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CORRECTORS FOR THE HOMOGENIZATION OF THE WAVE AND HEAT EQUATIONS

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ABSTRACT. — This paper is mainly devoted to the study of the corrector for the homogenization of the wave equation

$$\rho^\varepsilon u^\varepsilon_{tt} - \operatorname{div}(A^\varepsilon \operatorname{grad} u^\varepsilon) = 0 \quad \text{in } \Omega \times (0, T),$$
$$u^\varepsilon = 0 \quad \text{on } \partial\Omega \times (0, T),$$
$$u^\varepsilon(0) = a^\varepsilon, \qquad u^\varepsilon_t(0) = b^\varepsilon \quad \text{in } \Omega.$$

A by now standard argument permits to pass to the limit in this equation and to obtain the homogenized equation satisfed by the limit u of u^ε. Note however that the energy E^ε corresponding to u^ε, defined by

$$E^\varepsilon = \frac{1}{2}\int_\Omega [\rho^\varepsilon |u^\varepsilon_t|^2 + A^\varepsilon \operatorname{grad} u^\varepsilon \operatorname{grad} u^\varepsilon](x, t)\,dx = \frac{1}{2}\int_\Omega [\rho^\varepsilon |b^\varepsilon|^2 + A^\varepsilon \operatorname{grad} a^\varepsilon \operatorname{grad} a^\varepsilon](x)\,dx$$

does not in general converge to the energy corresponding to u.

We thus partition u^ε into a sum of two terms $u^\varepsilon = \tilde{u}^\varepsilon + v^\varepsilon$. The first term $\tilde{u}^\varepsilon$ solves the same wave equation with initial conditions $\tilde{a}^\varepsilon$ and $\tilde{b}^\varepsilon$ designed in a manner such that the energy $\tilde{E}^\varepsilon$ corresponding to $\tilde{u}^\varepsilon$ converges to E^0. A corrector result for $\tilde{u}^\varepsilon$ can thus be proved, namely,

$$\tilde{u}^\varepsilon_t - u_t \to 0 \quad \text{strongly in } C^0([0, T]; L^2(\Omega)),$$
$$\operatorname{grad} \tilde{u}^\varepsilon - P^\varepsilon \operatorname{grad} u \to 0 \quad \text{strongly in } C_0([0, T]; (L^1(\Omega))^N).$$

As far as v^ε is concerned, we prove that v^ε tends to zero weakly-$*$ in $L^\infty(0, T; H^1_0(\Omega)) \cap W^{1,\infty}(0, T; L^2(\Omega))$. This convergence is strong if and only if $a^\varepsilon - \tilde{a}^\varepsilon$ and $b^\varepsilon - \tilde{b}_\varepsilon$ tend strongly to zero in $H^1_0(\Omega)$ and in $L^2(\Omega)$ respectively. If such is not the case $(1/2)\int_\varepsilon \rho^\varepsilon |v^\varepsilon|^2(x, t)\,dx$ and $(1/2)\int_\Omega (A^\varepsilon \operatorname{grad} v^\varepsilon \operatorname{grad} v^\varepsilon)(x, t)\,dx$ converge [in the weak-$*$ topology of $L^\infty(0, T)$] to a positive constant. Thus v^ε is a perturbation which permeates all times.

The corrector problem for the heat equation is also investigated in this paper, in which case v^ε is proved to be an initial-boundary layer concentrated about the time $t = 0$.

RÉSUMÉ. — Dans cet article, nous étudions principalement le problème des correcteurs pour l'homogénéisation de l'équation des ondes :

$$\rho^\varepsilon u^\varepsilon_{tt} - \operatorname{div}(A^\varepsilon \operatorname{grad} u^\varepsilon) = 0 \quad \text{dans } \Omega \times (0, T),$$
$$u^\varepsilon = 0 \quad \text{sur } \partial\Omega \times (0, T),$$
$$u^\varepsilon(0) = a^\varepsilon, \qquad u^\varepsilon_t(0) = b^\varepsilon \quad \text{dans } \Omega.$$

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Il est facile (et maintenant classique) de passer à la limite dans cette équation et d'obtenir l'équation homogénéisée satisfaite par la limite u des u^ε. Mais l'énergie E^ε correspondant à u^ε, définie par :

$$E^\varepsilon=\frac{1}{2}\int_\Omega [\rho^\varepsilon |u_t^\varepsilon|^2+A^\varepsilon \operatorname{grad} u^\varepsilon \operatorname{grad} u^\varepsilon](x, t)\,dx=\frac{1}{2}\int_\Omega [\rho^\varepsilon |b^\varepsilon|^2+A^\varepsilon \operatorname{grad} a^\varepsilon \operatorname{grad} a^\varepsilon](x)\,dx$$

ne converge pas, en général, vers l'énergie E^0 correspondant à u.

Pour cette raison, nous décomposons u^ε en une somme de deux termes $u^\varepsilon=\tilde{u}^\varepsilon+v^\varepsilon$, où $\tilde{u}^\varepsilon$ est une solution de la même équation des ondes, mais avec des conditions initiales $\tilde{a}^\varepsilon$ et $\tilde{b}^\varepsilon$, que nous choisissons de manière à ce que l'énergie $\tilde{E}^\varepsilon$ correspondant à $\tilde{u}^\varepsilon$ converge vers E^0. Nous démontrons alors un résultat de correcteur pour $\tilde{u}^\varepsilon$:

$$\tilde{u}_t^\varepsilon-u_t\to 0 \quad \text{dans } C^0([0, T]; L^2(\Omega)) \text{ fort,}$$

$$\operatorname{grad} \tilde{u}^\varepsilon-P^\varepsilon \operatorname{grad} u\to 0 \quad \text{dans } C_0([0, T]; (L^1(\Omega))^N) \text{ fort.}$$

En ce qui concerne v^ε, nous démontrons qu'il converge faible-étoile vers zéro dans $L^\infty(0, T; H_0^1(\Omega))\cap W^{1,\infty}(0, T; L^2(\Omega))$. Cette convergence n'est forte que si $a^\varepsilon-\tilde{a}^\varepsilon$ *et* $b^\varepsilon-\tilde{b}^\varepsilon$ tendent fortement vers zéro dans $H_0^1(\Omega)$ et dans $L^2(\Omega)$ respectivement. Si tel n'est pas le cas, $(1/2)\int_\varepsilon \rho^\varepsilon|v^\varepsilon|^2(x, t)\,dx$ et $(1/2)\int_\Omega (A^\varepsilon \operatorname{grad} v^\varepsilon \operatorname{grad} v^\varepsilon)(x, t)\,dx$ convergent (dans $L^\infty(0, T)$ faible-étoile) vers une constante non nulle, ce qui montre que v^ε est une perturbation qui perdure pour tout temps $t>0$.

Nous étudions également dans cet article le problème des correcteurs pour l'équation de la chaleur. Dans ce cas v^ε est une couche limite concentrée autour du temps $t=0$.

1. Introduction

This paper is devoted to the study of correctors for the homogenization of the wave and heat equations. The following problems are investigated for an arbitrary bounded domain Ω of $\mathbb{R}^N$ and an arbitrary positive time T:

$$(1.1)\quad \begin{cases} \rho^\varepsilon\dfrac{\partial^2 u^\varepsilon}{\partial t^2}-\operatorname{div}(A^\varepsilon \operatorname{grad} u^\varepsilon)=f \text{ in } \Omega\times(0, T),\\ u^\varepsilon=0 \quad \text{on } \partial\Omega\times(0, T),\\ u^\varepsilon(0)=a^\varepsilon \quad \text{in } \Omega,\\ \dfrac{\partial u^\varepsilon}{\partial t}(0)=b^\varepsilon \quad \text{in } \Omega,\end{cases}$$

$$(1.2)\quad \begin{cases} \beta^\varepsilon\dfrac{\partial \tau^\varepsilon}{\partial t}-\operatorname{div}(K^\varepsilon \operatorname{grad} \tau^\varepsilon)=g \quad \text{in } \Omega\times(0, T),\\ \tau^\varepsilon=0 \quad \text{on } \partial\Omega\times(0, T),\\ \tau^\varepsilon(0)=c^\varepsilon \quad \text{in } \Omega.\end{cases}$$

The unknown functions are u^ε and τ^ε. The coefficients ρ^ε, A^ε, β^ε, K^ε are assumed to be independent of t, uniformly bounded in $L_\infty(\Omega)$, and uniformly bounded away from zero (or uniformly coercive). The initial conditions a^ε, b^ε, c^ε are taken to be bounded in $H_0^1(\Omega)$, $L_2(\Omega)$ and $L_2(\Omega)$ respectively.

The homogenization of (1.1), or (1.2) can be labeled as classical (*see* e.g. [BeLP], [Sa]) although a careful study of the associated correctors has not yet been performed, at least as far as the wave equation is concerned. A corrector result is available in the case of the heat equation (*see* [BeLP]).

The main issue and the most profound difference with the elliptic case is immediately perceived through the following simple considerations.

Assume that $f=0$, $a^\varepsilon=a^0$ and $b^\varepsilon=0$ in (1.1). The solution u^ε of (1.1) satisfies the principle of conservation of energy, namely,

$$(1.3)\qquad \frac{1}{2}\int_\Omega \left[\rho^\varepsilon\left(\frac{\partial u^\varepsilon}{\partial t}\right)^2 + A^\varepsilon \operatorname{grad} u^\varepsilon \operatorname{grad} u^\varepsilon\right](x, t)\,dx = \frac{1}{2}\int_\Omega A^\varepsilon \operatorname{grad} a^0 \operatorname{grad} a^0\,dx.$$

The weak limit u of u^ε, which is the solution of the homogenized equation associated to (1.1), *i.e.* (*cf.* Section 3),

$$(1.4)\qquad \begin{aligned} &\bar{\rho}\frac{\partial^2 u}{\partial t} - \operatorname{div}(A^0 \operatorname{grad} u) = 0 \quad \text{in } \Omega\times(0, T),\\ &u = 0 \quad \text{on } \partial\Omega\times(0, T),\\ &u(0) = a^0 \quad \text{in } \Omega,\\ &\frac{\partial u}{\partial t}(0) = 0 \quad \text{in } \Omega, \end{aligned}$$

also satisfies the principle of conservation of energy,

$$(1.5)\qquad \frac{1}{2}\int_\Omega \left[\bar{\rho}\left(\frac{\partial u}{\partial t}\right)^2 + A^0 \operatorname{grad} u \operatorname{grad} u\right](x, t)\,dx = \frac{1}{2}\int_\Omega A^0 \operatorname{grad} a^0 \operatorname{grad} a^0\,dx.$$

Since in general

$$(1.6)\qquad \lim_{\varepsilon\to 0} \frac{1}{2}\int_\Omega A^\varepsilon \operatorname{grad} a^0 \operatorname{grad} a^0\,dx > \frac{1}{2}\int_\Omega A^0 \operatorname{grad} a^0 \operatorname{grad} a^0\,dx$$

the convergence of the energy does not occur. But such a convergence is at the root of construction of correctors. We thus partition u^ε into two parts $\tilde{u}^\varepsilon$ and v^ε.

The first part $\tilde{u}^\varepsilon$ will solve the wave equation (1.1) with a^ε and b^ε replaced by $\tilde{a}^\varepsilon$ and $\tilde{b}^\varepsilon$, which are themselves designed to achieve the convergence of the energy. In the case of interest to us here, $\tilde{b}^\varepsilon$ is null and $\tilde{a}^\varepsilon$ is chosen such that

$$(1.7)\qquad \begin{aligned} &\tilde{a}^\varepsilon \rightharpoonup a^0 \quad \text{weakly in } H_0^1(\Omega),\\ &\frac{1}{2}\int_\Omega A^\varepsilon \operatorname{grad} \tilde{a}^\varepsilon \operatorname{grad} \tilde{a}^\varepsilon\,dx \to \frac{1}{2}\int_\Omega A^0 \operatorname{grad} a^0 \operatorname{grad} a^0\,dx. \end{aligned}$$

A corrector result is obtained for $\tilde{u}^\varepsilon$ (*cf.* Theorem 4.1) in the spirit of the elliptic case, namely it is proved that

$$(1.8)\qquad \begin{aligned} &\frac{\partial \tilde{u}^\varepsilon}{\partial t}-\frac{\partial u}{\partial t}\to 0 \quad \text{strongly in } \mathscr{C}^0([0, T]; L_2(\Omega)),\\ &\text{grad}\, \tilde{u}^\varepsilon - P^\varepsilon \,\text{grad}\, u \to 0 \quad \text{strongly in } \mathscr{C}^0([0, T]; [L_1(\Omega)]^N),\end{aligned}$$

where P^ε is a matrix which only depends on the sequence A^ε (*cf.* Definition 2.2).

The part v^ε is easily proved to converge to zero in the appropriate weak topologies, namely

$$(1.9)\qquad \begin{aligned} &\frac{\partial v^\varepsilon}{\partial t}\rightharpoonup 0 \quad \text{weak-}\ast \text{ in } L_\infty(0, T; L_2(\Omega)),\\ &\text{grad}\, v^\varepsilon \rightharpoonup 0 \quad \text{weak-}\ast \text{ in } L_\infty(0, T; [L_2(\Omega)]^N).\end{aligned}$$

The convergences are proved to be strong if and only if

$$(1.10)\qquad \begin{aligned} &a^\varepsilon - \tilde{a}^\varepsilon \to 0 \quad \text{strongly in } H_0^1(\Omega),\\ &b^\varepsilon - \tilde{b}^\varepsilon \to 0 \quad \text{strongly in } L_2(\Omega).\end{aligned}$$

Such a condition cannot be satisfied in the above considered example unless A^ε happens to strongly converge to A^0.

The term v^ε acts as a perturbation originating in the incompatible character of the initial conditions a^ε, b^ε with the oscillating structure of the matrix A^ε. If (1.10) is not satisfied this perturbation is found to permeate all times; indeed it is proved (*cf.* Theorem 4.3) that the kinetic and potential energy terms, namely $(1/2)\int_\Omega [\rho^\varepsilon((\partial v^\varepsilon/\partial t))^2](x, t)\,dx$ and $(1/2)\int_\Omega [A^\varepsilon \,\text{grad}\, v^\varepsilon \,\text{grad}\, v^\varepsilon](x, t)\,dx$, admit the same constant limit as ε tends to zero.

Convergences (1.8) and (1.9) provide a description of the behaviour of $u^\varepsilon = \tilde{u}^\varepsilon + v^\varepsilon$ (*cf.* Theorem 4.4).

A similar approach is used to study the heat equation (1.2). The solution τ^ε is partitioned into two parts $\tilde{\tau}^\varepsilon$ and θ^ε. Although most of the obtained results can be found in [BeLP] our analysis delivers a detailed picture of θ^ε which behaves as an authentic initial-boundary layer concentrated about the time $t=0$ in the adequate strong topologies (*cf.* Theorem 7.2).

