RANDOM MEDIA AND COMPOSITES

Edited by Robert V. Kohn
and Graeme W. Milton
Courant Institute of Mathematical
Sciences, New York University

siam

Philadelphia

CONTENTS

LIST OF CONTRIBUTORS

James G. Berryman, University of California, Lawrence Livermore National Laboratory, Livermore, California 94550

Safla Brahim-Otsmane, Laboratoire d'Analyse Numérique, Université Paris VI et CNRS, Tour 55-65, 75252 Paris Cedex 05, France

P. M. Duxbury, Department of Physics and Astronomy and Center for Fundamental Materials Research, Michigan State University, East Lansing, Michigan 48824-1116

Kenneth Golden, Department of Mathematics, Princeton University, Princeton, New Jersey 08544

Gilles A. Francfort, Laboratoire Central des Ponts et Chaussées, 75732 Paris Cedex 15, France

Y. S. Li, Department of Physics and Astronomy and Center for Fundamental Materials Research, Michigan State University, East Lansing, Michigan 48824-1116

C. J. Lobb, Division of Applied Sciences and Department of Physics, Harvard University, Cambridge, Massachusetts 02138

Francois Murat, Laboratoire d'Analyse Numérique, Université Paris VI et CNRS, Tour 55-56, 75252 Paris Cedex 05, France

L. Poladian, Sir Frank Packer Theoretical Physics Department, University of Sydney, NSW, 2006, Australia.

Sidney Redner, Center for Polymer Studies and Department of Physics, Boston University, Boston, Massachusetts 02215

Jacob Rubinstein, Department of Mathematics, Technion-Israel Institute of Technology, 32000 Haifa, Israel

William W. Symes, Department of Mathematical Sciences, Rice University, Houston, Texas 77005

Luc Tartar, Department of Mathematics, Carnegie Mellon University, Pittsburgh, Pennsylvania 15213-3890

Sal Torquato, Department of Mechanical and Aerospace Engineering, North Carolina State University, Raleigh, North Carolina 27695

J. R. Willis, School of Mathematical Sciences, University of Bath, Bath BA2 7AY, United Kingdom

CHAPTER 2

Homogenization in Thermoelasticity

Safia Brahim-Otsmane*
Gilles A. Francfort†
François Murat*

ABSTRACT.

This paper is concerned with the homogenization of a system of partial differential equations which describes the thermomechanical behaviour of an heterogeneous medium. The unknowns are the linearized displacement u^ε and temperature increment τ^ε, where ε denotes the typical size of the heterogeneities.

After recalling results of existence and uniqueness, we pass to the limit as ε tends to zero. The limit (u, τ) of $(u^\varepsilon, \tau^\varepsilon)$ happens to be the solution of a system of the same type. The specific heat coefficient is however modified in an unexpected manner. More surprizing is the fact that the initial value for the temperature is also altered.

The changes are analyzed through a study of correctors in the last part of the paper.

* Laboratoire d'Analyse Numérique, Université Paris VI et CNRS, Tour 55-65, 75252 PARIS CEDEX 05, France.

† Laboratoire Central des Ponts et Chaussées, 58 Boulevard Lefebvre, 75732 PARIS CEDEX 15, France.

1. SETTING OF THE PROBLEM.

The following system is investigated:

$$\rho^\varepsilon \frac{\partial^2 u^\varepsilon}{\partial t^2} - div(A^\varepsilon\, grad\, u^\varepsilon - \gamma^\varepsilon \tau^\varepsilon) = f \quad \text{in} \quad \Omega \times (0,T) \tag{1.1}$$

$$\beta^\varepsilon \frac{\partial \tau^\varepsilon}{\partial t} - div(K^\varepsilon\, grad\, \tau^\varepsilon) + \gamma^\varepsilon\, grad \frac{\partial u^\varepsilon}{\partial t} = g \quad \text{in} \quad \Omega \times (0,T) \tag{1.2}$$

$$u^\varepsilon(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,T) \tag{1.3}$$

$$\tau^\varepsilon(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,T) \tag{1.4}$$

$$u^\varepsilon(x,0) = a^\varepsilon(x) \quad \text{in} \quad \Omega \tag{1.5}$$

$$\frac{\partial u^\varepsilon}{\partial t}(x,0) = b^\varepsilon(x) \quad \text{in} \quad \Omega \tag{1.6}$$

$$\tau^\varepsilon(x,0) = c^\varepsilon(x) \quad \text{in} \quad \Omega. \tag{1.7}$$

This system is a scalar version of the usually accepted equation modeling linear thermoelasticity (see [13], [4]) and consists, besides the boundary and initial conditions (1.3)-(1.7) of the coupling of a hyperbolic equation with a parabolic one. The displacement field is thus scalar valued which reduces the cumbersome notation of elasticity. Such a simplification does not alter the mathematical treatment of the problem (see [5] for the case where u^ε is a vector valued displacement field).

