac{\partial\tilde{\omega}}{\partial x_i}dx.$$

Repeating the procedure of (1.22) but with μ equal to 0 and w_k^ε replaced by g^ε, we determine ν to be:

$$\tag{1.26} \nu=\overline{\left(a_{ij}\alpha_j-a_{ij}\frac{\partial\Psi}{\partial y_j}\right)}\frac{\partial u}{\partial x_i}+\overline{\left(a_{kj}\alpha_j\frac{\partial\Psi}{\partial y_k}\right)}\tau.$$

Defining a_{ij}, A_i, B_i, λ_{ij}, γ_i, σ to be

$$\tag{1.27} \begin{aligned} a_{ij} &= \overline{a_{ij}-a_{kj}\frac{\partial\chi_i}{\partial y_k}}, & \lambda_{ij} &= \overline{\lambda_{ij}-\lambda_{kj}\frac{\partial\Theta_i}{\partial y_k}},\\ A_i &= \overline{a_{ij}\alpha_j-a_{kj}\alpha_j\frac{\partial\chi_i}{\partial y_k}}, & \gamma_i &= \overline{a_{ij}\alpha_j}-A_i,\\ B_i &= \overline{a_{ij}\alpha_j-a_{ij}\frac{\partial\Psi}{\partial y_j}}, & \sigma &= \overline{a_{kj}\alpha_j\frac{\partial\Psi}{\partial y_k}}, \end{aligned}$$

it can be shown, using (1.19), that a_{ij} and λ_{ij} are symmetric positive definite hence invertible, that A_i and B_i are equal and that σ is positive.

We set:

$$\tag{1.28} \alpha_i=a_{ik}^{-1}A_k=a_{ik}^{-1}B_k.$$

Recalling (1.18), (1.23), (1.26)–(1.28) yields:

$$\tag{1.29} \begin{aligned} &\bar{\rho}\lambda^2u-a_{ij}\left(\frac{\partial^2u}{\partial x_i\partial x_j}-\alpha_j\frac{\partial\tau}{\partial x_i}\right)=\bar{\rho}(\lambda f+g),\\ &(\bar{\beta}+\sigma)\lambda\tau-\lambda_{ij}\frac{\partial^2\tau}{\partial x_i\partial x_j}+\lambda a_{ij}\alpha_j\frac{\partial u}{\partial x_i}=\bar{\beta}k+\overline{a_{ij}(y)\alpha_j(y)}\frac{\partial f}{\partial x_i}, \end{aligned}$$

and, in view of the properties of the a_{ij}'s and λ_{ij}'s, the Dirichlet form associated with (1.29) is strictly coercive on $(H_0^1(\Omega))^2$, hence (1.29) admits a unique solution in $(H_0^1(\Omega))^2$.

Then, using (1.14), we obtain the following proposition:

PROPOSITION 1.2. *$R_\lambda(A_\varepsilon)F$ converges weakly in $(H_0^1(\Omega))^3$ to the unique solution in $(H_0^1(\Omega))^3$ of*:

$$
\begin{aligned}
&\lambda u-u_t=f,\\
&\lambda\bar{\rho}u_t-a_{ij}\left(\frac{\partial^2 u}{\partial x_i\partial x_j}-\alpha_j\frac{\partial\tau}{\partial x_i}\right)=\bar{\rho}g,\\
&\lambda(\bar{\beta}+\sigma)\tau-\lambda_{ij}\frac{\partial^2\tau}{\partial x_i\partial x_j}+a_{ij}\alpha_j\frac{\partial u_t}{\partial x_i}=\bar{\beta}k+\gamma_i\frac{\partial f}{\partial x_i}.
\end{aligned}
\tag{1.30}
$$

We then define A to be:

$$
A=\begin{pmatrix}
0 & 1 & 0\\
\frac{1}{\bar{\rho}}a_{ij}\frac{\partial^2}{\partial x_i\partial x_j} & 0 & -\frac{1}{\bar{\rho}}a_{ij}\alpha_j\frac{\partial}{\partial x_i}\\
0 & -\frac{1}{\bar{\beta}+\sigma}a_{ij}\alpha_j\frac{\partial}{\partial x_i} & \frac{1}{\bar{\beta}+\sigma}\lambda_{ij}\frac{\partial^2}{\partial x_i\partial x_j}
\end{pmatrix}.
\tag{1.31}
$$