This paper is to be related to our paper on thermoelasticity [BrFMu] (*see* also [F]) in which a coupling between (1.1) and (1.2) is introduced through the first equations of both (1.1) and (1.2). The results obtained there are less satisfactory because of our failure to provide a detailed analysis of the corresponding v^ε and θ^ε. Let us emphasize however the qualitative jump in the degree of intricacy involved in dealing with the system of linear thermoelasticity.

The present paper is organized as follows. Section 2 recalls known corrector results in the elliptic case. Section 3 is devoted to the study of existence and to the homogenization of the wave equation. Section 4 states the corrector result for (1.1) while Section 5 addresses the proofs of the result anounced in Section 4. Section 6 is concerned with the homogenization of the heat equation while Section 7 deals with the corresponding corrector result.

The original parts of this paper will be found in Sections 4, 5 and 7. Corrector results for the solutions u^ε, τ^ε of (1.1), (1.2) are stated in Theorems 4.4 and 7.3.

2. A review of homogenization and corrector results in the elliptic case

This section briefly recalls basic results pertaining to the homogenization of a scalar second order elliptic equation in divergence form. The results presented here are at the root of the subsequent study. In particular, the notion of corrector matrix firstly introduced by L. Tartar [T1] plays an essential role in our analysis.

Throughout the paper Ω denotes a bounded open domain of $\mathbb{R}^N$ with boundary $\partial\Omega$, T is a strictly positive real number, ε belongs to a sequence of strictly positive real numbers that converge to zero, α, β are two real numbers satisfying $0<\alpha<\beta$ and

$$(2.1)\quad \mathcal{M}(\alpha, \beta; \Omega)=\{A(x)\in L_\infty(\Omega; \mathbb{R}^N\times\mathbb{R}^N) \mid A(x)\xi\xi\geqq\alpha|\xi|^2 \text{ and } A^{-1}(x)\xi\xi\geqq\beta^{-1}|\xi|^2 \text{ for almost every } x \text{ of } \Omega \text{ and every } \xi \text{ of } \mathbb{R}^N\}.$$

Remark 2.1. — Any element A of $\mathcal{M}(\alpha, \beta; \Omega)$ satisfies

$$(2.2)\quad \begin{aligned} &A(x)\xi\xi\geqq\alpha|\xi|^2,\\ &|A(x)\xi|\leqq\beta'|\xi|, \end{aligned}$$

with $\beta'=\beta$ for almost any x of Ω and every ξ of $\mathbb{R}^N$. Conversely a matrix $A(x)$ satisfying (2.2) belongs to $\mathcal{M}(\alpha, \alpha/(\beta')^2; \Omega)$. ●

DEFINITION 2.1. — *A sequence* A^ε *of* $\mathcal{M}(\alpha, \beta; \Omega)$ *is said to* H*-converge to a matrix* A^0 *of* $\mathcal{M}(\alpha, \beta; \Omega)$ *if and only if for any* f *in* $H^{-1}(\Omega)$ *the sequence* v^ε *of solutions of*

$$(2.3)\quad \begin{aligned} &-\operatorname{div}(A^\varepsilon \operatorname{grad} v^\varepsilon)=f, \quad \text{in } \Omega,\\ &v^\varepsilon=0 \quad \text{on } \partial\Omega, \end{aligned}$$

satisfies

$$(2.4)\quad \begin{aligned} &v^\varepsilon \rightharpoonup v^0 \quad \text{weakly in } H_0^1(\Omega),\\ &A^\varepsilon \operatorname{grad} v^\varepsilon \rightharpoonup A^0 \operatorname{grad} v^0 \quad \text{weakly in } [L_2(\Omega)]^N, \end{aligned}$$

where v^0 *is the solution of*

$$(2.5)\quad \begin{aligned} &-\operatorname{div}(A^0 \operatorname{grad} v^0)=f, \quad \text{in } \Omega,\\ &v^0=0 \quad \text{on } \partial\Omega. \end{aligned}$$

This convergence will be denoted by $A^\varepsilon \overset{H}{\rightharpoonup} A^0$. ●

The above definition was introduced by S. Spagnolo [Sp] (under the name of G-convergence) in the case of symmetric matrices and by L. Tartar [T1] and F. Murat [Mu1] in the non symmetric case. An extensive litterature on the topic is now available; see e.g. the books of A. Bensoussan, J.-L. Lions, G. Papanicolaou [BeLP] and E. Sanchez-Palencia [Sa] which concern themselves with the important case of periodic coefficients, *i.e.*, of matrices A^ε of the form $A^\varepsilon(x)=A(x/\varepsilon)$ where $A(y)$ is a periodic matrix defined on $\mathbb{R}^N$; *see* also the survey paper of V. V. Zhikov, S. M. Kozlov, O.A. Oleinik, K. T. Ngoan [ZKON]. The motivation for such a definition lies in the following compactness result due to S. Spagnolo [Sp] and L. Tartar [T1]:

THEOREM 2.1. — *Any given sequence of* $\mathcal{M}(\alpha, \beta; \Omega)$ *admits a subsequence which* H-*converges to an element of* $\mathcal{M}(\alpha, \beta; \Omega)$ ●

In fact the notion of H-convergence does not hinge on any specific type of boundary conditions. Its local character is demonstrated in the following

THEOREM 2.2. — *Let* A^ε *be a* H-*converging sequence of elements of* $\mathcal{M}(\alpha, \beta; \Omega)$ *which* H-*converges to* A^0. *If* z^ε *and* f^ε *are such that*

$$\begin{gathered} z^\varepsilon \rightharpoonup z \quad \textit{weakly in } H^1_{loc}(\Omega), \\ f^\varepsilon \to f \quad \textit{strongly in } H^{-1}_{loc}(\Omega), \\ -\operatorname{div}(A^\varepsilon \operatorname{grad} z^\varepsilon)=f^\varepsilon \quad \textit{in } \Omega, \end{gathered} \tag{2.6}$$

then

$$A^\varepsilon \operatorname{grad} z^\varepsilon \rightharpoonup A^0 \operatorname{grad} z \quad \textit{weakly in } [L_2^{loc}(\Omega)]^N. \quad \bullet \tag{2.7}$$

We now recall L. Tartar's corrector result which describes the structure of the sequence grad z^ε in Theorem 2.2 (*cf.* e.g. [T1], [Mu1], [BeLP], [Sa], [ZKON]). To this effect, N functions $w_i^\varepsilon (1 \leqq i \leqq N)$ such that

$$\begin{gathered} \operatorname{grad} w_i^\varepsilon \rightharpoonup e_i \quad \text{weakly in } [L_2(\Omega)]^N, \\ -\operatorname{div}(A^\varepsilon \operatorname{grad} w_i^\varepsilon) = -\operatorname{div}(A^0 e_i) \quad \text{in } \Omega, \end{gathered} \tag{2.8}$$

are introduced. In (2.8) $e_i (1 \leqq i \leqq N)$ denotes a basis of $\mathbb{R}^N$. The existence of such functions is easily obtained through the solving of a Dirichlet problem on a domain that compactly contains Ω.

DEFINITION 2.2. — *The sequence* P^ε *of elements of* $L_2(\Omega; \mathbb{R}^N \times \mathbb{R}^N)$ *defined by*

$$P^\varepsilon e_i = \operatorname{grad} w_i^\varepsilon \quad \textit{in } \Omega, \qquad 1 \leqq i \leqq N, \tag{2.9}$$

is called a sequence of corrector matrices associated to A^ε. ●

Remark 2.2. — If P^ε and Q^ε are two such sequences, it can be proved that

$$P^\varepsilon - Q^\varepsilon \to 0 \quad \text{strongly in } [L_2^{loc}(\Omega)]^{N^2}. \quad \bullet \tag{2.10}$$

Remark 2.3. — In view of Theorem 2.2

$$(2.11)\qquad \begin{aligned} P^\varepsilon &\rightharpoonup I \quad \text{weakly in } [L_2(\Omega)]^{N^2}, \\ A^\varepsilon P^\varepsilon &\rightharpoonup A^0 \quad \text{weakly in } [L_2(\Omega)]^{N^2}. \end{aligned}$$ ●

With the help of the corrector matrices we are in the position to futher describe the structure of the gradients of local solutions.

THEOREM 2.3. — *Let* A^ε *be a* H-*converging sequence of elements of* $\mathcal{M}(\alpha, \beta; \Omega)$. *Denote by* A^0 *its* H-*limit and by* P^ε *an associated sequence of corrector matrices. Then, if* z^ε *and* f^ε *satisfy* (2.6),

$$(2.12)\qquad \begin{aligned} &\text{grad } z^\varepsilon = P^\varepsilon \text{ grad } z + r^\varepsilon, \\ &r^\varepsilon \to 0 \quad \textit{strongly in} \quad [L_1^{\text{loc}}(\Omega)]^N. \end{aligned}$$

If z belongs to $W_{\text{loc}}^{1,p}(\Omega)$, $2 \leqq p \leqq +\infty$, *and* P^ε *is bounded in* $[L_q(\Omega)]^{N^2}$, $2 \leqq q \leqq +\infty$, *then*

$$(2.13)\qquad r^\varepsilon \to 0 \quad \textit{strongly in } [L_{\text{loc}}^{s}(\Omega)]^N$$

with

$$(2.14)\qquad \frac{1}{s} = \max\left(\frac{1}{2}, \frac{1}{p} + \frac{1}{q}\right).$$

Finally, if z^ε *and z are elements of* $H^1(\Omega)$ *and satisfy* (2.6) *together with*

$$(2.15)\qquad \int_\Omega A^\varepsilon \text{ grad } z^\varepsilon \text{ grad } z^\varepsilon \, dx \to \int_\Omega A^0 \text{ grad } z \text{ grad } z \, dx,$$

the convergences (2.12), (2.13) *take place in* $[L_1(\Omega)]^N$ *and* $[L_s(\Omega)]^N$ *respectively.* ●

The proofs of the various results presented above will not be reproduced here. They are based on the proper use of the test functions φw_i^ε where φ is an element of $\mathscr{C}_0^\infty(\Omega)$ together with repeated use of the "div-curl lemma", the prototype of the theory of compensated compactness (*see* [T1], [T2], [Mu2], [Mu3]). The "div-curl lemma" is now stated in a time dependent form which will be of use later on.

THEOREM 2.4. — *Let* ξ^ε *and* g^ε *be two sequences of* $[L_2(\Omega \times (0, T))]^N$ *that satisfy*

$$(2.16)\qquad \begin{aligned} \xi^\varepsilon &\rightharpoonup \xi \quad \textit{weakly in } [L_2(\Omega \times (0, T))]^N, \\ g^\varepsilon &\rightharpoonup g \quad \textit{weakly in } [L_2(\Omega \times (0, T))]^N, \end{aligned}$$

while

$$(2.17)\qquad \begin{aligned} &\text{div } \xi^\varepsilon \quad \textit{lies in a compact subset of } H_{\text{loc}}^{-1}(\Omega \times (0, T)), \\ &\text{curl } g^\varepsilon \quad \textit{lies in a compact subset of } [H_{\text{loc}}^{-1}(\Omega \times (0, T))]^{N^2}, \\ &\frac{\partial \xi^\varepsilon}{\partial t} \textit{ or } \frac{\partial g^\varepsilon}{\partial t} \quad \textit{lies in a compact subset of } [H_{\text{loc}}^{-1}(\Omega \times (0, T))]^N, \end{aligned}$$

then

$$\xi^\varepsilon g^\varepsilon \rightharpoonup \xi g \quad in \ \mathscr{D}'(\Omega \times (0, T)). \quad \bullet$$

3. Homogenization of the wave equation

This section is devoted to the study of the homogenization of the wave equation. After a brief review of a few results pertaining to existence and uniqueness and a careful examination of the sense in which the initial conditions are satisfied the homogenization result is stated and proved through a reduction of the problem to the elliptic setting.

The following wave equation with Dirichlet boundary conditions is investigated:

$$(3.1) \qquad \rho^\varepsilon \frac{\partial^2 u^\varepsilon}{\partial t^2} - \operatorname{div}(A^\varepsilon \operatorname{grad} u^\varepsilon) = f \quad \text{in } \Omega \times (0, T),$$

$$(3.2) \qquad u^\varepsilon = 0 \quad \text{on } \partial\Omega \times (0, T),$$

$$(3.3) \qquad u^\varepsilon(0) = a^\varepsilon \quad \text{in } \Omega,$$

$$(3.4) \qquad \frac{\partial u^\varepsilon}{\partial t}(0) = b^\varepsilon \quad \text{in } \Omega.$$

In (3.1)-(3.4) the scalar valued function u^ε is the unknown, whereas the other quantities are given data of the problem. They are assumed to satisfy the following hypotheses:

$$(3.5) \qquad \begin{gathered} \rho^\varepsilon \in L_\infty(\Omega), \\ \rho^\varepsilon \rightharpoonup \bar{\rho} \quad \text{weak-}* \text{ in } L_\infty(\Omega), \\ \lambda_1 \leqq \rho^\varepsilon(x) \leqq \lambda_2, \qquad \text{almost everywhere in } \Omega, \end{gathered}$$

where λ_1, λ_2 are two strictly positive real numbers ($\lambda_1 < \lambda_2$),

$$(3.6) \qquad \begin{gathered} A^\varepsilon \in \mathscr{M}(\alpha, \beta; \Omega), \\ A^\varepsilon \stackrel{H}{\rightharpoonup} A^0, \\ {}^t A^\varepsilon = A^\varepsilon, \end{gathered}$$

$$(3.7) \qquad f \in L_2(0, T; L_2(\Omega)),$$

$$(3.8) \qquad \begin{gathered} a^\varepsilon \in H_0^1(\Omega), \\ a^\varepsilon \rightharpoonup a^0 \quad \text{weakly in } H_0^1(\Omega), \end{gathered}$$

$$(3.9) \qquad \begin{gathered} b^\varepsilon \in L_2(\Omega), \\ b^\varepsilon \rightharpoonup \bar{b} \quad \text{weakly in } L_2(\Omega), \\ \rho^\varepsilon b^\varepsilon \rightharpoonup \overline{\rho b} \quad \text{weakly in } L_2(\Omega), \\ b^0 = \overline{\rho b}/\bar{\rho}. \end{gathered}$$

Remark 3.1. — Since A^ε is symmetric, its H-limit A^0 is also symmetric. ●

The problem (3.1)-(3.4) is well known (*cf.* e.g. [LMa] Ch. II.8) to yield a unique solution. Specifically, the following theorem holds true:

THEOREM 3.1. — *Under hypotheses* (3.5)-(3.9), *there exists a solution of* (3.1)-(3.4) *satisfying*

(3.10)
$$u^\varepsilon \in \mathscr{C}^0([0, T]; H_0^1(\Omega)), \qquad \frac{\partial u^\varepsilon}{\partial t} \in \mathscr{C}^0([0, T]; L_2(\Omega)).$$

Furthermore uniqueness holds in the larger class

(3.11)
$$u^\varepsilon \in L_\infty(0, T; H_0^1(\Omega)), \qquad \frac{\partial u^\varepsilon}{\partial t} \in L_\infty(0, T; L_2(\Omega)).$$

Finally setting

(3.12)
$$e^\varepsilon(t) = \frac{1}{2}\int_\Omega \left[\rho^\varepsilon \left(\frac{\partial u^\varepsilon}{\partial t}\right)^2 + A^\varepsilon \operatorname{grad} u^\varepsilon \operatorname{grad} u^\varepsilon\right](x, t)\, dx,$$
$$E^\varepsilon = \frac{1}{2}\int_\Omega [\rho^\varepsilon (b^\varepsilon)^2 + A^\varepsilon \operatorname{grad} a^\varepsilon \operatorname{grad} a^\varepsilon](x)\, dx,$$

one has

(3.13)
$$e^\varepsilon(t) = E^\varepsilon + \int_0^t \int_\Omega f \frac{\partial u^\varepsilon}{\partial t}\, dx\, ds \quad in\ [0, T]. \quad ●$$

Remark 3.2. — If f is identically zero, (3.13) is precisely the statement that the energy is conserved. We will refer loosely to (3.13) as "the conservation of energy" even when $f \not\equiv 0$. ●

Remark 3.3. — The uniqueness result, which is stated for the "L^∞ in time" class (3.11) of functions necessitates to further analyze the meaning of the initial conditions (3.3), (3.4). To this effect a few definitions are to be recalled.