In this system the unknowns are the linearized displacement $u^\varepsilon(x,t)$ and temperature increment $\tau^\varepsilon(x,t)$. All other quantities are data of the problem : Ω is a bounded open subset of $\mathbb{R}^N$ with boundary $\partial\Omega$ (not necessarily smooth) ; T is a positive real number. The coefficients ρ^ε, β^ε, A^ε, K^ε and γ^ε are assumed to depend only on $x \in \Omega$, to be measurable and bounded and to satisfy

$$\begin{cases} \rho^\varepsilon \in L^\infty(\Omega) \quad , \quad \beta^\varepsilon \in L^\infty(\Omega) \\ \lambda_1 \le \rho^\varepsilon(x) \le \lambda_2 \quad , \quad \lambda_1 \le \beta^\varepsilon(x) \le \lambda_2 \quad \text{a.e.} \quad x \in \Omega \end{cases} \tag{1.8}$$

(1.9)

$$\begin{cases} A^\varepsilon \in (L^\infty(\Omega))^{N^2} \quad , \quad K^\varepsilon \in (L^\infty(\Omega))^{N^2} \\ A^\varepsilon = {}^tA^\varepsilon \quad , \quad K^\varepsilon = {}^tK^\varepsilon \\ \lambda_1|\zeta|^2 \le A^\varepsilon(x)\zeta\zeta \le \lambda_2|\zeta|^2 \, , \, \lambda_1|\zeta|^2 \le K^\varepsilon(x)\zeta\zeta \le \lambda_2|\zeta|^2 \, , \text{ a.e. } x \in \Omega \, , \, \forall \zeta \in \mathbb{R}^N \end{cases}$$

$$\begin{cases} \gamma^\varepsilon \in (L^\infty(\Omega))^N \\ |\gamma^\varepsilon(x)| \le \lambda_2 \quad \text{a.e.} \quad x \in \Omega \end{cases} \tag{1.10}$$

where $0 < \lambda_1 < \lambda_2$ are given real numbers. See [13], [4] for the me cance of these quantities. The loading f and heat source g are assı

$$f \in L^2(0,T;L^2(\Omega)) \quad , \quad g \in L^2(0,T;H^{-1}(\Omega)) \tag{1.11}$$

(more general right hand sides could be considered), and the initial are such that

$$\begin{cases} a^\varepsilon \in H_0^1(\Omega) \quad , \quad \|a^\varepsilon\|_{H_0^1(\Omega)} \le \lambda_3 \\ b^\varepsilon \in L^2(\Omega) \quad , \quad \|b^\varepsilon\|_{L^2(\Omega)} \le \lambda_3 \\ c^\varepsilon \in L^2(\Omega) \quad , \quad \|c^\varepsilon\|_{L^2(\Omega)} \le \lambda_3 \end{cases} \tag{1.12}$$

for some positive real number λ_3.

Finally ε denotes a parameter which takes its values in a seque 0. This parameter describes the typical size of the heterogeneities

In this paper we revisit results from the theses of Gilles A. Fra Brahim-Otsmane (see [2], [4], [5], [6]). The three last references are a periodic setting where ρ^ε, β^ε, A^ε, K^ε and γ^ε are given by form $\rho^\varepsilon(x) = \rho(x/\varepsilon)$ with ρ periodic. A few of the results presentec paper are new, for example the fact that K belongs to $L^\infty(\Omega)$ and correctors in spaces of time-continuous functions.