It is simply a matter of reproducing the proof of Proposition 1.1, but with constant coefficients this time, to show that A *generates a semigroup of operators* $S(t)$ such that

$$
\|S(t)\|\leq\alpha' \quad \text{for any } t\geq 0.
\tag{1.32}
$$

Renaming α^{-1} the maximum of α' and α^{-1}, we deduce from Proposition 1.2 and (1.32) the following corollary:

COROLLARY 1.2. *$R_\lambda(A_\varepsilon)F$ converges weakly in $(H_0^1(\Omega))^3$ to $R_\lambda(A)\underset{\sim}{F}$ where*:

$$
\underset{\sim}{F}=\left(f,g,\frac{\bar{\beta}k+\gamma_i\frac{\partial f}{\partial x_i}}{\bar{\beta}+\sigma}\right).
\tag{1.33}
$$

Now, (1.8) implies that, for any U, there is a bounded subsequence of $S_\varepsilon(t)U$ that converges weak$*$ in $L_\infty(\mathbb{R}_+,H)$ to $\mathcal{G}(t)$ an element of $L_\infty(\mathbb{R}_+,H)$. This is a direct consequence of the separability of $L_1(\mathbb{R}_+,H)$ and of Banach–Alaoglu's theorem (Rudin [8, Chap. 3]). Still identifying a sequence with its subsequences, we get that, for any V in H,

$$
\int_0^\infty e^{-\lambda t}(S_\varepsilon(t)U,V)_H\,dt \overset{\varepsilon\to 0}{\to} \int_0^\infty e^{-\lambda t}(\mathcal{G}_{(t)},V)_H\,dt
\tag{1.34}
$$

where $(\cdot,\cdot)_H$ is the natural inner product on H. But the resolvent of the generator of a semigroup applied on a vector U is equal to the Laplace transform of the semigroup acting on U (Yosida [8, Chap. 9]), thus:

$$
\int_0^\infty e^{-\lambda t}(S_\varepsilon(t)U,V)_H\,dt=(R_\lambda(A_\varepsilon)U,V)_H,
\tag{1.35}
$$

which itself converges to

$$(R_\lambda(A)\underset{\sim}{U},V)_H=\int_0^\infty e^{-\lambda t}(S(t)\underset{\sim}{U},V)_H dt. \tag{1.36}$$

Since V is arbitrary, we finally obtain, using the uniqueness of Laplace transforms of scalar functions:

$$\mathcal{G}_{(t)}=S(t)\underset{\sim}{U} \qquad (t\geq 0). \tag{1.37}$$

We have proved in this section the following theorem:

THEOREM. *The generalized solution of* (1.2) *with Dirichlet boundary conditions and initial conditions* (f,g,k) *in* H *converges weak* in* $L_\infty(\mathbb{R}_+,H)$ *to the generalized solution of*:

$$\begin{aligned}\bar{\rho}\frac{\partial^2 u}{\partial t^2}&=a_{ij}\left(\frac{\partial^2 u}{\partial x_i\partial x_j}-\alpha_j\frac{\partial \tau}{\partial x_i}\right),\\ (\bar{\beta}+\sigma)\frac{\partial \tau}{\partial t}&=\lambda_{ij}\frac{\partial^2 \tau}{\partial x_i\partial x_j}-a_{ij}\alpha_j\frac{\partial^2 u}{\partial t\partial x_i}\end{aligned} \tag{1.38}$$

with Dirichlet boundary conditions and initial conditions

$$\left(f,g,\frac{\bar{\beta}k+\gamma_i\dfrac{\partial f}{\partial x_i}}{\bar{\beta}+\sigma}\right). \tag{1.39}$$

Before concluding this section, let us emphasize once more the rather unusual change in initial temperature in (1.39).

2. Fast oscillations of the temperature field. Since, through a L_∞ weak* type of convergence, a rapidly oscillating function (like $e^{it/\varepsilon}$) goes to 0, it is fairly natural to expect a t/ε dependence of u^ε and τ^ε. This kind of problem is most easily addressed using asymptotic expansion techniques. We have already mentioned the semiheuristic character of this section, so that we will not dwell on the restrictions to the problem that would make the argument totally rigorous.