If X and Y are two Banach spaces with continuous embedding of X into Y, $\mathscr{C}_s^0([0, T]; X)$ is defined as the space of X-valued functions v defined on [0, T] such that the real valued function $\langle h, v(t)\rangle_{X', X}$ is continuous on [0, T] for any h in the dual space X' of X. If X is a reflexive Banach Space then (*see* e.g. [LMa], lemma 8.1, p. 297)

(3.14)
$$L_\infty(0, T; X) \cap \mathscr{C}^0([0, T]; Y) \subset \mathscr{C}_s^0([0, T]; X).$$

Define further the following subspace of $\mathscr{C}^0([0, T]; Y)$:

$$W(0, T; X, Y)=\left\{v\in L_\infty(0, T; X)\,\Big|\,\frac{\partial v}{\partial t}\in L_2(0, T; Y)\right\}. \tag{3.15}$$

Consider now a solution u^ε of (3.1), (3.2) with the "L_∞ in time" regularity; u^ε belongs to $W(0, T; H_0^1(\Omega), L_2(\Omega))$, thus to $\mathscr{C}^0([0, T]; L_2(\Omega))$ and (3.3) has a meaning. Further $\rho^\varepsilon(\partial u^\varepsilon/\partial t)$ belongs to $W(0, T; L_2(\Omega), H^{-1}(\Omega))$, thus to $\mathscr{C}^0([0, T]; H^{-1}(\Omega))$ and in view of (3.14) to $\mathscr{C}_s^0([0, T]; L_2(\Omega))$. Consequently the function

$$t\to\int_\Omega h\frac{\partial u^\varepsilon}{\partial t}dx=\int_\Omega\left(\frac{h}{\rho^\varepsilon}\right)\rho^\varepsilon\frac{\partial u^\varepsilon}{\partial t}dx \tag{3.16}$$

is continuous on [0, T] for any h in $L_2(\Omega)$, *i. e.*,

$$\frac{\partial u^\varepsilon}{\partial t}\in\mathscr{C}_s^0([0, T]; L_2(\Omega)) \tag{3.17}$$

and the initial condition (3.4) has a meaning. This initial condition is exactly (and not only formally) equivalent to the initial condition on $\rho^\varepsilon(\partial u^\varepsilon/\partial t)$, namely

$$\left(\rho^\varepsilon\frac{\partial u^\varepsilon}{\partial t}\right)(0)=\rho^\varepsilon b^\varepsilon\quad\text{in }\Omega. \tag{3.18}$$ ●

The conservation of energy (3.13) together with (3.5)-(3.9) immediately imply that

$$\begin{aligned}&u^\varepsilon\ \text{ is bounded in }\ L_\infty(0, T; H_0^1(\Omega)),\\&\frac{\partial u^\varepsilon}{\partial t}\ \text{ is bounded in }\ L_\infty(0, T; L_2(\Omega)).\end{aligned} \tag{3.19}$$

Let us introduce the "homogenized" wave equation

$$\begin{aligned}&\bar\rho\frac{\partial^2 u}{\partial t^2}-\operatorname{div}(A^0\operatorname{grad} u)=f\quad\text{in }\Omega\times(0, T),\\&u=0\quad\text{on }\partial\Omega\times(0, T),\\&u(0)=a^0\quad\text{in }\Omega,\\&\frac{\partial u}{\partial t}(0)=b^0\quad\text{in }\Omega.\end{aligned} \tag{3.20}$$

The following homogenization result holds true:

THEOREM 3.2. — *The solution u^ε of* (3.1)-(3-4) *converges to the solution u of* (3.20) *in the following sense:*

$$\begin{aligned}&u^\varepsilon\rightharpoonup u\quad\textit{weak-}\ast\ \textit{in}\ L_\infty(0, T; H_0^1(\Omega)),\\&\frac{\partial u^\varepsilon}{\partial t}\rightharpoonup\frac{\partial u}{\partial t}\quad\textit{weak-}\ast\ \textit{in}\ L_\infty(0, T; L_2(\Omega)).\end{aligned} \tag{3.21}$$ ●

The proof of Theorems 3.2 can be found in [BeLP], Chapt. 2, p. 301 or [Sa], Chapt. 5, Theorem 6.3, p. 67. It is presented here for the sake of completeness.

Proof of Theorem 3.2. – The proof consists in reducing the problem to an elliptic setting. Such a reduction can be performed through Laplace transformation or through multiplication of the equation by a test function φ in $\mathscr{C}_0^\infty(0, T)$. Our preference goes to the latter technique and we set, for any w in $L_2(0, T; L_2(\Omega))$,

$$\hat{w}(x)=\int_0^T w(x, t)\varphi(t)\,dt, \qquad \check{w}(x)=\int_0^T w(x, t)\frac{\partial^2\varphi}{\partial t^2}(t)\,dt. \tag{3.22}$$

By virtue of (3.19) we are at liberty to extract a subsequence ε' such that

$$\begin{aligned} &u^{\varepsilon'} \rightharpoonup u^* \quad \text{weak-}* \text{ in } L_\infty(0, T; H_0^1(\Omega)),\\ &\frac{\partial u^{\varepsilon'}}{\partial t} \rightharpoonup \frac{\partial u^*}{\partial t} \quad \text{weak-}* \text{ in } L_\infty(0, T; L^2(\Omega)). \end{aligned} \tag{3.23}$$

Then

$$\begin{aligned} &\hat{u}^{\varepsilon'} \rightharpoonup \hat{u}^* \quad \text{weakly in } H_0^1(\Omega),\\ &\check{u}^{\varepsilon'} \rightharpoonup \check{u}^* \quad \text{weakly in } H_0^1(\Omega). \end{aligned} \tag{3.24}$$

Futher $\hat{u}^{\varepsilon'}$, $\check{u}^{\varepsilon'}$ satisfy, in view of (3.1),

$$\begin{aligned} -\operatorname{div}(A^{\varepsilon'} \operatorname{grad} \hat{u}^{\varepsilon'}) &= \hat{f} - \rho^\varepsilon \check{u}^{\varepsilon'} \quad \text{in } \Omega,\\ \hat{u}^{\varepsilon'} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.25}$$

The very definition of H-convergence (Definition 2.1, Theorem 2.2) implies that $\hat{u}^*$ and $\check{u}^*$ satisfy

$$-\operatorname{div}(A^0 \operatorname{grad} \hat{u}^*) = \hat{f} - \bar{\rho}\,\check{u}^* \quad \text{in } \Omega. \tag{3.26}$$

Since φ is arbitrary, (3.26) yields in turn

$$\bar{\rho}\frac{\partial^2 u^*}{\partial t^*} - \operatorname{div}(A^0 \operatorname{grad} u^*) = f \text{ in } \mathscr{D}'(\Omega \times (0, T)). \tag{3.27}$$

Let us now investigate the initial conditions. By virtue of (3.23),

$$u^{\varepsilon'} \to u^* \quad \text{strongly in } \mathscr{C}^0([0, T]; L_2(\Omega)), \tag{3.28}$$

because

$$\begin{aligned} &W(0, T; X, Y) \text{ is compactly embedded in } \mathscr{C}^0([0, T]; Y)\\ &\qquad \text{if X is compactly embedded in Y,} \end{aligned} \tag{3.29}$$

(*cf.* e.g. [Si], Corollary 4, p. 85).

Thus the initial condition (3.3) passes to the limit and we obtain

$$u^*(0)=a^0 \quad \text{in } \Omega. \tag{3.30}$$

In view of (3.23) and equation (3.1), $\rho^\varepsilon(\partial u^\varepsilon/\partial t)$ is bounded in $W(0, T; L_2(\Omega), H^{-1}(\Omega))$; thus application of (3.29) yields

$$\rho^{\varepsilon'}\frac{\partial u^{\varepsilon'}}{\partial t} \to \bar{\rho}\frac{\partial u^*}{\partial t} \quad \text{strongly in } \mathscr{C}^0([0, T]; H^{-1}(\Omega)), \tag{3.31}$$

which permits to pass to the limit in the initial condition

$$\left(\rho^{\varepsilon'}\frac{\partial u^{\varepsilon'}}{\partial t}\right)(0)=\rho^{\varepsilon'}b^{\varepsilon'}. \tag{3.32}$$

Hence

$$\left(\bar{\rho}\frac{\partial u^*}{\partial t}\right)(0)=\overline{\rho b}. \tag{3.33}$$

Thus u^*, which belongs to the "L_∞ in time" class defined by (3.11) satisfies (3.27), (3.30), (3.33). Remark 3.3 applies to u^* and yields

$$\frac{\partial u^*}{\partial t}(0)=\frac{\overline{\rho b}}{\bar{\rho}}=b^0. \tag{3.34}$$

The "L^∞ in time" uniqueness result for the solution of the wave equation implies that $u^*=u$. Since the limit is uniquely defined the whole sequence u^ε converges to u and the proof of Theorem 3.2 is complete. ●

Remark 3.4. — In the context of Theorem 3.2, the following statements of uniform convergence in time hold true, for any elements k and l of $H^{-1}(\Omega)$ and $L_2(\Omega)$ respectively:

$$\begin{aligned} &\langle u^\varepsilon(t), k\rangle_{H_0^1(\Omega), H^{-1}(\Omega)} \to \langle u(t), k\rangle_{H_0^1(\Omega), H^{-1}(\Omega)} \\ &\qquad \text{strongly in } \mathscr{C}^0([0, T]), \\ &\int_\Omega \rho^\varepsilon(x)\frac{\partial u^\varepsilon}{\partial t}(x, t)\, l(x)\, dx \to \int_\Omega \bar{\rho}(x)\frac{\partial u}{\partial t}(x, t)\, l(x)\, dx, \\ &\qquad \text{strongly in } \mathscr{C}^0([0, T]), \end{aligned} \tag{3.35}$$

as ε tends to zero.

In concrete terms the statements of convergence (3.35) demonstrate that the possible time oscillations of the quantities u^ε and $\rho^\varepsilon(\partial u^\varepsilon/\partial t)$ disappear upon spatial averaging. In particular the first convergence in (3.35) should be compared to that obtained in (3.21). Note however that the second convergence in (3.35) does not imply strong convergence of the spatial average of $\partial u^\varepsilon/\partial t$.

The convergences in (3.35) are easily deduced from (3.28) and (3.31) respectively upon approximating k and l by elements of $L_2(\Omega)$ and $H_0^1(\Omega)$ respectively. In fact, the same argument yields the "compactness" of the embedding (3.14) whenever X is compactly embedded in Y. ●

4. Statement of the corrector result for the wave equation

This section states and comments the corrector result for the wave equation (3.1)-(3.4). The solution u^ε is split into two terms $\tilde{u}^\varepsilon$ and v^ε. The energy associated with $\tilde{u}^\varepsilon$ is designed in a manner such that a corrector result can be obtained for $\tilde{u}^\varepsilon$. The term v^ε converges weakly *but not* strongly to zero and its energy is lost by the homogenization process. The obtained results are summed up in Theorem 4.4. The proofs are given in Section 5.

Note that a corrector result pertaining to the wave equation is derived in [BeLP], Chap. 2, Section 3.6 under the implicit assumption that all initial conditions are null; then the term v^ε can be done away with. Such is not generally the case. This remark seems to be new.