The plan is as follows : Section 2 briefly recalls existence an sults. The question of the sense of the initial conditions is empha plays an important role in the change of the initial condition for t Section 3 defines the quantities appearing in the homogenized sy the homogenization result for (1.1)-(1.7). With respect to (1.1)-(1 nized system is of the same type as the original one (which is in its feature). It exhibits however two essential differences : on the one conduction coefficient is modified ; on the other hand, the initial c temperature is completely changed. Section 4 gives the proofs of tion result, essentially by a reduction process leading to an elliptic a carefull study of the sense of the initial conditions. In Section are investigated through a study of the correctors : the solution into two parts $(\tilde{u}^\varepsilon, \tilde{\tau}^\varepsilon)$ and $(v^\varepsilon, \theta^\varepsilon)$. The first part $(\tilde{u}^\varepsilon, \tilde{\tau}^\varepsilon)$ conve limit as $(u^\varepsilon, \tau^\varepsilon)$ but with a "correct energy" which gives rise to a There is no change in the initial conditions for $(\tilde{u}^\varepsilon, \tilde{\tau}^\varepsilon)$. The seco converges weakly to $(0,0)$ and dissipates the part of the energy whi the "correct energy" of $(\tilde{u}^\varepsilon, \tilde{\tau}^\varepsilon)$. Proofs of the corrector results Section 6. They use the same ideas as for the elliptic setting and wave equations (see [3]), but many details are more delicate to ha

2. EXISTENCE AND UNIQUENESS RESULTS.

The main feature of system (1.1)-(1.7) resides in the followi mally obtained by multiplication of the equations (1.1), (1.2) by respectively, integration over Ω and integration by parts using t boundary conditions (1.3), (1.4) :

$$(2.1)\quad \begin{cases} \frac{1}{2}\frac{d}{dt}\int_\Omega \rho^\varepsilon |\frac{\partial u^\varepsilon}{\partial t}|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_\Omega A^\varepsilon\, grad\, u^\varepsilon\, grad\, u^\varepsilon\,dx \\ -\int_\Omega \gamma^\varepsilon \tau^\varepsilon\, grad\,\frac{\partial u^\varepsilon}{\partial t}\,dx = \int_\Omega f\,\frac{\partial u^\varepsilon}{\partial t}\,dx \end{cases}$$

$$(2.2)\quad \begin{cases} \frac{1}{2}\frac{d}{dt}\int_\Omega \beta^\varepsilon|\tau^\varepsilon|^2\,dx + \int_\Omega K^\varepsilon\, grad\,\tau^\varepsilon\, grad\,\tau^\varepsilon\,dx \\ +\int_\Omega \gamma^\varepsilon\, grad\,\frac{\partial u^\varepsilon}{\partial t}\,\tau^\varepsilon\,dx = < g\,,\tau^\varepsilon > \end{cases}$$

where $<,>$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Addition of these two equations miraculously eliminates the term $\int_\Omega \gamma^\varepsilon\tau^\varepsilon grad(\partial u^\varepsilon/\partial t)\,dx$ and integration in time yields

$$(2.3)\quad \begin{cases} \frac{1}{2}\int_\Omega [\rho^\varepsilon\,|\frac{\partial u^\varepsilon}{\partial t}|^2 + A^\varepsilon\, grad\, u^\varepsilon\, grad\, u^\varepsilon + \beta^\varepsilon\,|\tau^\varepsilon|^2](x,t)dx \\ +\int_0^t\int_\Omega K^\varepsilon\, grad\,\tau^\varepsilon\, grad\,\tau^\varepsilon\,dx\,ds \\ = \frac{1}{2}\int_\Omega [\rho^\varepsilon\,|b^\varepsilon|^2 + A^\varepsilon\, grad\, a^\varepsilon\, grad\, a^\varepsilon + \beta^\varepsilon\,|c^\varepsilon|^2]\,dx \\ +\int_0^t\int_\Omega f\frac{\partial u^\varepsilon}{\partial t}\,dx\,ds + \int_0^t < g\,,\tau^\varepsilon > ds. \end{cases}$$

In view of the coercivity and boundedness assumptions (1.8)-(1.12), (2.3) implies that

$$(2.4)\quad \begin{cases} u^\varepsilon \text{ is bounded in } L^\infty(0,T;H_0^1(\Omega)) \\ \frac{\partial u^\varepsilon}{\partial t} \text{ is bounded in } L^\infty(0,T;L^2(\Omega)) \\ \tau^\varepsilon \text{ is bounded in } L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega)) \end{cases}$$

independently of ε.

This formal remark is at the root of the following existence and uniqueness result :

THEOREM 2.1. *Let the assumptions (1.8)-(1.12) be satisfied. Then for any fixed ε there exists a solution $(u^\varepsilon\,,\tau^\varepsilon)$*

$$(2.5)\quad \begin{cases} u^\varepsilon \in C^0(0,T;H_0^1(\Omega)) \\ \frac{\partial u^\varepsilon}{\partial t} \in C^0(0,T;L^2(\Omega)) \\ \tau^\varepsilon \in C^0(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega)) \end{cases}$$

of the system (1.1)-(1.7). Uniqueness holds in the (larger) class define The solution satisfies equality (2.3) for any t in $[0,T]$. Moreover the esti holds true independently of ε.