Recalling (1.2) we now suppose that u^ε and τ^ε are functions of *both* t and $\delta=t/\varepsilon$; ∂_t becomes $\partial_t+\frac{1}{\varepsilon}\partial_\delta$. We then Laplace transform (1.2) with respect to *both* t and δ, the dual variables being respectively ζ and μ. From now on:

$\hat{}$ will denote the t-Laplace transform,

$\tilde{}$ will denote the δ-Laplace transform,

$\check{}$ will denote $\hat{\tilde{}}$ or $\tilde{\hat{}}$.

In order to be able to perform these transformations, we need to impose initial conditions on both t and δ. We will set:

$$\begin{aligned}u^\varepsilon(x;0,\delta)&=f(x), & u^\varepsilon(x;t,0)&=p(x,t),\\ \frac{\partial u^\varepsilon}{\partial t}(x;0,\delta)&=g(x), & \frac{\partial u^\varepsilon}{\partial \delta}(x;t,0)&=q(x,t),\\ \tau^\varepsilon(x;0,\delta)&=k(x), & \tau^\varepsilon(x;t,0)&=\Theta(x,t),\end{aligned} \tag{2.1}$$

where f, g, k are as before and p, q, Θ are unknown. We obtain:

(2.2)

$$\rho\left(\frac{x}{\varepsilon}\right)\left\{\left(\zeta^2\check{u}^\varepsilon-\frac{\zeta f}{\mu}-\frac{g}{\mu}\right)+\frac{2}{\varepsilon}(\zeta\mu\check{u}^\varepsilon-\zeta\hat{p})+\frac{1}{\varepsilon^2}(\mu^2\check{u}^\varepsilon-\mu\hat{p}-\hat{q})\right\}$$
$$=\frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\varepsilon}\right)\left(\frac{\partial\check{u}^\varepsilon}{\partial x_j}-\alpha_j\left(\frac{x}{\varepsilon}\right)\check{\tau}^\varepsilon\right)\right),$$
$$\beta\left(\frac{x}{\varepsilon}\right)\left\{\left(\zeta\check{\tau}^\varepsilon-\frac{k}{\mu}\right)+\frac{1}{\varepsilon}(\mu\check{\tau}^\varepsilon-\hat{\Theta})\right\}$$
$$=\frac{\partial}{\partial x_i}\left(\lambda_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial\check{\tau}^\varepsilon}{\partial x_j}\right)-a_{ij}\left(\frac{x}{\varepsilon}\right)\alpha_j\left(\frac{x}{\varepsilon}\right)\left\{\frac{\partial}{\partial x_i}\left(\zeta\check{u}^\varepsilon-\frac{f}{\mu}\right)+\frac{1}{\varepsilon}\frac{\partial}{\partial x_i}(\mu\check{u}^\varepsilon-\hat{p})\right\}.$$

We seek an expansion of u^ε and τ^ε in the form

$$u^\varepsilon=\sum\varepsilon^i u_i(x,y,t,\delta),\qquad \tau^\varepsilon=\sum\varepsilon^i\tau_i(x,y,t,\delta)\quad\text{where } y=\frac{x}{\varepsilon}. \tag{2.3}$$

The dependence of the u_i's and τ_i's on y is taken to be *Y-periodic*. This is always what is assumed when performing double scaling in space in problems related to homogenization.

We also need to control the fast time behavior of u_i and τ_i. Since we would like them to be oscillating in δ, or, at least, to be such that

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T u_i(x,t,y,\delta)\,d\delta\ (\text{respectively } \tau_i) \tag{2.4}$$

exist and be finite, we are led through Wiener's Tauberian theorem (Rudin [6, Chap. 9]) to suppose that

$$\lim_{\mu\to 0}\mu\check{u}_i\ (\text{respectively } \mu\check{\tau}_i)\ \text{exists and is finite,} \tag{2.5}$$

and we will furthermore assume that this limit is to be taken pointwise in x and weakly in $H^1(Y)$ with regard to the y dependence.