The energies e^0 and E^0 associated to the solution u of the homogenized wave equation (3.20) are defined as

$$(4.1)\qquad \begin{aligned} e^0(t) &= \frac{1}{2}\int_\Omega \left[\bar{\rho}\left(\frac{\partial u}{\partial t}\right)^2 + A^0 \operatorname{grad} u \operatorname{grad} u\right](x, t)\,dx, \\ E_0 &= \frac{1}{2}\int_\Omega [\bar{\rho}(b^0)^2 + A^0 \operatorname{grad} a^0 \operatorname{grad} a^0](x)\,dx. \end{aligned}$$

Then, as in Theorem 3.1,

$$(4.2)\qquad e^0(t) = E^0 + \int_0^t \int_\Omega f \frac{\partial u}{\partial t}\,dx\,ds \quad \text{in } [0, T].$$

The boundedness of the initial conditions and of u^ε implies that $e^\varepsilon(t)$ and E^ε [defined in (3.12)] are bounded in $L_\infty(0, T)$ and $\mathbb{R}$ respectively. For a subsequence ε' of ε, one has

$$(4.3)\qquad \begin{aligned} e^{\varepsilon'} &\rightharpoonup e \quad \text{weak-}* \quad \text{in } L_\infty(0, T); \\ E^{\varepsilon'} &\to E \quad \text{in } \mathbb{R}. \end{aligned}$$

Remark 4.1. — The subsequence $e^{\varepsilon'}$ may be proved to converge to e in the strong topology of $\mathscr{C}^0([0, T])$ [*cf.* (4.14), (4.26) below]. ●

The main difficulty resides in the fact in general

$$(4.4)\qquad e(t) \neq e^0(t) \quad \text{and} \quad E \neq E^0;$$

Theorem 4.3 below will detail the behaviour of e^ε and E^ε. This observation is at the root of the introduction of a solution $\widetilde{u}^\varepsilon$ of the wave equation with initial conditions $\widetilde{a}^\varepsilon$ and $\widetilde{b}^\varepsilon$ defined by

$$\text{(4.5)} \quad \begin{gathered} -\operatorname{div}(A^\varepsilon \operatorname{grad} \widetilde{a}^\varepsilon) = -\operatorname{div}(A^0 \operatorname{grad} a^0) \quad \text{in } \Omega, \\ \widetilde{a}^\varepsilon = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

$$\text{(4.6)} \quad \widetilde{b}^\varepsilon = b^0.$$

In (4.5)-(4.6), a^0 and b^0 are the functions defined in (3.8), (3.9). Define further $\widetilde{u}^\varepsilon$ and v^ε as the solutions of the following wave equations:

$$\text{(4.7)} \quad \begin{gathered} \rho^\varepsilon \frac{\partial^2 \widetilde{u}^\varepsilon}{\partial t^2} - \operatorname{div}(A^\varepsilon \operatorname{grad} \widetilde{u}^\varepsilon) = f \quad \text{in } \Omega \times (0, T), \\ \widetilde{u}^\varepsilon = 0 \quad \text{on } \partial\Omega \times (0, T), \\ \widetilde{u}^\varepsilon(0) = \widetilde{a}^\varepsilon \quad \text{in } \Omega, \\ \frac{\partial \widetilde{u}^\varepsilon}{\partial t}(0) = \widetilde{b}^\varepsilon = b^0 \quad \text{in } \Omega, \end{gathered}$$

$$\text{(4.8)} \quad \begin{gathered} \rho^\varepsilon \frac{\partial^2 v^\varepsilon}{\partial t^2} - \operatorname{div}(A^\varepsilon \operatorname{grad} v^\varepsilon) = 0 \quad \text{in } \Omega \times (0, T), \\ v^\varepsilon = 0 \quad \text{on } \partial\Omega \times (0, T), \\ v^\varepsilon(0) = a^\varepsilon - \widetilde{a}^\varepsilon \quad \text{in } \Omega, \\ \frac{\partial v^2}{\partial t}(0) = b^\varepsilon - \widetilde{b}^\varepsilon = b^\varepsilon - b^0 \quad \text{in } \Omega. \end{gathered}$$

Introduce the corresponding energies, *i.e.*,

$$\text{(4.9)} \quad \begin{gathered} \widetilde{e}^\varepsilon(t) = \frac{1}{2} \int_\Omega \left[\rho^\varepsilon \left(\frac{\partial \widetilde{u}^\varepsilon}{\partial t} \right)^2 + A^\varepsilon \operatorname{grad} \widetilde{u}^\varepsilon \operatorname{grad} \widetilde{u}^\varepsilon \right](x, t)\, dx, \\ \widetilde{E}^\varepsilon = \frac{1}{2} \int_\Omega [\rho^\varepsilon (b^0)^2 + A^\varepsilon \operatorname{grad} \widetilde{a}^\varepsilon \operatorname{grad} \widetilde{a}^\varepsilon](x)\, dx, \end{gathered}$$

$$\text{(4.10)} \quad \begin{gathered} \eta^\varepsilon(t) = \frac{1}{2} \int_\Omega \left[\rho^\varepsilon \left(\frac{\partial v^\varepsilon}{\partial t} \right)^2 + A^\varepsilon \operatorname{grad} v^\varepsilon \operatorname{grad} v^\varepsilon \right](x, t)\, dx, \\ H^\varepsilon = \frac{1}{2} \int_\Omega [\rho^\varepsilon (b^\varepsilon - b^0)^2 + A^\varepsilon \operatorname{grad}(a^\varepsilon - \widetilde{a}^\varepsilon) \operatorname{grad}(a^\varepsilon - \widetilde{a}^\varepsilon)](x)\, dx. \end{gathered}$$

They satisfy

$$\text{(4.11)} \quad \begin{gathered} \widetilde{e}^\varepsilon(t) = \widetilde{E}^\varepsilon + \int_0^t \int_\Omega f \frac{\partial \widetilde{u}^\varepsilon}{\partial t}\, dx\, ds \quad \text{in } [0, T], \\ \eta^\varepsilon(t) = H^\varepsilon \quad \text{in } [0, T]. \end{gathered}$$

Obviously

(4.12) $$u^\varepsilon = \tilde{u}^\varepsilon + v^\varepsilon.$$

The functions $\tilde{u}^\varepsilon$ and v^ε are successively investigated and the obtained results are summed up in Theorem 4.4.

As far as $\tilde{u}^\varepsilon$ is concerned we obtain the

THEOREM 4.1. — *The following convergences hold true:*

(4.13) $$\tilde{a}^\varepsilon \rightharpoonup a^0 \quad \textit{weakly in } H_0^1(\Omega),$$

(4.14) $$\tilde{e}^\varepsilon \to e^0 \quad \textit{strongly in } \mathscr{C}^0([0, T]),$$

(4.15) $$\tilde{E}^\varepsilon \to E^0 \quad \textit{in } \mathbb{R},$$

(4.16) $$\begin{aligned} &\tilde{u}^\varepsilon \rightharpoonup u \quad \textit{weak-}* \textit{ in } L_\infty(0, T; H_0^1(\Omega)),\\ &\frac{\partial \tilde{u}^\varepsilon}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \quad \textit{weak-}* \textit{ in } L_\infty(0, T; L_2(\Omega)), \end{aligned}$$

where u is the solution of the homogenized wave equation (3.20) *and a^0, e^0, E^0 are the corresponding initial condition and energies* [cf. (4.1)].

Furthermore, the following corrector result holds true:

(4.17) $$\frac{\partial \tilde{u}^\varepsilon}{\partial t} \to \frac{\partial u}{\partial t} \quad \textit{strongly in } \mathscr{C}^0([0, T]; L_2(\Omega)),$$

(4.18) $$\begin{aligned} &\operatorname{grad} \tilde{u}^\varepsilon = P^\varepsilon \operatorname{grad} u + R^\varepsilon,\\ &R^\varepsilon \to 0 \quad \textit{strongly in } \mathscr{C}^0([0, T]; [L_1(\Omega)]^N), \end{aligned}$$

where P^ε is the corrector matrix associated to the sequence A^ε. If u belongs to $\mathscr{C}^0([0, T]; W_0^{1,p}(\Omega))$, $2 \leqq p \leqq +\infty$ and P^ε is bounded in $[L_q(\Omega)]^{N^2}$, $2 \leqq q \leqq +\infty$, then the convergence of R^ε in (4.18) *takes place in $\mathscr{C}^0([0, T]; [L_s(\Omega)]^N)$ with*

(4.19) $$\frac{1}{s} = \max\left(\frac{1}{2}, \frac{1}{p} + \frac{1}{q}\right). \quad \bullet$$

Remark 4.2. — A result in the spirit of that of Theorem 4.1 is given in [BeLP], Chapt. 2, Section 3.6 in a periodic setting when ρ^ε is identically equal to 1, $A^\varepsilon(x) = A(x/\varepsilon)$ with A periodically defined on $\mathscr{M}(\alpha, \beta; \mathbb{R}^N)$ and u is sufficiently smooth. Most important however is the implicit assumption made in Section 3.6 of [BeLP] that the initial conditions are equal to zero. For this reason the result obtained there is similar to Theorem 4.1 above and not to Theorem 4.4 below. ●

Remark 4.3. — Assume that the data are smooth enough for all computations in this remark to be legitimate. The time derivative $\partial \tilde{u}^\varepsilon/\partial t$ satisfies the wave equation

$$
(4.20)\qquad
\begin{aligned}
&\rho^\varepsilon \frac{\partial^2}{\partial t^2}\left(\frac{\partial \tilde{u}^\varepsilon}{\partial t}\right) - \operatorname{div}\left(A^\varepsilon \operatorname{grad}\left(\frac{\partial \tilde{u}^\varepsilon}{\partial t}\right)\right) = \frac{\partial f}{\partial t} \quad \text{in } \Omega \times (0, T),\\
&\frac{\partial \tilde{u}^\varepsilon}{\partial t} = 0 \quad \text{on } \partial\Omega \times (0, T),\\
&\frac{\partial \tilde{u}^\varepsilon}{\partial t}(0) = b^0 \quad \text{in } \Omega,\\
&\left(\rho^\varepsilon \frac{\partial}{\partial t}\left(\frac{\partial \tilde{u}^\varepsilon}{\partial t}\right)\right)(0) = \operatorname{div}(A^\varepsilon \operatorname{grad} \tilde{u}^\varepsilon(0)) + f(0),\\
&\qquad = \operatorname{div}(A^0 \operatorname{grad} a^0) + f(0) \quad \text{in } \Omega.
\end{aligned}
$$

Note that the initial condition on $\partial/\partial t(\partial \tilde{u}^\varepsilon/\partial t)$ is given as an initial condition on $\rho^\varepsilon(\partial/\partial t(\partial \tilde{u}^\varepsilon/\partial t))$ in accordance with Remark 3.3. Because the initial conditions on $\partial \tilde{u}^\varepsilon/\partial t$, $\rho^\varepsilon(\partial/\partial t(\partial \tilde{u}^\varepsilon/\partial t))$ are fixed Theorem 3.2 is applicable. It yields the weak convergence of $\partial \tilde{u}^\varepsilon/\partial t$ to $\partial u/\partial t$ in $W(0, T; H_0^1(\Omega), L_2(\Omega))$ and, with the help of (3.29), the strong convergence of $\partial \tilde{u}^\varepsilon/\partial t$ in $\mathscr{C}^0([0, T]; L_2(\Omega))$. This result is identical to (4.17).

Note finally that similar considerations would be doomed in the case of u^ε since $\operatorname{div}(A^\varepsilon \operatorname{grad} a^\varepsilon) + f(0)$ is, in general, only bounded in $H^{-1}(\Omega)$. ●

As far as v^ε is concerned we obtain the

THEOREM 4.2. — *The following convergences hold true:*

$$
(4.21)\qquad
\begin{aligned}
&v^\varepsilon \rightharpoonup 0 \quad \textit{weak-}* \textit{ in } L_\infty(0, T, H_0^1(\Omega)),\\
&\frac{\partial v^\varepsilon}{\partial t} \rightharpoonup 0 \quad \textit{weak-}* \textit{ in } L_\infty(0, T; L_2(\Omega)).
\end{aligned}
$$

The convergences in (4.21) *are strong if and only if*

$$
(4.22)\qquad
\begin{aligned}
&a^\varepsilon - \tilde{a}^\varepsilon \to 0 \quad \textit{strongly in } H_0^1(\Omega),\\
&b^\varepsilon - b^0 \to 0 \quad \textit{strongly in } L_2(\Omega). \quad ●
\end{aligned}
$$

The following theorem details the behavior of the various energies.

THEOREM 4.3. — *One has*

$$
(4.23)\qquad
\begin{aligned}
&\frac{1}{2}\int_\Omega \rho^\varepsilon \left[\frac{\partial \tilde{u}^\varepsilon}{\partial t}\right]^2 dx \to \frac{1}{2}\int_\Omega \bar{\rho}\left[\frac{\partial u}{\partial t}\right]^2 dx\\
&\textit{strongly in } C^0([0, T]),\\
&\frac{1}{2}\int_\Omega A^\varepsilon \operatorname{grad} \tilde{u}^\varepsilon \operatorname{grad} \tilde{u}^\varepsilon\, dx \to \frac{1}{2}\int_\Omega A^0 \operatorname{grad} u \operatorname{grad} u\, dx\\
&\textit{strongly in } C^0([0, T]);
\end{aligned}
$$

if ε' *denotes a subsequence of* ε *such that* $H^{\varepsilon'}$ *converges to* H *in* $\mathbb{R}$ (*note that* H^{ε} *is a bounded sequence of real numbers*), *then*

$$(4.24)\qquad \begin{aligned} &\frac{1}{2}\int_{\Omega} \rho^{\varepsilon'}\left[\frac{\partial v^{\varepsilon'}}{\partial t}\right]^2 dx \rightharpoonup \frac{1}{2}\mathrm{H} \quad \textit{weak-}* \textit{ in } \mathrm{L}_\infty(0, \mathrm{T}),\\ &\frac{1}{2}\int_{\Omega} \mathrm{A}^{\varepsilon'} \operatorname{grad} v^{\varepsilon'} \operatorname{grad} v^{\varepsilon'} dx \rightharpoonup \frac{1}{2}\mathrm{H} \quad \textit{weak-}* \textit{ in } \mathrm{L}_\infty(0, \mathrm{T}),\end{aligned}$$

that is the energy associated to $v^{\varepsilon'}$ *satisfies an equipartition principle.*

For the subsequence ε',

$$(4.25)\qquad \begin{aligned} &\frac{1}{2}\int_{\Omega} \rho^{\varepsilon'}\left[\frac{\partial u^{\varepsilon'}}{\partial t}\right]^2 dx \rightharpoonup \frac{1}{2}\left(\mathrm{H}+\int_{\Omega}\bar{\rho}\left[\frac{\partial u}{\partial t}\right]^2 dx\right)\\ &\qquad \textit{weak-}* \textit{ in } \mathrm{L}_\infty(0, \mathrm{T}),\\ &\frac{1}{2}\int_{\Omega} \mathrm{A}^{\varepsilon'} \operatorname{grad} u^{\varepsilon'} \operatorname{grad} u^{\varepsilon'} dx \rightharpoonup \frac{1}{2}\left(\mathrm{H}+\int_{\Omega}\mathrm{A}^0 \operatorname{grad} u \operatorname{grad} u\, dx\right)\\ &\qquad \textit{weak-}* \textit{ in } \mathrm{L}_\infty(0, \mathrm{T}).\end{aligned}$$

Finally the following convergences also hold true:

$$(4.26)\qquad e^{\varepsilon}-\tilde{e}^{\varepsilon}-\mathrm{H}^{\varepsilon}\to 0 \quad \textit{strongly in } \mathscr{C}^0([0, \mathrm{T}]),$$

$$(4.27)\qquad \mathrm{E}^{\varepsilon}-\tilde{\mathrm{E}}^{\varepsilon}-\mathrm{H}^{\varepsilon}\to 0 \quad \textit{in } \mathbb{R},$$

and

$$(4.28)\qquad \underline{\lim}\, \mathrm{E}^{\varepsilon} \geqq \mathrm{E}^0,$$

while

(4.29) E^{ε} *tends to* E^0 *if and only if* (4.22) *holds true.*

Results similar to (4.28), (4.29) *also hold true for* e^{ε}, e^0. ●

The results obtained for $\tilde{u}^{\varepsilon}$ and v^{ε} can be brought together in a statement on the intimate behavior of u^{ε} itself.