REMARK 2.1. Note that the regularity (2.5) (and even the "L^∞ in time" defined by (2.4)) is sufficient for each term in (1.1), (1.2) to be meani example $\gamma^\varepsilon grad(\partial u^\varepsilon/\partial t)$ can be written as $\partial/\partial t(\gamma^\varepsilon grad\, u^\varepsilon)$ which has tional meaning since γ^ε belongs to $(L^\infty(\Omega))^N$ and $grad\, u^\varepsilon$ to $L^\infty(0,T;($ the same is true for the other terms.

On the other hand, the sense of the term $\int_\Omega \gamma^\varepsilon\tau^\varepsilon grad(\partial u^\varepsilon/\partial t)\,dx$ i in the above formal computation (2.1), (2.2). Nevertheless equality (2 every term has a meaning) holds true for the solution $(u^\varepsilon\,,\tau^\varepsilon)$ of the sy

The proof of Theorem 2.1 can be considered as routine work, ever times rather delicate : see [4], [6], [9] for a proof using semigroup theory case one has to carefully examine the meaning of the equations, the the domain of the operators, and the question of uniqueness) and [2] f by Galerkin's method (in which case one has to cautiously track the se initial conditions, the continuous character of u^ε, $\partial u^\varepsilon/\partial t$ and τ^ε, and th of uniqueness).

When the regularity (2.5) is met by $(u^\varepsilon\,,\partial u^\varepsilon/\partial t\,,\tau^\varepsilon)$ the sense of conditions (1.5), (1.6), (1.7) is clear. We shall see in the following Propo these initial conditions are equally meaningful under the weaker hypot whenever $(u^\varepsilon\,,\tau^\varepsilon)$ is a solution of (1.1), (1.2). In this statement $C_s^0(0,T$ space of functions v such that the real function $t \to < h, v(t) >_{X',X}$ is on $[0,T]$ for any $h\in X'$. This Proposition will be used in Section 4.

PROPOSITION 2.1. *Consider $(u^\varepsilon\,,\tau^\varepsilon)$ which belongs to the class "L^∞ in fined by (2.4) and is a solution of (1.1), (1.2) in a distributional sense. any fixed ε*

$$(2.6)\quad \begin{cases} u^\varepsilon \in C_s^0(0,T;H_0^1(\Omega)) \\ \frac{\partial u^\varepsilon}{\partial t} \in C_s^0(0,T;L^2(\Omega)) \\ \tau^\varepsilon \in C_s^0(0,T;L^2(\Omega)) \end{cases}$$

which lends a meaning to the initial conditions (1.5), (1.6), (1.7). More entropy s^ε is defined as

$$(2.7)\quad s^\varepsilon = \beta^\varepsilon\tau^\varepsilon + \gamma^\varepsilon\, grad\, u^\varepsilon$$

then

$$(2.8)\quad \begin{cases} u^\varepsilon \in C^0(0,T;L^2(\Omega)) \\ \rho^\varepsilon\frac{\partial u^\varepsilon}{\partial t} \in C^0(0,T;H^{-1}(\Omega)) \\ s^\varepsilon \in C^0(0,T;H^{-1}(\Omega)). \end{cases}$$

Finally the initial conditions

(2.9) $$u^\varepsilon(x,0) = a^\varepsilon(x) \quad in \quad \Omega$$

(2.10) $$(\rho^\varepsilon \frac{\partial u^\varepsilon}{\partial t})(x,0) = \rho^\varepsilon(x)\, b^\varepsilon(x) \quad in \quad \Omega$$

(2.11) $$s^\varepsilon(x,0) = \beta^\varepsilon(x)\, c^\varepsilon(x) + \gamma^\varepsilon(x)\, grad\, a^\varepsilon(x) \quad in \quad \Omega$$

are equivalent to the initial conditions (1.5), (1.6), (1.7). □

PROOF OF PROPOSITION 2.1. Recall that if X and Y are two Banach spaces such that X is reflexive and continuously imbedded in Y, then

$$L^\infty(0,T;X) \cap C^0(0,T;Y) \subset C^0_s(0,T;X)$$

(see e.g. [8] lemme 8.1 p. 297). In view of (2.4), u^ε belongs to $L^\infty(0,T;H^1_0(\Omega))$ with $\partial u^\varepsilon/\partial t$ in $L^\infty(0,T;L^2(\Omega))$, and thus it belongs to $C^0_s(0,T;H^1_0(\Omega))$.