With these considerations in mind we can proceed to replace $\partial/\partial x_i$ by $\partial/\partial x_i + \frac{1}{\varepsilon}\partial/\partial y_i$ and u^ε and τ^ε by their expansions in (2.2).

We obtain two "series" in ascending powers of ε starting at ε^{-2}; we successively identify the factors of each of these powers to 0. As factor of ε^{-2} we get:

$$\begin{aligned}&\rho(y)(\mu^2\check{u}_0-\mu\hat{p}-\hat{q})=\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial\check{u}_0}{\partial y_j}\right),\\&\frac{\partial}{\partial y_i}\left(\lambda_{ij}(y)\frac{\partial\check{\tau}_0}{\partial y_j}\right)-a_{ij}(y)\alpha_j(y)\frac{\partial}{\partial y_i}(\mu\check{u}_0-\hat{p})=0.\end{aligned} \tag{2.6}$$

Since the Dirichlet form associated to the operator

$$D=\rho(y)\mu^2\cdot-\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial\cdot}{\partial y_j}\right) \tag{2.7}$$

is strictly coercive on the subspace of $H^1(Y)$ consisting of Y-periodic functions, the first equation of (2.6) has a unique solution there; thus $\hat{p}/\mu+\hat{q}/\mu^2$ is the solution. Hence

$$\check{u}_0=\frac{\hat{p}}{\mu}+\frac{\hat{q}}{\mu^2}. \tag{2.8}$$

But, in view of (2.5), (2.8) implies that $\hat{q}=0$, thus $\check{u}_0$ is equal to $\hat{p}/\mu$ and *does not depend on* y. Inverting (2.8) we obtain that u_0 *does not depend on* δ *either*;

$$u_0(x,t)=p(x,t). \tag{2.9}$$

Then, from the second equation of (2.6),

$$\check{\tau}_0=\check{\tau}_0(x,\mu), \tag{2.10}$$

since the only periodic solution of that equation is a constant with respect to y.

As factor of ε^{-1} we get, using (2.8), (2.10):

$$\begin{aligned} D\check{u}_1&=\frac{\partial a_{ij}}{\partial y_i}(y)\frac{\partial \check{u}_0}{\partial x_j}-\frac{\partial}{\partial y_i}\big(a_{ij}(y)\alpha_j(y)\big)\check{\tau}_0,\\ \beta(y)(\mu\check{\tau}_0-\hat{\Theta})&=\frac{\partial}{\partial y_i}\left(\lambda_{ij}(y)\frac{\partial\check{\tau}_1}{\partial y_j}\right)+\frac{\partial\lambda_{ij}}{\partial y_i}(y)\frac{\partial\check{\tau}_0}{\partial x_j}-\mu a_{ij}(y)\alpha_j(y)\frac{\partial\check{u}_1}{\partial y_i}. \end{aligned} \tag{2.11}$$

Defining χ_i^μ and Ψ^μ to be the unique periodic solutions in $H^1(Y)$ of

$$\begin{aligned} \mu^2\chi_i^\mu-\frac{\partial}{\partial y_k}\left(a_{kj}(y)\frac{\partial\chi_i^\mu}{\partial y_j}\right)&=-\frac{\partial a_{ki}}{\partial y_k}(y),\\ \mu^2\Psi^\mu-\frac{\partial}{\partial y_k}\left(a_{kj}(y)\frac{\partial\Psi^\mu}{\partial y_j}\right)&=-\frac{\partial}{\partial y_k}\big(a_{kj}(y)\alpha_j(y)\big), \end{aligned} \tag{2.12}$$

we obtain from the first equation of (2.11):

$$\check{u}_1=-\chi_j^\mu\frac{\partial\check{u}_0}{\partial x_j}+\Psi^\mu\check{\tau}_0. \tag{2.13}$$