THEOREM 4.4. — *The solution* u^{ε} *of* (3.1)-(3.4) *can be decomposed as follows:*

$$(4.30)\qquad \begin{aligned} &\frac{\partial u^{\varepsilon}}{\partial t}=\frac{\partial u}{\partial t}+\frac{\partial v^{\varepsilon}}{\partial t}+r^{\varepsilon},\\ &\operatorname{grad} u^{\varepsilon}=\mathrm{P}^{\varepsilon} \operatorname{grad} u+\operatorname{grad} v^{\varepsilon}+\mathrm{R}^{\varepsilon},\end{aligned}$$

where P^ε *is the corrector matrix associated to the sequence* A^ε *and* u *is the solution of the homogenized wave equation* (3.20); v^ε *is the solution of* (4.8) *and satisfies*

$$(4.31)\qquad \frac{\partial v^\varepsilon}{\partial t} \rightharpoonup 0 \quad \textit{weak-}* \textit{ in } L_\infty(0, T; L_2(\Omega)),$$
$$\operatorname{grad} v^\varepsilon \rightharpoonup 0 \quad \textit{weak-}* \textit{ in } L_\infty(0, T; [L_2(\Omega)]^N),$$

while

$$(4.32)\qquad r^\varepsilon \rightharpoonup 0 \quad \textit{strongly in } \mathscr{C}^0([0, T]; L_2(\Omega)),$$
$$R^\varepsilon \to 0 \quad \textit{strongly in } \mathscr{C}^0([0, T]; [L_1(\Omega)]^N).$$

Convergence of R^ε *will take place in better spaces if additional regularity is met by* u *and* P^ε (cf. *Theorem* 4.1).

Futhermore the convergence in (4.31) *is strong is and only if*

$$(4.33)\qquad a^\varepsilon - \tilde{a}^\varepsilon \to 0 \quad \textit{strongly in } H_0^1(\Omega),$$
$$b^\varepsilon - b^0 \to 0 \quad \textit{strongly in } L_2(\Omega),$$

in which case $\partial v^\varepsilon/\partial t$ *and* $\operatorname{grad} v^\varepsilon$ *do not appear in* (4.30). *If* (4.33) *is not satisfied both* $\int_\Omega (\partial v^\varepsilon/\partial t)^2\,dx$ *and* $\int_\Omega (\operatorname{grad} v^\varepsilon)^2\,dx$ *converge (weak-* $*$ *in* $L_\infty(0, T)$) *to strictly positive functions.* ●

Remark 4.4. — The presence of the terms $\partial v^\varepsilon/\partial t$ and $\operatorname{grad} v^\varepsilon$ is in general unavoidable in the corrector result (4.30), which is one of the main results of this paper. ●

Remark 4.5. — A more detailed study of the structure of $\partial v^\varepsilon/\partial t$ and $\operatorname{grad} v^\varepsilon$ has yet to be performed. The new concept of H-measures has been recently proposed by L. Tartar [T 3], so as to analyze such problems. ●

Remark 4.6. — In the spirit of Remark 4.5 the analogue of (3.1)-(3.4) where the Dirichlet boundary conditions have been replaced by periodic boundary conditions is of interest. In this setting the bounded open domain Ω is replaced by the unit torus $\mathscr{C}$, $1/\varepsilon$ is an integer and all data are defined on $\mathscr{C}$.

Consider the example where f is taken to be zero, while

$$(4.34)\qquad A^\varepsilon(x) = A(x/\varepsilon), \qquad \rho^\varepsilon(x) = \rho(x/\varepsilon),$$
$$a^\varepsilon(x) = \varepsilon\, a(x/\varepsilon), \qquad b^\varepsilon(x) = b(x/\varepsilon),$$

with A in $L_\infty(\mathscr{C}, \mathbb{R}^N \times \mathbb{R}^N)$, ρ in $L_\infty(\mathscr{C})$, α in $H^1(\mathscr{C})$ and β in $L_2(\mathscr{C})$ with

$$(4.35)\qquad \int_{\mathscr{C}} \rho(y)\,\beta(y)\,dy = 0.$$

The periodic analogue of (3.1)-(3.4) reads as

$$(4.36)\qquad \begin{aligned} &\rho\left(\frac{x}{\varepsilon}\right)\frac{\partial^2 u^\varepsilon}{\partial t^2}-\operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\operatorname{grad} u^\varepsilon\right)=0 \quad \text{in } \mathscr{C}\times(0,\infty),\\ &u^\varepsilon(x,0)=a^\varepsilon(x) \quad \text{in } \mathscr{C},\\ &\frac{\partial u^\varepsilon}{\partial t}(x,0)=b^\varepsilon(x) \quad \text{in } \mathscr{C}. \end{aligned}$$

Note that since the spatial reference domain is the unit torus the formulation (4.36) implicitly implies periodic boundary conditions. The same remark applies to (4.38) below.

Under hypotheses (4.34), (4.35), it is easily seen that, in the notation adopted in the present paper, u, $\tilde{a}^\varepsilon$, $\tilde{b}^\varepsilon$, $\tilde{u}^\varepsilon$ are identically zero. Thus u^ε coincides with v^ε which is further given by the quasi explicit formula

$$(4.37)\qquad v^\varepsilon(x,t)=\varepsilon V\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon}\right),$$

where V is the solution of

$$(4.38)\qquad \begin{aligned} &\rho(y)\frac{\partial^2 V}{\partial s^2}-\operatorname{div}_y(A(y)\operatorname{grad}_y V)=0 \quad \text{in } \mathscr{C}x(0,\infty),\\ &V(y,0)=\alpha(y) \quad \text{in } \mathscr{C},\\ &\frac{\partial V}{\partial s}(y,0)=\beta(y) \quad \text{in } \mathscr{C}. \end{aligned}$$

The above description (4.37) of v^ε is structurally more accurate than that of Theorem 4.2. Let us however emphasize that an ansatz similar to (4.37) is doomed as soon as one departs from the above described example (4.34); the case of initial data that exhibit an additional variation in x cannot be tackled in this manner even for constant A's and ρ's.

Finally note that the solution V of (4.38) is *not* periodic in s. This has to be related to the study performed in [F] where an asymptotic expansion for the system of linear thermoelasticity in the periodic case leads to a double scaling in both space and time with no periodicity with respect to the fast time variable. ●

5. Proof of the corrector result for the wave equation

This section is devoted to the proofs of the various results announced in Section 4.

Proof of Theorem 4.1. – The proof is divided into four steps.

First step. — Firstly (4.13)-(4.16) are proved. In view of the definition (4.5) of $\tilde{a}^\varepsilon$, (4.13) is a mere consequence of the Definition 2.1 of H-convergence and Theorem 3.1 applied to (4.7) implies (4.16).

Convergence (4.15) is immediate in view of the Definitions (3.5), (3.9) of $\bar{\rho}$ and b^0 together with the convergence of $\int_\Omega A^\varepsilon \operatorname{grad} \tilde{a}^\varepsilon \operatorname{grad} \tilde{a}^\varepsilon \, dx$ to $\int_\Omega A^0 \operatorname{grad} a^0 \operatorname{grad} a^0 \, dx$ which is in turn easily obtained upon multiplication of (4.5) by $\tilde{a}^\varepsilon$ and integration by parts.

The first equality in (4.11) together with (4.15), (4.16) imply the weak-$*$ convergence in $L_\infty(0, T)$ of $\tilde{e}^\varepsilon$ to e^0. The statement (4.14) of strong convergence results from a straightforward application of Arzela-Ascoli's theorem upon remarking that for any positive real number h

$$(5.1)\qquad \left|\int_t^{t+h}\int_\Omega f \frac{\partial \tilde{u}^\varepsilon}{\partial t} dx\, ds\right| \leq \left\|\frac{\partial \tilde{u}^\varepsilon}{\partial t}\right\|_{L_2(0,\, T;\, L_2(\Omega))} \|f\|_{L_2(t,\, t+h;\, L_2(\Omega))}.$$

Second step. — Following the method of proof devised in [T1], [Mu1] for the elliptic case, we consider the quantity $\tilde{X}^\varepsilon$ defined for any Φ in $\mathscr{C}^\infty([0, T]; \mathscr{C}_0^\infty(\Omega))$ by

$$(5.2)\qquad \tilde{X}^\varepsilon = \frac{1}{2}\int_\Omega \left[\rho^\varepsilon\left(\frac{\partial \tilde{u}^\varepsilon}{\partial t} - \frac{\partial \Phi}{\partial t}\right)^2 + A^\varepsilon(\operatorname{grad} \tilde{u}^\varepsilon - P^\varepsilon \operatorname{grad} \Phi)(\operatorname{grad} \tilde{u}^\varepsilon - P^\varepsilon \operatorname{grad} \Phi)\right](x, t)\, dx.$$

Developing $\tilde{X}^\varepsilon(t)$ yields

$$(5.3)\qquad \tilde{X}^\varepsilon(t) = I^\varepsilon(t) + II^\varepsilon(t) + III^\varepsilon(t),$$

where

$$(5.4)\qquad \begin{aligned} I^\varepsilon(t) &= \frac{1}{2}\int_\Omega \left[\rho^\varepsilon\left(\frac{\partial \tilde{u}^\varepsilon}{\partial t}\right)^2 + A^\varepsilon \operatorname{grad} \tilde{u}^\varepsilon \operatorname{grad} \tilde{u}^\varepsilon\right](x, t)\, dx = \tilde{e}^\varepsilon(t), \\ II^\varepsilon(t) &= -\int_\Omega \left[\rho^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial t}\frac{\partial \Phi}{\partial t} + A^\varepsilon P^\varepsilon \operatorname{grad} \Phi \operatorname{grad} \tilde{u}^\varepsilon\right](x, t)\, dx, \\ III^\varepsilon(t) &= \frac{1}{2}\int_\Omega \left[\rho^\varepsilon\left(\frac{\partial \Phi}{\partial t}\right)^2 + A^\varepsilon P^\varepsilon \operatorname{grad} \Phi\, P^\varepsilon \operatorname{grad} \Phi\right](x, t)\, dx. \end{aligned}$$

We propose to compute the limits of I^ε, III^ε and II^ε successively.

The limit of I^ε has already been computed in the first step; the result is stated in (4.14). The quantity $III^\varepsilon(t)$, as well as its time derivative, is bounded in $L_\infty(0, T)$ because of the smoothness of Φ. Thus $III^\varepsilon(t)$ converges strongly in $\mathscr{C}^0([0, T])$ to a limit which is easily identified through a direct application of the "div-curl lemma" (Theorem 2.4); we obtain

$$(5.5)\qquad III^\varepsilon(t) \to III(t) = \frac{1}{2}\int_\Omega \left[\bar{\rho}\left(\frac{\partial \Phi}{\partial t}\right)^2 + A^0 \operatorname{grad} \Phi \operatorname{grad} \Phi\right](x, t)\, dx,$$

strongly in $\mathscr{C}^0([0, T])$.

The second term $II^\varepsilon(t)$ is clearly bounded in $L_\infty(0, T)$. Rewriting the term $\rho^\varepsilon(\partial\tilde{u}^\varepsilon/\partial t)$ as $(\partial/\partial t)(\rho^\varepsilon \tilde{u}^\varepsilon)$ and applying the time dependent version of the "div-curl lemma" (Theorem 2.4) to the other term enables us to identify the weak-$*$ limit of $II^\varepsilon(t)$ as $II(t)$ defined as

$$(5.6)\qquad II(t) = -\int_\Omega \left[\bar{\rho}\frac{\partial u}{\partial t}\frac{\partial \Phi}{\partial t} + A^0 \operatorname{grad}\Phi \operatorname{grad} u\right](x, t)\,dx.$$

We will prove in the fourth step that the afore mentioned convergence actually takes place in the strong topology of $\mathscr{C}^0([0, T])$.

Recalling (4.1), (4.14), (5.5), (5.6), we conclude that

$$(5.7)\qquad \tilde{X}^\varepsilon \to \frac{1}{2}\int_\Omega \left[\bar{\rho}\left(\frac{\partial u}{\partial t} - \frac{\partial \Phi}{\partial t}\right)^2 + A^0(\operatorname{grad} u - \operatorname{grad}\Phi)(\operatorname{grad} u - \operatorname{grad}\Phi)(x, t)\right]dx$$

strongly in $\mathscr{C}^0([0, T])$.

Thus

$$(5.8)\qquad \overline{\lim}\left\{\lambda_1 \left\|\frac{\partial \tilde{u}^\varepsilon}{\partial t} - \frac{\partial \Phi}{\partial t}\right\|^2_{\mathscr{C}^0([0, T]; L_2(\Omega))} + \alpha \|\operatorname{grad}\tilde{u}^\varepsilon - P^\varepsilon \operatorname{grad}\Phi\|^2_{\mathscr{C}^0([0, T]; [L_2(\Omega)]^N)}\right\}$$
$$\leq \lambda_2 \left\|\frac{\partial u}{\partial t} - \frac{\partial \Phi}{\partial t}\right\|^2_{\mathscr{C}^0([0, T]; L_2(\Omega))} + \beta \|\operatorname{grad} u - \operatorname{grad}\Phi\|^2_{\mathscr{C}^0([0, T]; [L_2(\Omega)]^N)}.$$

Third step. — In view of the regularity properties of u, namely,

$$(5.9)\qquad u \in \mathscr{C}^0([0, T]; H_0^1(\Omega)),\quad \frac{\partial u}{\partial t} \in \mathscr{C}^0([0, T]; L_2(\Omega)),$$

the function Φ (and $\partial\Phi/\partial t$) can be chosen so as to be arbitrarily close to u (and $\partial u/\partial t$) in the topology of $\mathscr{C}^0([0, T]; H_0^1(\Omega))$ (and $\mathscr{C}^0([0, T]; L_2(\Omega))$). Decomposition of $\partial\tilde{u}^\varepsilon/\partial t - \partial u/\partial t$ into the sum $(\partial\tilde{u}^\varepsilon/\partial t - \partial\Phi/\partial t) + (\partial\Phi/\partial t - \partial u/\partial t)$ and of $\operatorname{grad}\tilde{u}^\varepsilon - P^\varepsilon \operatorname{grad} u$ into the sum $(\operatorname{grad}\tilde{u}^\varepsilon - P^\varepsilon \operatorname{grad}\Phi) + P^\varepsilon(\operatorname{grad}\Phi - \operatorname{grad} u)$ and application of Hölder's inequality to the last terms yields, with the help of (5.8).

$$(5.10)\qquad \overline{\lim}\left\{\lambda_1 \left\|\frac{\partial \tilde{u}^\varepsilon}{\partial t} - \frac{\partial u}{\partial t}\right\|^2_{\mathscr{C}^0([0, T]; L_2(\Omega))} + \alpha \|\operatorname{grad}\tilde{u}^\varepsilon - P^\varepsilon \operatorname{grad} u\|^2_{\mathscr{C}^0([0, T]; [L_1(\Omega)]^N)}\right\}$$

$$\leq C_1 \left\| \frac{\partial \tilde{u}}{\partial t} - \frac{\partial \Phi}{\partial t} \right\|^2_{\mathscr{C}^0([0, T]; L_2(\Omega))} + C_2 \| \operatorname{grad} u - \operatorname{grad} \Phi \|^2_{\mathscr{C}^0([0, T]; [L_2(\Omega)]^N)},$$

where

$$C_1 = 2[\lambda_1 + \sup(1, \operatorname{meas} \Omega) \lambda_2]$$

and

$$C_2 = 2[\alpha \overline{\lim} \| P^\varepsilon \|^2_{[L_2(\Omega)]^{N^2}} + \beta \sup(1, \operatorname{meas} \Omega)].$$

Inequality (5.10) implies (4.17), (4.18) because of the bounded character of P^ε in $[L_2(\Omega)]^{N^2}$.