By virtue of (2.4) together with equation (1.1), $\rho^\varepsilon(\partial u^\varepsilon/\partial t)$ belongs to $L^\infty(0,T;L^2(\Omega))$ with $\partial/\partial t\,(\rho^\varepsilon(\partial u^\varepsilon/\partial t))$ in $L^2(0,T;H^{-1}(\Omega))$. Thus $\rho^\varepsilon(\partial u^\varepsilon/\partial t)$ belongs to $C^0(0,T;H^{-1}(\Omega))$ and to $C^0_s(0,T;L^2(\Omega))$, which implies that

$$t \to \int_\Omega h \frac{\partial u^\varepsilon}{\partial t}\, dx = \int_\Omega (\frac{h}{\rho^\varepsilon})(\rho^\varepsilon \frac{\partial u^\varepsilon}{\partial t})\, dx \quad \text{is continuous on } [0,T]$$

for any $h \in L^2(\Omega)$. Thus $\partial u^\varepsilon/\partial t$ belongs to $C^0_s(0,T;L^2(\Omega))$, the initial condition on $\rho^\varepsilon(\partial u^\varepsilon/\partial t)$ is meaningful, (2.10) is satisfied and is exactly (and not only formally) equivalent to (1.6) whenever $(u^\varepsilon, \tau^\varepsilon)$ belongs to the class "L^∞ in time" defined by (2.4). Note that this equivalence is straightforward when the continuity (2.5) holds true.

Finally let us consider the entropy s^ε defined by (2.7). From (2.4) and equation (1.2) we deduce that s^ε belongs to $L^\infty(0,T;L^2(\Omega))$ with $\partial s^\varepsilon/\partial t$ in $L^2(0,T;H^{-1}(\Omega))$. Thus s^ε belongs to $C^0(0,T;H^{-1}(\Omega))$, which gives a sense to the initial condition on s^ε. On the other hand s^ε belongs to $C^0_s(0,T;L^2(\Omega))$. Since u^ε lies in $C^0_s(0,T;H^1_0(\Omega))$ the function

(2.12) $$\begin{cases} t \to \int_\Omega h\,\beta^\varepsilon\,\tau^\varepsilon\, dx = \int_\Omega h\, s^\varepsilon\, dx - \int_\Omega h\,\gamma^\varepsilon\, grad\, u^\varepsilon\, dx \\ \text{is continuous on } [0,T] \text{ for any } h \in L^2(\Omega). \end{cases}$$

Taking $h' = h/\beta^\varepsilon$ for test function in (2.12) proves that τ^ε belongs to $C^0_s(0,T;L^2(\Omega))$. Finally (2.12) also proves that (2.11) holds true and that (1.5), (1.7) is equivalent to (2.9), (2.11) whenever $(u^\varepsilon, \tau^\varepsilon)$ belongs to the class "L^∞ in time" defined by (2.4). Note that this equivalence is straightforward when the continuity (2.5) holds true. □

3. STATEMENT OF THE HOMOGENIZATION RESULT.

Consider a sequence of data which meet the assumptions (1.8)-(1.12). In this setting, Theorem 2.1 asserts that the unique solution $(u^\varepsilon, \tau^\varepsilon)$ of the thermoelasticity problem (1.1)-(1.7) is bounded as ε tend to 0 (see (2.4)). Our goal in this Section is to pass to the limit in ε.

THEOREM 3.1. *If the assumptions (1.8)-(1.12) are satisfied, there exist* *quence $(u^\varepsilon, \tau^\varepsilon)$ (still indexed by the same superscript ε), which converge in the following sense*

(3.0) $$\begin{cases} u^\varepsilon \rightharpoonup u \text{ weak} * \text{ in } L^\infty(0,T;H^1_0(\Omega)) \\ \frac{\partial u^\varepsilon}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \text{ weak} * \text{ in } L^\infty(0,T;L^2(\Omega)) \\ \tau^\varepsilon \rightharpoonup \tau \text{ weak} * \text{ in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega)). \end{cases}$$

In (3.0) (u, τ) is the unique solution (given by Theorem 2.1) of the follow *genized thermoelasticity system :*