Then, integrating the second equation of (2.11) with respect to y and defining γ_i^μ and σ^μ to be the analogues of γ_i and σ for χ_i^μ and Ψ^μ as in (1.27),

$$\check{\tau}_0=\frac{1}{\mu}\,\frac{\bar{\beta}\hat{\Theta}+\gamma_i^\mu\dfrac{\partial\hat{p}}{\partial x_i}}{\bar{\beta}+\sigma^\mu}\overset{\text{(def.)}}{=}\frac{\hat{\eta}^\mu}{\mu} \tag{2.14}$$

where $\overline{}$ denotes the Y-average $\int_Y dy$. We introduce Λ_k^μ and H^μ to be the unique periodic solutions in $H^1(Y)$, up to a constant, of:

$$\begin{aligned} -\frac{\partial}{\partial y_i}\left(\lambda_{ij}(y)\frac{\partial\Lambda_k^\mu}{\partial y_j}\right)&=-\left(a_{ij}(y)\alpha_j(y)\frac{\partial\chi_k^\mu}{\partial y_i}-\frac{\beta(y)}{\bar{\beta}}\gamma_k^\mu\right),\\ -\frac{\partial}{\partial y_i}\left(\lambda_{ij}(y)\frac{\partial H^\mu}{\partial y_j}\right)&=-\left(a_{ij}(y)\alpha_j(y)\frac{\partial\Psi^\mu}{\partial y_i}-\frac{\beta(y)}{\bar{\beta}}\sigma^\mu\right). \end{aligned} \tag{2.15}$$

The equations (2.15) are well posed since the Y-averages of the right-hand sides do vanish by definition of γ_i^μ and σ^μ.

Recalling $\Theta_j(y)$, we obtain for $\check{\tau}_1$:

$$\check{\tau}_1=-\frac{1}{\mu}\Theta_j(y)\frac{\partial\hat{\eta}^\mu}{\partial x_j}-\Lambda_j^\mu\frac{\partial\hat{p}}{\partial x_j}+H^\mu\hat{\eta}_\mu+\text{ an arbitrary function of } x \text{ only.} \tag{2.16}$$

Finally, as factor of ε^0, we get:

(2.17)

$$\begin{aligned}
&\rho(y)\left(\zeta^2\check{u}_0-\zeta\frac{f}{\mu}-\frac{g}{\mu}+2\zeta\mu\check{u}_1\right)+D\check{u}_2\\
&\quad=\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial\check{u}_1}{\partial x_j}\right)+a_{ij}(y)\frac{\partial^2\check{u}_0}{\partial x_i\partial x_j}+a_{ij}(y)\frac{\partial^2\check{u}_1}{\partial x_i\partial y_j}\\
&\qquad-a_{ij}(y)\alpha_j(y)\frac{\partial\check{\tau}_0}{\partial x_i}-\frac{\partial}{\partial y_i}\left(a_{ij}(y)\alpha_j(y)\check{\tau}_1\right),\\
&\beta(y)\left(\zeta\check{\tau}_0-\frac{k}{\mu}+\mu\check{\tau}_1\right)=\frac{\partial}{\partial y_i}\left(\lambda_{ij}(y)\frac{\partial\check{\tau}_2}{\partial y_j}\right)+\frac{\partial}{\partial y_i}\left(\lambda_{ij}(y)\frac{\partial\check{\tau}_1}{\partial x_j}\right)\\
&\qquad+\lambda_{ij}(y)\frac{\partial^2\check{\tau}_1}{\partial x_i\partial y_j}+\lambda_{ij}(y)\frac{\partial^2\check{\tau}_0}{\partial x_i\partial x_j}\\
&\qquad-a_{ij}(y)\alpha_j(y)\frac{\partial}{\partial x_i}\left(\zeta\check{u}_0-\frac{f}{\mu}\right)-a_{ij}(y)\alpha_j(y)\frac{\partial}{\partial y_i}(\zeta\check{u}_1)\\
&\qquad-a_{ij}(y)\alpha_j(y)\frac{\partial}{\partial x_i}(\mu\check{u}_1)-a_{ij}(y)\alpha_j(y)\frac{\partial}{\partial y_i}(\mu\check{u}_2).
\end{aligned}$$