The convergence of $(\operatorname{grad} u^\varepsilon - P^\varepsilon \operatorname{grad} u)$ in better spaces stated at the end of Theorem 4.1 is obtained, for any finite p, by approximating $\operatorname{grad} u$ by $\operatorname{grad} \Phi$ in $C^0([0, T]; [L_p(\Omega)]^N)$ (in place of $C^0([0, T]; [L_2(\Omega)]^N)$) and using the assumed $[L_q(\Omega)]^{N^2}$-bound on P^ε. Whenever p is infinite and q is strictly greater than two the approximation of $\operatorname{grad} u$ is performed in $C^0([0, T]; [L_{\bar{p}}(\Omega)]^N)$ where $(1/\bar{p}) + (1/q) = (1/2)$; the difference $P^\varepsilon(\operatorname{grad} u - \operatorname{grad} \Phi)$ is then controlled in $C^0([0, T]; [L_2(\Omega)]^N)$. The remaining case (*i.e.*, $p = +\infty$ and $q = 2$) never occurs since Meyer's regularity result (*cf.* [Me]) ensures the boundedness of P^ε in $[L_q(\Omega)]^{N^2}$ with $q > 2$.

Fourth step. — We finally prove the uniform convergence of $\Pi^\varepsilon(t)$ in $\mathscr{C}^0([0, T])$. To this effect we first note that, by virtue of (4.7),

$$(5.11) \quad \frac{\partial}{\partial t}\left\{ \int_\Omega \left[\rho^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial t} \frac{\partial \Phi}{\partial t} \right](x, t)\, dx \right\} = \int_\Omega \left[\rho^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial t} \frac{\partial^2 \Phi}{\partial t^2} \right](x, t)\, dx + \int_\Omega \left[f \frac{\partial \Phi}{\partial t} \right](x, t)\, dx - \int_\Omega \left[A^\varepsilon \operatorname{grad} \tilde{u}^\varepsilon \operatorname{grad} \frac{\partial \Phi}{\partial t} \right](x, t)\, dx,$$

is bounded in $L_2(0, T)$. Thus

$$(5.12) \quad \int_\Omega \left[\rho^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial t} \frac{\partial \Phi}{\partial t} \right](x, t)\, dx \text{ is relatively compact in } \mathscr{C}^0([0, T]).$$

On the other hand, with the help of the Definition 2.2 of P^ε,

$$(5.13) \quad \int_\Omega [A^\varepsilon P^\varepsilon \operatorname{grad} \Phi \operatorname{grad} \tilde{u}^\varepsilon](x, t)\, dx = \sum_{i=1}^N \int_\Omega \left[A^\varepsilon \operatorname{grad} w_i^\varepsilon \frac{\partial \Phi}{\partial x_i} \operatorname{grad} \tilde{u}^\varepsilon \right](x, t)\, dx$$

$$= \sum_{i=1}^{N} \left\langle -\operatorname{div}(A^0 e_i), \frac{\partial \Phi}{\partial x_i} \tilde{u}^\varepsilon \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}(t)$$

$$- \sum_{i=1}^{N} \int_\Omega \left[A^\varepsilon \operatorname{grad} w_i^\varepsilon \operatorname{grad} \frac{\partial \Phi}{\partial x_i} \tilde{u}^\varepsilon \right](x, t)\, dx$$

$$= \langle k^\varepsilon, \tilde{u}^\varepsilon \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}(t),$$

where k^ε is the sum of a fixed function in $\mathscr{C}^0([0, T]; H^{-1}(\Omega))$ and of a bounded sequence in $\mathscr{C}^1([0, T]; L_2(\Omega))$. Thus, at the possible expense of extracting a subsequence ε', we may assume that

(5.14) $$k^{\varepsilon'} \to k \quad \text{strongly in } \mathscr{C}^0([0, T]; H^{-1}(\Omega)).$$

Consider h in $\mathscr{C}^0([0, T]; L_2(\Omega))$ and write

(5.15) $$\langle k^{\varepsilon'}, \tilde{u}^{\varepsilon'} \rangle - \langle k, u \rangle = \langle k^{\varepsilon'} - k, \tilde{u}^{\varepsilon'} \rangle + \langle k - h, \tilde{u}^{\varepsilon'} \rangle + \langle h, \tilde{u}^{\varepsilon'} - u \rangle + \langle h - k, u \rangle.$$

In (5.15), $\langle\ ,\ \rangle$ has to be understood as $\langle\ ,\ \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}(t)$.

Then

(5.16) $$\| \langle k^{\varepsilon'}, \tilde{u}^{\varepsilon'} \rangle - \langle k, u \rangle \|_{\mathscr{C}^0([0, T])} \leqq \{ \| k^{\varepsilon'} - k \|_{\mathscr{C}^0([0, T]; H^{-1}(\Omega))} + \| k - h \|_{\mathscr{C}^0([0, T]; H^{-1}(\Omega))} \} \| \tilde{u}^{\varepsilon'} \|_{L_\infty(0, T; H_0^1(\Omega))} + \| h \|_{\mathscr{C}^0([0, T]; L_2(\Omega))} \| \tilde{u}^{\varepsilon'} - u \|_{\mathscr{C}^0([0, T]; L_2(\Omega))} + \| k - h \|_{\mathscr{C}^0([0, T]; H^{-1}(\Omega))} \| u \|_{L_\infty(0, T; H_0^1(\Omega))}.$$

Choosing h in $\mathscr{C}^0([0, T]; L_2(\Omega))$ that approximates k in $\mathscr{C}^0([0, T]; H^{-1}(\Omega))$, using (5.14) and the convergence of $\tilde{u}^{\varepsilon'}$ to u in $\mathscr{C}^0([0, T]; L_2(\Omega))$ [*cf.* (3.28)] yields the strong convergence of $\langle k^{\varepsilon'}, \tilde{u}^{\varepsilon'} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}(t)$ to $\langle k, u \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}(t)$. Thus

(5.17) $$\int_\Omega [A^\varepsilon P^\varepsilon \operatorname{grad} \Phi \operatorname{grad} \tilde{u}^\varepsilon](x, t)\, dx \text{ is relatively compact in } \mathscr{C}^0([0, T]).$$

Statements (5.12) and (5.17) prove the strong convergence of $II^\varepsilon(t)$ in $\mathscr{C}^0([0, T])$. The proof of Theorem 4.1 is complete. ●

Proof of Theorem 4.2. — The decomposition (4.12) together with the convergence result (4.16) immediately imply (4.21). Note that (4.21) can also be obtained through application of Theorem 3.2 to (4.8). The statement of conservation of the energy associated to v^ε, *i.e.*, the second equality of (4.11), together with the Definitions (4.10) of η^ε and H^ε imply that the strong convergences in (4.21) are equivalent to (4.22). ●

Proof of Theorem 4.3. — The proof is divided into four steps. The first step is a remark pertaining to the convergence of the energy densities. The second step is concerned with the very proof of (4.24). The third step is devoted to the proof of

(4.23) while the fourth step successively proves (4.27) [and (4.26)], (4.28), the equivalence statement (4.29) and finally (4.25).

First step. — The equipartition of the kinetic and potential energy densities associated to v^ε is a consequence of the "div-curl lemma". Specifically the vector $(\rho^\varepsilon(\partial v^\varepsilon/\partial t), -A^\varepsilon \operatorname{grad} v^\varepsilon)$ is a divergence free vector in t and x which converges weakly in $[L_2(\Omega\times(0, T))]^{N+1}$ to 0, while the vector $(\partial v^\varepsilon/\partial t, \operatorname{grad} v^\varepsilon)$ is a gradient in t and x that converges weakly to 0 in $[L_2(\Omega\times(0, T))]^{N+1}$. Thus

$$(5.18)\qquad \rho^\varepsilon\left(\frac{\partial v^\varepsilon}{\partial t}\right)^2 - A^\varepsilon \operatorname{grad} v^\varepsilon \operatorname{grad} v^\varepsilon \rightharpoonup 0 \quad \text{weak-}* \text{ in } \mathscr{D}'(\Omega\times(0, T)).$$

The energy $(1/2)[\rho^\varepsilon(\partial v^\varepsilon/\partial t)^2 + A^\varepsilon \operatorname{grad} v^\varepsilon \operatorname{grad} v^\varepsilon]$ is bounded in $L_1(\Omega\times(0, T))$ and thus its admits a weak-$*$ converging subsequence in the vague topology of measures. If h denotes the limit measure (which is a positive bounded measure on Ω), we conclude with the help of (5.18) that

$$(5.19)\qquad \begin{aligned} &\frac{1}{2}\rho^{\varepsilon'}\left(\frac{\partial v^{\varepsilon'}}{\partial t}\right)^2 \rightharpoonup \frac{1}{2}h \quad \text{weak-}* \text{ in } \mathscr{D}'(\Omega\times(0, T)),\\ &\frac{1}{2}A^{\varepsilon'} \operatorname{grad} v^{\varepsilon'} \operatorname{grad} v^{\varepsilon'} \rightharpoonup \frac{1}{2}h \quad \text{weak-}* \text{ in } \mathscr{D}'(\Omega\times(0, T)).\end{aligned}$$

The convergences in (5.19) express a statement of equipartition of the energy density.

Second step. — It is actually possible to extend this result up to the boundary of Ω and to obtain (4.24) because of the homogeneous character of the boundary conditions on v^ε. Since v^ε lies in $L_2(0, T; H_0^1(\Omega))$, the function φv^ε is a licit test function in the first equation of (4.8) for any φ in $\mathscr{C}_0^\infty(0, T)$. Upon integrating the resulting expression by parts we obtain

$$(5.20)\qquad \int_0^T\left(\int_\Omega \rho^\varepsilon\left(\frac{\partial v^\varepsilon}{\partial t}\right)^2 dx\right)\varphi(t)\,dt - \int_0^T\left(\int_\Omega A^\varepsilon \operatorname{grad} v^\varepsilon \operatorname{grad} v^\varepsilon dx\right)\varphi(t)\,dt = -\int_0^T\int_\Omega \rho^\varepsilon \frac{\partial v^\varepsilon}{\partial t} v^\varepsilon \frac{\partial \varphi}{\partial t}\,dx\,dt.$$

The right hand side of (5.20) tends to zero since $\rho^\varepsilon(\partial v^\varepsilon/\partial t)$ is bounded in $L_\infty(0, T; L_2(\Omega))$ while v^ε converges strongly to zero in $\mathscr{C}^0([0, T]; L_2(\Omega))$. Since φ is arbitrary,

$$(5.21)\qquad \int_\Omega\left[\rho^\varepsilon\left(\frac{\partial v^\varepsilon}{\partial t}\right)^2\right](x, t)\,dx - \int_\Omega [A^2 \operatorname{grad} v^\varepsilon \operatorname{grad} v^\varepsilon](x, t)\,dx \rightharpoonup 0 \quad \text{weak-}* \text{ in } L_\infty(0, T),$$

while the conservation of energy (4.11) implies that for a subsequence $H^{\varepsilon'}$ such that $H^{\varepsilon'}$ converges to H in $\mathbb{R}$

$$(5.22)\quad \frac{1}{2}\int_{\Omega}\left[\rho^{\varepsilon'}\left(\frac{\partial v^{\varepsilon'}}{\partial t}\right)^2\right](x, t)\,dx + \frac{1}{2}\int_{\Omega}[A^{\varepsilon'}\operatorname{grad} v^{\varepsilon'}\operatorname{grad} v^{\varepsilon'}](x, t)\,dx \to H \quad \text{strongly in } \mathscr{C}^0([0, T]).$$

Convergences (5.21) and (5.22) yield convergence (4.24).

Third step. — By virtue of the strong convergence (4.17) of $\partial\widetilde{u}^\varepsilon/\partial t$, the inequality

$$(5.23)\quad \left|\int_{\Omega}\left[\rho^\varepsilon\left(\frac{\partial\widetilde{u}^\varepsilon}{\partial t}\right)^2\right](x, t+h)\,dx - \int_{\Omega}\left[\rho^\varepsilon\left(\frac{\partial\widetilde{u}^\varepsilon}{\partial t}\right)^2\right](x, t)\,dx\right| \leqq 2\lambda_2\left\|\frac{\partial\widetilde{u}^\varepsilon}{\partial t}\right\|_{L_\infty(0, T; L_2(\Omega))}\left\|\frac{\partial\widetilde{u}^\varepsilon}{\partial t}(t+h) - \frac{\partial\widetilde{u}^\varepsilon}{\partial t}(t)\right\|_{L_2(\Omega)},$$

together with Ascoli-Arzela's theorem implies the first part of (4.23). The second part of (4.23) is deduced from the first part of (4.23) and the uniform convergence (4.14) of $\widetilde{e}^\varepsilon$ to e^0.

Fourth step. — Setting $a^\varepsilon = \widetilde{a}^\varepsilon + (a^\varepsilon - \widetilde{a}^\varepsilon)$; $b^\varepsilon = b^0 + (b^\varepsilon - b^0)$ in the expression (3.12) for E^ε yields,

$$(5.24)\quad E^\varepsilon = \widetilde{E}^\varepsilon + H^\varepsilon + F^\varepsilon,$$

where $\widetilde{E}^\varepsilon$ is given in (4.9), H^ε in (4.10) and

$$(5.25)\quad F^\varepsilon = \int_{\Omega}[\rho^\varepsilon b^0(b^\varepsilon - b^0) + A^\varepsilon\operatorname{grad}\widetilde{a}^\varepsilon\operatorname{grad}(a^\varepsilon - \widetilde{a}^\varepsilon)]\,dx.$$

By virtue of the Definitions (3.5), (3.9) of $\bar{\rho}$ and b^0 and by application of the "div-cul lemma" to the vectors $A^\varepsilon\operatorname{grad}\widetilde{a}^\varepsilon$ and $\operatorname{grad}(a^\varepsilon - \widetilde{a}^\varepsilon)$ the term F^ε is seen to converge to zero, which proves (4.27). Convergence (4.26) is an immediate consequence of (4.27) in view of (3.13), (4.11) and Ascoli-Arzela's theorem which yields [*cf.* (5.1)],

$$(5.26)\quad \int_0^t\int_{\Omega}\left[f(x, s)\frac{\partial}{\partial t}(u^\varepsilon - \widetilde{u}^\varepsilon)\right](x, s)\,dx\,ds \to 0 \quad \text{strongly in } \mathscr{C}^0([0, T]).$$

Since $\widetilde{E}^\varepsilon$ converges to E^0 [*cf.* (4.15)] while H^ε remains positive, (4.28) is a consequence of (4.27). The statement (4.29) is also a consequence of (4.27).