We integrate both equations of (2.17) with respect to y; making use of all the previous results of this section, we obtain:

$$\begin{aligned}
&\bar{\rho}(\zeta^2\hat{p}-\zeta f-g)+\mu^3\overline{\rho\check{u}_2}=a_{ij}^\mu\frac{\partial^2\hat{p}}{\partial x_i\partial x_j}-a_{ij}^\mu\alpha_j^\mu\frac{\partial\hat{\eta}^\mu}{\partial x_i},\\
&\bar{\beta}\,(\zeta\hat{\eta}^\mu-k)+\mu^2\overline{\beta\check{\tau}_1}\\
&\quad=\lambda_{ij}\frac{\partial\hat{\eta}^\mu}{\partial x_i\partial x_j}-\mu\overline{\lambda_{ij}(y)\frac{\partial\Lambda_k^\mu}{\partial y_j}}\frac{\partial^2\hat{p}}{\partial x_i\partial x_k}+\mu\overline{\lambda_{ij}(y)\frac{\partial H^\mu}{\partial y_j}}\frac{\partial\hat{\eta}_\mu}{\partial x_i}\\
&\qquad-\zeta a_{ij}^\mu\alpha_j^\mu\frac{\partial\hat{p}}{\partial x_i}+\overline{a_{ij(y)}\alpha_{j(y)}}\frac{\partial f}{\partial x_i}-\zeta\sigma^\mu\hat{\eta}^\mu\\
&\qquad-\mu^2\overline{a_{ij}(y)\alpha_j(y)\frac{\partial\check{u}_1}{\partial x_i}}-\mu^2\overline{a_{ij}(y)\alpha_j(y)\frac{\partial\check{u}_2}{\partial y_i}},
\end{aligned} \tag{2.18}$$

where a_{ij}^μ, α_j^μ are to χ_j^μ and Ψ^μ what a_{ij} and α_j are to χ_j and Ψ in (1.27).

We now consider the limit of (2.18) as μ goes to 0. The following result holds:

PROPOSITION 2.1. *χ_k^μ, Ψ^μ, $\mu\Lambda_k^\mu$, μH^μ go respectively to χ_i, Ψ, 0 and 0 strongly in $H^1(Y)/\mathbb{R}$ as μ goes to 0. Hence a_{ij}^μ, α_j^μ, γ_i^μ, σ^μ go to a_{ij}, α_j, γ_i, σ.*

The proof of this proposition, which involves some basic estimates in $H^1(Y)/\mathbb{R}$, will not be given here; refer to Francfort [4] for the details.

Proposition 2.1 together with (2.5) enables us to perform the limiting process. Upon doing so, we come up with a set of two equations for $\hat{p}$ and $\hat{\Theta}$ which, together with the limit of (2.14), can be interpreted in the time dependent domain. p and Θ

satisfy:

$$
\begin{aligned}
&\Theta(x,t)=\frac{(\bar{\beta}+\sigma)\eta(x,t)-\gamma_i\dfrac{\partial p}{\partial x_i}(x,t)}{\bar{\beta}},\\
&\bar{\rho}\frac{\partial^2 p}{\partial t^2}=a_{ij}\frac{\partial^2 p}{\partial x_i\partial x_j}-a_{ij}\alpha_j\frac{\partial \eta}{\partial x_i},\\
&(\bar{\beta}+\sigma)\frac{\partial \eta}{\partial t}=\lambda_{ij}\frac{\partial^2 \eta}{\partial x_i\partial x_j}-a_{ij}\alpha_j\frac{\partial^2 p}{\partial t\partial x_i},\\
&p(x,0)=f,\quad \frac{\partial p}{\partial t}(x,0)=g,\quad \eta(x,0)=\frac{\bar{\beta}k+\gamma_i\dfrac{\partial f}{\partial x_i}}{\bar{\beta}+\sigma},
\end{aligned}
\tag{2.19}
$$

where $\hat{\eta}$ is the limit of $\hat{\eta}^\mu$ as μ goes to 0.