Finally (4.25) is deduced from (4.26), (4.23) and from the following convergence result

$$(5.27)\quad \frac{1}{2}\int_{\Omega}\left[\rho^{\varepsilon'}\left(\frac{\partial u^{\varepsilon'}}{\partial t}\right)^2 - A^{\varepsilon'}\operatorname{grad} u^{\varepsilon'}\operatorname{grad} u^{\varepsilon'}\right](x,t)\,dx$$

$$\rightharpoonup \frac{1}{2}\int_{\Omega}\left[\bar{\rho}\left(\frac{\partial u}{\partial t}\right)^2 - A^0\operatorname{grad} u\operatorname{grad} u\right](x,t)\,dx \quad \text{weak-}* \text{ in } L_\infty(0,T).$$

Indeed upon writing u^ε as $\tilde{u}^\varepsilon + v^\varepsilon$, convergence (5.27) is deduced from (4.23), (4.24) since it is easily shown, through multiplication of the first equality of (4.7) by φv^ε ($\varphi \in \mathscr{C}_0^\infty(0,T)$) and integration by parts, that

$$(5.28)\quad \int_{\Omega}\left[\rho^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial t}\frac{\partial v^\varepsilon}{\partial t} - A^\varepsilon \operatorname{grad}\tilde{u}^\varepsilon \operatorname{grad} v^\varepsilon\right]dx \rightharpoonup 0 \quad \text{weak-}* \text{ in } L_\infty(0,T). \quad \bullet$$

6. Homogenization of the heat equation

This short section is devoted to the study of the homogenization of the heat equation. The proofs are very similar to those of Section 3 and will merely be sketched here. Attention is focused on the study of the strong convergence of the solutions in $L_2(0, T; L_2(\Omega))$ and in $C^0([0, T]; L_2(\Omega))$.

Consider the following heat equation:

$$(6.1)\quad \beta^\varepsilon \frac{\partial \tau^\varepsilon}{\partial t} - \operatorname{div}(K^\varepsilon \operatorname{grad}\tau^\varepsilon) = g \quad \text{in } \Omega\times(0,T),$$

$$(6.2)\quad \tau^\varepsilon = 0 \quad \text{on } \partial\Omega\times(0,T),$$

$$(6.3)\quad \tau^\varepsilon(0) = c^\varepsilon \quad \text{in } \Omega,$$

where the scalar function τ^ε is the unknown while the other quantities are given data of the problem. They are assumed to satisfy the following hypotheses:

$$(6.4)\quad \begin{gathered} \beta^\varepsilon \in L_\infty(\Omega),\\ \beta^\varepsilon \rightharpoonup \bar{\beta} \quad \text{weak-}* \text{ in } L_\infty(\Omega),\\ \lambda_1 \leqq \beta^\varepsilon(x) \leqq \lambda_2\ (0<\lambda_1<\lambda_2), \quad \text{almost everywhere in } \Omega, \end{gathered}$$

$$(6.5)\quad \begin{gathered} K^\varepsilon \in \mathscr{M}(\alpha, \beta; \Omega),\\ K^\varepsilon \overset{H}{\rightharpoonup} K^0, \end{gathered}$$

but K^ε is not necessarily symmetric,

$$(6.6)\quad g \in L_2(0, T; H^{-1}(\Omega)),$$

$$\begin{aligned}(6.7)\qquad & c^\varepsilon \in L_2(\Omega),\\ & c^\varepsilon \rightharpoonup \overline{c} \quad \text{weakly is } L_2(\Omega),\\ & \beta^\varepsilon c^\varepsilon \rightharpoonup \overline{\beta c} \quad \text{weakly in } L_2(\Omega),\\ & c^0 = \overline{\beta c}/\overline{\beta}.\end{aligned}$$

The equations (6.1)-(6.3) are well known to yield a unique solution; specifically, the following theorem holds true:

THEOREM 6.1. – *Under hypotheses* (6.4)-(6.7), *there exists a solution* τ^ε *of* (6.1)-(6.3) *satisfying*

$$(6.8)\qquad \tau^\varepsilon \in \mathscr{C}^0([0, T]; L_2(\Omega)) \cap L_2(0, T; H_0^1(\Omega)).$$

Uniqueness holds in the (*larger*) *class*

$$(6.9)\qquad \tau^\varepsilon \in L_\infty(0, T; L_2(\Omega)) \cap L_2(0, T; H_0^1(\Omega)).$$

Finally if

$$(6.10)\qquad \begin{aligned} d^\varepsilon(t) &= \frac{1}{2}\int_\Omega [\beta^\varepsilon(\tau^\varepsilon)^2](x, t)\,dx + \int_0^t\int_\Omega [K^\varepsilon \operatorname{grad}\tau^\varepsilon \operatorname{grad}\tau^\varepsilon](x, s)\,dx\,ds\\ D^\varepsilon &= \frac{1}{2}\int_\Omega [\beta^\varepsilon(c^\varepsilon)^2](x)\,dx,\end{aligned}$$

then

$$(6.11)\qquad d^\varepsilon(t) = D^\varepsilon + \int_0^t \langle g, \tau^\varepsilon\rangle_{H^{-1}(\Omega),\, H_0^1(\Omega)}(s)\,ds. \quad \bullet$$

Remark 6.1. – In strict parallel with Remark 3.3, consider an element τ^ε of $L_2(0, T; H_0^1(\Omega))$ which is also a weak solution of (6.1). Then $\beta^\varepsilon \tau^\varepsilon$ is shown to belong to $\mathscr{C}^0([0, T]; H^{-1}(\Omega)) \cap \mathscr{C}_s^0([0, T]; L_2(\Omega))$ and the initial condition (6.3) for τ^ε is exactly equivalent to the initial condition

$$(6.12)\qquad (\beta^\varepsilon\tau^\varepsilon)(0) = \beta^\varepsilon c^\varepsilon \quad \text{in } \Omega. \quad \bullet$$

By virtue of (6.10), (6.11) together with (6.4)-(6.7) the following estimate holds for τ^ε:

$$(6.13)\qquad \tau^\varepsilon \quad \text{is bounded in } L_\infty(0, T; L_2(\Omega)) \cap L_2(0, T; H_0^1(\Omega)).$$

Let us introduce the homogenized heat equation

$$(6.14)\qquad \begin{aligned} &\overline{\beta}\frac{\partial\tau}{\partial t} - \operatorname{div}(K^0 \operatorname{grad}\tau) = g \quad \text{in } \Omega\times(0, T),\\ &\tau = 0 \quad \text{on } \partial\Omega\times(0, T),\\ &\tau(0) = c^0 \quad \text{in } \Omega.\end{aligned}$$

The following homogenization result holds true:

THEOREM 6.2. — *The solution τ^ε of* (6.1)-(6.3) *converges to the solution τ of* (6.14) *in the following sense :*

$$(6.15)\qquad \tau^\varepsilon \rightharpoonup \tau \quad \textit{weak-}\ast \textit{ in } L_\infty(0, T; L_2(\Omega)) \cap L_2(0, T; H_0^1(\Omega)). \quad \bullet$$

The proof is very close to that of Theorem 3.2 (*cf.* [BeLP], Chapt. 2, p. 242, [Sa], Chapt. 5, Theorem 6.2). In particular the strong convergence of $\beta^{\varepsilon'}\tau^{\varepsilon'}$ to $\bar{\beta}\tau$ in $\mathscr{C}^0([0, T]; H^{-1}(\Omega))$ is used in a crucial way. It will not be reproduced here.

A further convergence property is more specific to the heat equation, namely the

THEOREM 6.3. — *In the context of Theorem* 6.2,

$$(6.16)\qquad \tau^\varepsilon \rightharpoonup \tau \quad \textit{strongly in } L_2(0, T; L_2(\Omega)). \quad \bullet$$

Proof of Theorem 6.3. — The proof of Theorem 6.3 is a direct consequence of the Lemma 6.1 below, in view of (6.13) together with the fact that

$$(6.17)\qquad \beta^\varepsilon \frac{\partial \tau^\varepsilon}{\partial t} \quad \text{in bounded in } L_2(0, T; H^{-1}(\Omega)).$$

LEMMA 6.1. — *Consider a sequence z^ε such that*

$$(6.18)\qquad \begin{aligned} &z^\varepsilon \quad \textit{is bounded in } L_2(0, T; H_0^1(\Omega)),\\ &\beta^\varepsilon \frac{\partial z^\varepsilon}{\partial t} \quad \textit{is bounded in } L_2(0, T; H^{-1}(\Omega)), \end{aligned}$$

where β^ε satisfies (6.4). *Then z^ε is relatively compact in* $L_2(0, T; L_2(\Omega))$. $\bullet$

Proof of lemma 6.1. — Extract from ε a subsequence ε' such that $z^{\varepsilon'}$ converges weakly to some $\bar{z}$ in $L_2(0, T; H_0^1(\Omega))$ while $\beta^{\varepsilon'} z^{\varepsilon'}$ converges to $\bar{\beta}\bar{z}$ weakly in $L_2(0, T; L_2(\Omega))$ and strongly in $L_2(0, T; H^{-1}(\Omega))$. Because

$$(6.19)\qquad \int_0^T\!\!\int_\Omega \beta^{\varepsilon'}(z^{\varepsilon'}-\bar{z})^2\,dx\,dt = \int_0^T \langle \beta^{\varepsilon'} z^{\varepsilon'}, z^{\varepsilon'}-\bar{z}\rangle_{H^{-1}(\Omega),\,H_0^1(\Omega)}(t)\,dt + \int_0^T\!\!\int_\Omega \beta^{\varepsilon'}\bar{z}^2\,dx\,dt - \int_0^T\!\!\int_\Omega \beta^{\varepsilon'} z^{\varepsilon'}\bar{z}\,dx\,dt,$$

and because the right hand side of (6.19) is seen to converge to zero, z^ε is relatively compact in $L_2(0, T; L_2(\Omega))$. $\bullet$

Remark 6.2. — The statement that

$$(6.20)\qquad \tau^\varepsilon \to \tau^0 \quad \text{strongly in } \mathscr{C}^0([0, T]; L_2(\Omega)),$$

is generally false, even if $\beta^\varepsilon \equiv 1$, $K^\varepsilon \equiv I$, and $g \equiv 0$ as shown by Counterexample 6.1. Note however that it is proved in the next section that

$$t^\kappa \tau^\varepsilon \to t^\kappa \tau \quad \text{strongly in } \mathscr{C}^0([0, T]; L_2(\Omega)) \tag{6.21}$$

for any strictly positive real number κ. ●

Counterexample 6.1. – Consider a sequence λ^ε of eigenvalues of the Dirichlet problem for the Laplace operator that tends to infinity, and let y^ε denote the corresponding eigenfunction, *i. e.*,

$$\begin{aligned} &-\Delta y^\varepsilon = \lambda^\varepsilon y^\varepsilon \quad \text{in } \Omega, \\ &y^\varepsilon = 0 \quad \text{on } \partial\Omega, \\ &\| y^\varepsilon \|_{L_2(\Omega)} = 1. \end{aligned} \tag{6.22}$$

Then z^ε defined by

$$z^\varepsilon(x, t) = e^{-\lambda^\varepsilon t} y^\varepsilon(x) \tag{6.23}$$

satisfies

$$\begin{aligned} &\frac{\partial z^\varepsilon}{\partial t} - \Delta z^\varepsilon = 0 \quad \text{in } \Omega \times (0, \infty), \\ &z^\varepsilon = 0 \quad \text{on } \partial\Omega \times (0, \infty), \\ &z^\varepsilon(0) = y^\varepsilon \quad \text{in } \Omega. \end{aligned} \tag{6.24}$$

It is easily seen that

$$\begin{aligned} &z^\varepsilon \rightharpoonup 0 \quad \text{weak-}* \text{ in } L_\infty(0, T; L_2(\Omega)) \cap L_2(0, T; H_0^1(\Omega)), \\ &z^\varepsilon \to 0 \quad \text{strongly in } L_2(0, T; L_2(\Omega)), \end{aligned} \tag{6.25}$$

while

$$\| z^\varepsilon \|_{C^0([0, T]; L_2(\Omega))} = \| y^\varepsilon \|_{L_2(\Omega)} = 1. \quad ● \tag{6.26}$$

7. The corrector result for the heat equation

This section is devoted to the study of the corrector for the heat equation (6.1)-(6.3). As in the case of the wave equation the solution τ^ε is split into two terms $\tilde{\tau}^\varepsilon$ and θ^ε. A corrector result is derived on the term $\tilde{\tau}^\varepsilon$. The term θ^ε is shown to be a initial layer in both $\mathscr{C}^0([0, T]; L_2(\Omega))$ and $L_2(0, T; H_0^1(\Omega))$ topologies, in contrast with the perennity and the equipartition of the energies demonstrated in the case of the wave equation. The proofs are only sketched since they closely follow those of Section 5. Only the initial layer is investigated in a detailed manner.

The bulk of the results presented here can be found in [BeLP], Chapt. 2, Section 2.11, although we believe that our approach is complementary especially because the treatment of the initial layer θ^ε is original.

The analogues $d^0(t)$ and D^0 of $d^\varepsilon(t)$ and D^ε are defined as

$$(7.1)\qquad \begin{gathered} d^0(t)=\frac{1}{2}\int_\Omega [\beta(\tau)^2](x,t)\,dx+\int_0^t\int_\Omega [K^0 \operatorname{grad}\tau \operatorname{grad}\tau](x,s)\,dx\,ds,\\ D^0=\frac{1}{2}\int_\Omega [\beta(c^0)^2](x)\,dx. \end{gathered}$$

Then, as in Theorem 6.1,

$$(7.2)\qquad d^0(t)=D^0+\int_0^t \langle g,\tau\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}(s)\,ds.$$

As in the case of the wave equation, it is possible to extract a subsequence ε' of ε such that

$$(7.3)\qquad \begin{gathered} d^{\varepsilon'}\rightharpoonup d \quad \text{weak-}* \text{ in } L_\infty(0,T),\\ D^{\varepsilon'}\to D \quad \text{in } \mathbb{R}. \end{gathered}$$

Remark 7.1. – The subsequence $d^{\varepsilon'}$ can be proved to converge strongly to d in $\mathscr{C}^0([0,T])$. ●

In general

$$(7.4)\qquad d(t)\neq d^0(t) \qquad \text{and} \qquad D\neq D^0,$$

(*cf.* Proposition 7.1 for further details).

We define $\tilde\tau^\varepsilon$ and θ^ε as the solutions of

$$(7.5)\qquad \begin{gathered} \beta^\varepsilon\frac{\partial\tilde\tau^\varepsilon}{\partial t}-\operatorname{div}(K^\varepsilon \operatorname{grad}\tilde\tau^\varepsilon)=g \quad \text{in } \Omega\times(0,T),\\ \tilde\tau^\varepsilon=0 \quad \text{on } \partial\Omega\times(0,T),\\ \tilde\tau^\varepsilon(0)=c^0 \quad \text{in } \Omega, \end{gathered}$$

$$(7.6)\qquad \begin{gathered} \beta^\varepsilon\frac{\partial\theta^\varepsilon}{\partial t}-\operatorname{div}(K^\varepsilon \operatorname{grad}\theta^\varepsilon)=0 \quad \text{in } \Omega\times(0,T),\\ \theta^\varepsilon=0 \quad \text{on } \partial\Omega\times(0,T),\\ \theta^\varepsilon(0)=c^\varepsilon-c^0 \quad \text{in } \Omega, \end{gathered}$$

and introduce the corresponding quantities

$$(7.7)\qquad \begin{gathered} \tilde d^\varepsilon(t)=\frac{1}{2}\int_\Omega [\beta^\varepsilon(\tilde\tau^\varepsilon)^2](x,t)\,dx+\int_0^t\int_\Omega [K^\varepsilon \operatorname{grad}\tilde\tau^\varepsilon \operatorname{grad}\tilde\tau^\varepsilon](x,s)\,dx\,ds,\\ \tilde D^\varepsilon=\frac{1}{2}\int_\Omega [\beta^\varepsilon(c^0)^2](x)\,dx, \end{gathered}$$

$$\delta^\varepsilon(t)=\frac{1}{2}\int_\Omega [\beta^\varepsilon(\theta^\varepsilon)^2](x, t)\,dx+\int_0^t\int_\Omega [K^\varepsilon \operatorname{grad}\theta^\varepsilon \operatorname{grad}\theta^\varepsilon](x, s)\,dx\,ds,$$

(7.8)

$$\Delta^\varepsilon=\frac{1}{2}\int_\Omega [\beta^\varepsilon(c^\varepsilon-c^0)^2](x)\,dx.$$

Then

(7.9)
$$\tilde{d}^\varepsilon(t)=\tilde{D}^\varepsilon+\int_0^t \langle g, \tilde{\tau}^\varepsilon\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}(s)\,ds \quad \text{on } [0, T],$$
$$\delta^\varepsilon(t)=\Delta^\varepsilon \quad \text{on } [0, T].$$

Obviously

(7.10)
$$\tau^\varepsilon=\tilde{\tau}^\varepsilon+\theta^\varepsilon.$$

The respective behaviours of the functions $\tilde{\tau}^\varepsilon$ and θ^ε are successively investigated and the obtained results are summed up in Theorem 7.3.

As far as $\tilde{\tau}^\varepsilon$ is concerned we obtain the

THEOREM 7.1. — *The following convergences hold true:*

(7.11)
$$\tilde{d}^\varepsilon \to d^0 \quad \textit{strongly in } \mathscr{C}^0([0, T]),$$

(7.12)
$$\tilde{D}^\varepsilon \to D^0 \quad \textit{in } \mathbb{R},$$

(7.13)
$$\tilde{\tau}^\varepsilon \rightharpoonup \tau \quad \textit{weak-* in } L_\infty(0, T; L_2(\Omega))\cap L_2(0, T; H_0^1(\Omega)),$$

where τ is the solution of the homogenized heat equation (6.14).

Furthermore, the following corrector result holds true:

(7.14)
$$\tilde{\tau}^\varepsilon \to \tau \quad \textit{strongly in } \mathscr{C}^0([0, T]; L_2(\Omega)),$$

(7.15)
$$\operatorname{grad}\tilde{\tau}^\varepsilon=Q^\varepsilon \operatorname{grad}\tau+R^\varepsilon,$$
$$R^\varepsilon \to 0 \quad \textit{strongly in } L_2(0, T; [L_1(\Omega)]^N),$$

where Q^ε is the corrector matrix associated to the sequence K^ε. ●

Remark 7.2. — As far as R^ε is concerned results of convergence in better spaces can be obtained in a manner similar to that of the wave equation. ●

Sketch of the proof of Theorem 7.1. — The proof of Theorem 7.1 closely follows that of Theorem 4.1. The quantity under investigation is

(7.16)
$$\tilde{Y}^\varepsilon(t)=\frac{1}{2}\int_\Omega [\beta^\varepsilon(\tilde{\tau}^\varepsilon-\Phi)^2](x, t)\,dx$$
$$+\int_0^t\int_\Omega [K^\varepsilon(\operatorname{grad}\tilde{\tau}^\varepsilon-Q^\varepsilon \operatorname{grad}\Phi)(\operatorname{grad}\tilde{\tau}^\varepsilon-Q^\varepsilon \operatorname{grad}\Phi)](x, s)\,dx\,ds.$$

One has to pay due attention to the lack of symmetry of K^ε in developing $\tilde{Y}^\varepsilon(t)$. ●

As far as θ^ε is concerned we obtain the

THEOREM 7.2. — *The following convergences hold true:*

$$\theta^\varepsilon \rightharpoonup 0 \quad \textit{weak-}* \textit{ in } L_\infty(0, T; L_2(\Omega)) \cap L_2(0, T; H_0^1(\Omega)). \tag{7.17}$$

The convergence in (7.17) *is strong if and only if*

$$c^\varepsilon - c^0 \to 0 \quad \textit{strongly in } L_2(\Omega). \tag{7.18}$$

Finally, for each strictly positive real number κ,

$$t^\kappa \theta^\varepsilon \to 0 \quad \text{strongly in } \mathscr{C}^0([0, T]; L_2(\Omega)) \cap L_2(0, T; H_0^1(\Omega)). \tag{7.19}$$ ●

Proof of Theorem 7.2. — Convergence (7.17) follows through direct application of Theorem 6.2. The Definition (7.8) of Δ^ε together with the second equality of (7.9) imply the equivalence between (7.18) and the strong convergence in (7.17).

The function $t^\kappa \theta^\varepsilon$ ($\kappa > 0$) belongs to $\mathscr{C}^0([0, T]; L_2(\Omega))$ as well as to $L_2(0, T; H_0^1(\Omega))$ and satisfies

$$\begin{aligned} \beta^\varepsilon \frac{\partial}{\partial t}(t^\kappa \theta^\varepsilon) - \operatorname{div}(K^\varepsilon \operatorname{grad}(t^\kappa \theta^\varepsilon)) &= \kappa t^{\kappa-1} \beta^\varepsilon \theta^\varepsilon \quad \text{in } \Omega \times (0, T), \\ t^\kappa \theta^\varepsilon &= 0 \quad \text{on } \partial\Omega \times (0, T), \\ (t^\kappa \theta^\varepsilon)(0) &= 0 \quad \text{in } \Omega. \end{aligned} \tag{7.20}$$

Since θ^ε lies in $L_\infty(0, T; L_2(\Omega))$.

$$\kappa t^{\kappa-1} \beta^\varepsilon \theta^\varepsilon \in L_1(0, T; L_2(\Omega)), \tag{7.21}$$

and the multiplication of the first equation of (7.20) by $t^\kappa \theta^\varepsilon$ is licit. It yields, for every t in $[0, T]$,

$$\begin{aligned} \frac{1}{2} t^{2\kappa} \int_\Omega [\beta^\varepsilon (\theta^\varepsilon)^2](x, t)\, dx + \int_0^t s^{2\kappa} \int_\Omega [K^\varepsilon \operatorname{grad} \theta^\varepsilon \operatorname{grad} \theta^\varepsilon](x, s)\, ds \\ = \kappa \int_0^t s^{2\kappa-1} \int_\Omega [\beta^\varepsilon (\theta^\varepsilon)^2](x, s)\, ds. \end{aligned} \tag{7.22}$$

Theorem 6.3 immediately implies the convergence to zero of the right hand side of (7.22) for any positive t and strictly positive κ. The uniform coercivity and boundedness properties of β^ε and K^ε [*cf.* (6.4), (6.5)] together with (7.22) finally lead to the statement (7.19) of strong convergence. ●

The "heat equivalent" of Theorem 4.3 is now stated.

PROPOSITION 7.1. — *The following convergences hold true*

$$d^\varepsilon - \tilde{d}^\varepsilon - \Delta^\varepsilon \to 0 \quad \textit{strongly in } \mathscr{C}^0([0, T]), \tag{7.23}$$

$$D^\varepsilon - \tilde{D}^\varepsilon - \Delta^\varepsilon \to 0 \quad \text{in } \mathbb{R}, \tag{7.24}$$

and

(7.25) $\underline{\lim}\, D^\varepsilon \geq D^0$ *while* D^ε *tends to* D^0 *if and only if* (7.18) *holds true. Results similar to* (7.25) *also true for* d^ε, $\tilde{d}^\varepsilon$. ●

Remark 7.3. – In analogy with the case of the wave equation, each of the terms entering the various "energies" can be tracked. In particular

$$\frac{1}{2}\int_\Omega \beta^\varepsilon (\tilde{\tau}^\varepsilon)^2\, dx \to \frac{1}{2}\int_\Omega \bar{\beta}(\tau)^2\, dx \quad \text{strongly in } \mathscr{C}^0([0, T]),$$

$$(7.26)\quad \frac{1}{2}\int_\Omega \beta^\varepsilon (\theta^\varepsilon)^2\, dx \rightharpoonup 0 \quad \text{weak-}* \text{ in } L_\infty(0, T) \text{ and almost everywhere on } (0, T),$$

$$\frac{1}{2}\int_\Omega \beta^\varepsilon (\tau^\varepsilon)^2\, dx \rightharpoonup \frac{1}{2}\int_\Omega \bar{\beta}(\tau)^2\, dx \quad \text{weak-}* \text{ in } L_\infty(0, T). \quad ●$$

Proof of Proposition 7.1. – Setting $c^\varepsilon = c^0 + c^\varepsilon - c^0$ in the expression (6.10) for D^ε yields

$$(7.27)\qquad D^\varepsilon = \tilde{D}^\varepsilon + \Delta^\varepsilon + F^\varepsilon,$$

where $\tilde{D}^\varepsilon$ is given by (7.7), Δ^ε by (7.8) and

$$(7.28)\qquad F^\varepsilon = \int_\Omega \beta^\varepsilon c^0 (c^\varepsilon - c^0)\, dx.$$

By virtue of the Definition (6.7) of c^0, F^ε converges to zero which proves (7.24); convergence (7.23) follows immediately from (7.24) in view of (6.11), (7.9) and Ascoli-Arzela's theorem which yields

$$(7.29)\qquad \int_0^t \langle g, \tau^\varepsilon - \tilde{\tau}^\varepsilon \rangle_{H^{-1}(\Omega),\, H_0^1(\Omega)}(s)\, ds \to 0 \quad \text{strongly in } \mathscr{C}^0([0, T]).$$

The end of the proof of Proposition 7.1 is analogous to the case of the wave equation (*cf.* the proof of Theorem 4.3 in Section 5.) ●

The results obtained for $\tilde{\tau}^\varepsilon$ and θ^ε can be assembled in a statement concerning τ^ε itself.

THEOREM 7.3. – *The solution* τ^ε *of* (6.1)-(6.3) *can be decomposed as follows:*

$$(7.30)\qquad \begin{aligned} \tau^\varepsilon &= \tau + \theta^\varepsilon + r^\varepsilon, \\ \operatorname{grad} \tau^\varepsilon &= Q^\varepsilon \operatorname{grad} \tau + \operatorname{grad} \theta^\varepsilon + R^\varepsilon, \end{aligned}$$

where Q^ε *is the corrector matrix associated to the sequence* K^ε *and* τ *is the solution of the homogenized heat equation* (6.14); θ^ε *is the solution of* (7.6) *and satisfies*

$$(7.31)\qquad \theta^\varepsilon \rightharpoonup 0 \quad \text{weak-}* \text{ in } L_\infty(0, T; L_2(\Omega)) \cap L_2(0, T; H_0^1(\Omega)),$$

$$(7.32)\qquad t^\kappa \theta^\varepsilon \rightharpoonup 0 \quad \text{strongly in } \mathscr{C}^0[0, T]; L_2(\Omega)) \cap L_2(0, T; H_0^1(\Omega)),$$

for every strictly positive real number κ, *while*

$$(7.33)\qquad \begin{aligned} r^\varepsilon &\to 0 \quad \text{strongly in } \mathscr{C}^0([0, T]; L_2(\Omega)), \\ R^\varepsilon &\to 0 \quad \text{strongly in } L_2(0, T; [L_1(\Omega)]^N). \end{aligned}$$

Convergence of R^ε *will take place in better spaces if additional regularity is met by* τ *and* Q^ε.

Furthermore the convergence in (7.31) *is strong if and only if*

(7.34) $$c^{\varepsilon} - c^{0} \to 0 \quad \textit{strongly in } L_2(\Omega). \quad \bullet$$

Remark 7.4. – The field τ^{ε} experiences an initial layer in the topologies of $\mathscr{C}^0([0, T]; L_2(\Omega))$ of and of $L_2(0, T; H_0^1(\Omega))$ [*cf.* (7.31)–(7.32)]. Such a layer vanishes [*i.e.*, κ can be taken to equal to zero and the convergence in (7.31) is strong] if and only if (7.34) occurs, in which case θ^{ε} can be dropped all together from (7.30). ●

Note added in Proof. – In the context of Remark 4.5, the two last authors have computed the measure limit h [defined as the limit of $(1/2)\,[\rho^{\varepsilon}(\partial v^{\varepsilon}/\partial t)^2 + A^{\varepsilon}\,\mathrm{grad}\, v^{\varepsilon}\,\mathrm{grad}\, v^{\varepsilon}]$, *see* (5.19)] in the case where the only oscillations come from the initial conditions, *i.e.*, whenever

$$\rho^{\varepsilon}(x) = \rho(x), \qquad A^{\varepsilon}(x) = A(x),$$

with ρ and A smooth [see forthcoming paper entitled: "Oscillations and energy densities in the wave equation", to appear in: Communications in Partial Differential Equations]. The analysis is based on a determination of the H-measure associated to $(\partial v^{\varepsilon}/\partial t, \mathrm{grad}\, v^{\varepsilon})$ and it elaborates on the work, on that same topic, of L. Tartar [T3] and of P. Gérard ["Microlocal Analysis of compactness", to appear in: Non linear partial differential equations and their applications, *Collège de France Seminar* 1989-1990, Pitman Research Notes in Mathematics, Longman, Harlow].

The same problem of computation of h in the case where the coefficients periodically oscillate, *i.e.*,

$$\rho^{\varepsilon}(x) = \rho\left(\frac{x}{\varepsilon}\right), \qquad A^{\varepsilon}(x) = A\left(\frac{x}{\varepsilon}\right),$$

[*cf.* Remark 4.6] has recently been treated by P. Gérard [« Mesures semi-classiques et ondes de Bloch », to appear].

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