It is clear that $\eta(x,t)$ can be identified with $\tau(x,t)$, the homogenized temperature field, and $p(x,t)$ with $u(x,t)$, the homogenized displacement field. Replacing Θ by its value in (2.14) we also obtain an expression for the leading term of the asymptotic expansion of τ^ε, that is τ_0; its δ-Laplace transform satisfies:

$$
\tilde{\tau}_0=\frac{1}{\mu}\left(\frac{\bar{\beta}+\sigma}{\bar{\beta}+\sigma^\mu}\eta+\frac{(\gamma_i^\mu-\gamma_i)}{\bar{\beta}+\sigma^\mu}\frac{\partial p}{\partial x_i}\right).
\tag{2.20}
$$

This expression is not explicitly invertible in general, in view of the complicated dependence of γ_i^μ and σ^μ on μ. It is, however, possible to show from (2.20) that τ_0 is the solution of a Volterra integral equation of the second kind (see Francfort [4]). Such an equation does not provide more information about τ_0 than (2.20) itself and it is therefore of little value for our purpose. The following proposition holds:

PROPOSITION 2.2. *σ^μ and γ_i^μ go to zero as μ goes to $+\infty$.*

The proof of this last proposition uses the same estimates as the ones that establish Proposition 2.1.

Propositions 2.1 and 2.2 enable us to conclude that, as μ goes to 0, $\mu\tilde{\tau}_0$ goes to η, whereas as μ goes to $+\infty$, $\mu\tilde{\tau}_0$ goes to Θ. In a time dependent context, these facts translate into statements on the behavior of τ_0 near infinity and near the origin,

$$
\begin{aligned}
&\lim_{\delta\to+\infty}\frac{1}{\delta}\int_0^\delta \tau_0(x,t,\delta')\,d\delta'=\tau(x,t),\\
&\lim_{\delta\to 0+}\frac{1}{\delta}\int_0^\delta \tau_0(x,t,\delta')\,d\delta'=\Theta(x,t),
\end{aligned}
\tag{2.21}
$$

provided these limits exist. The second equation of (2.21) is consistent with our self-imposed δ-initial conditions. The first equation shows that the fast oscillations of the leading term τ_0 of the asymptotic expansion of τ^ε are centered about $\tau(x,t)$, the solution of the homogenized problem. The initial condition $\tau(x,0)$ is the initial average of the oscillating function τ_0. This average is generally different from the initial value of τ_0 (or τ^ε). In other words, the shift in initial conditions is necessary if the initial "phase" is not zero. This contrasts with the method of geometrical optics in which the initial phase is arbitrary. It appears anyway that a geometrical optics type ansatz in place of (2.1) will fail since, if the solutions of (2.12) are sums of terms of more than one frequency in δ, the fast oscillations need not be periodic in δ.

Note that the first equation of (2.21) is the only specific information that we managed to obtain about the large time behavior of τ_0. We do not know, in a general case, for how long a time our expression of the fast oscillations remains valid.

Note also that fast oscillations do not appear in the leading term of the asymptotic expansion of u^ε, which could have been foreseen in view of the convergence obtained for u^ε and u_t^ε in §1.

Conclusion. Numerical computations corroborate the results of §§1 and 2 and confirm that fast oscillations are indeed the phenomenon leading to this unusual change in initial data [4]. It is the first time, to the author's knowledge, that fast oscillations in time are evidenced in a homogenization problem with time independent coefficients.

If seeking a more physical explanation, one could examine the entropy associated with the problem:

$$s=\beta\left(\frac{x}{\varepsilon}\right)\tau^\varepsilon+a_{ij}\left(\frac{x}{\varepsilon}\right)\alpha_j\left(\frac{x}{\varepsilon}\right)\frac{\partial u^\varepsilon}{\partial x_i}.$$

It is fairly straightforward, using the results of §2 and some of the steps performed there, to show that there is no fast dependence in time of the space average of the leading term in the expansion of s^ε. That the macroscopic entropy of this body is a *slowly varying quantity* appears to be a sound idea and does fit our physical intuition. A fast oscillation in the temperature field is the effect that balances the space oscillations of the strains due to the inhomogeneities of the coefficients and allows the entropy to evolve slowly at its own pace. In this respect the unusual initial change in temperature is needed to insure that no fast change in entropy is taking place at time zero.

To conclude this study, let us point out that choosing the entropy as the natural variable in place of the temperature introduces space derivatives of the third order and thereby prohibits a rigorous analysis of the type performed in §1. A perturbation analysis using double scaling is feasible but eventually leads to reintroducing the temperature field as the proper variable.

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