(3.1) $$\bar\rho \frac{\partial^2 u}{\partial t^2} - div(A^0\, grad\, u - \gamma^0 \tau) = f \quad in \quad \Omega \times (0,T)$$

(3.2) $$(\bar\beta + \kappa)\frac{\partial \tau}{\partial t} - div(K^0\, grad\, \tau) + \gamma^0\, grad \frac{\partial u}{\partial t} = g \quad in \quad \Omega \times (0,T)$$

(3.3) $$u(x,t) = 0 \quad on \quad \partial\Omega \times (0,T)$$

(3.4) $$\tau(x,t) = 0 \quad on \quad \partial\Omega \times (0,T)$$

(3.5) $$u(x,0) = a^0(x) \quad in \quad \Omega$$

(3.6) $$\frac{\partial u}{\partial t}(x,0) = b^0(x) \quad in \quad \Omega$$

(3.7) $$\tau(x,0) = c^0(x) \quad in \quad \Omega$$

where the definitions of $\bar\rho$, $\bar\beta$, A^0, K^0, κ, a^0, b^0 and c^0 are given below.

In view of the bound (1.8), we are at liberty to extract a subsequen that

(3.8) $$\begin{cases} \rho^{\varepsilon'} \rightharpoonup \bar\rho\,,\ \beta^{\varepsilon'} \rightharpoonup \bar\beta \text{ weak} * \text{ in } L^\infty(\Omega) \\ \text{with } \lambda_1 \le \bar\rho \le \lambda_2\,,\ \lambda_1 \le \bar\beta \le \lambda_2 \text{ a.e. in } \Omega\,; \end{cases}$$

this defines $\bar\rho$ and $\bar\beta$.

The matrix A^0 (and in a similar manner the matrix K^0) is defin homogenized matrix obtained from a subsequence of A^ε. It is now w see [11], [16], [17] and also [1], [14] in a periodic setting) that one can subsequence ε' such that for a certain matrix A^0 which satisfies the same

(1.9) as A^ε, the following property holds true : for any sequence h^ε converging strongly to h in $H^{-1}(\Omega)$, the solutions $v^{\varepsilon'}$ of the elliptic problem

(3.9a) $$-div(A^{\varepsilon'}\, grad\, v^{\varepsilon'}) = h^{\varepsilon'} \text{ in } \Omega\,,\ v^\varepsilon \in H_0^1(\Omega).$$

satisfy

(3.9b) $$A^{\varepsilon'}\, grad\, v^{\varepsilon'} \rightharpoonup A^0\, grad\, v \text{ weakly in } (L^2(\Omega))^N$$

for v the unique solution of

(3.9c) $$-div(A^0\, grad\, v) = h \text{ in } \Omega\,,\ v \in H_0^1(\Omega).$$

If (3.9) holds true for any strongly converging sequence $h^{\varepsilon'}$ in $H^{-1}(\Omega)$, the sequence $A^{\varepsilon'}$ is said to $H-converge$ to A^0. The matrix A^0 represents in essence the effective medium associated to the heterogeneous medium A^ε.

The matrices A^0 and K^0 are thus defined.

For a sequence $A^{\varepsilon'}$ which H-converges to A^0, there exist N functions $w_i^{\varepsilon'}$ in $H^1(\Omega)$ $(1 \le i \le N)$ such that

(3.10a) $$\begin{cases} -\,div(A^{\varepsilon'}\, grad\, w_i^{\varepsilon'}) = -div(A^0\, e_i) \text{ in } \Omega \\ w_i^{\varepsilon'} \rightharpoonup (e_i\,,\, x) = x_i \text{ weakly in } H^1(\Omega) \end{cases}$$

where e_i denotes the i-th vector of some orthonormal basis of $\mathbb{R}^N$ (to define such w_i^ε it is for example sufficient to impose a convenient boundary condition on $\partial\Omega$, as in the proof of (3.13) below). We define the matrix $P^{\varepsilon'} \in (L^2(\Omega))^{N^2}$ by

(3.10b) $$P^{\varepsilon'}\, e_i = grad\, w_i^{\varepsilon'} \text{ in } \Omega.$$

If $v^{\varepsilon'}$ is the solution of (3.9) then ([11], [17])

(3.10c) $$\begin{cases} grad\, v^{\varepsilon'} = P^{\varepsilon'}\, grad\, v + r^{\varepsilon'} \\ \text{with } r^{\varepsilon'} \to 0 \text{ strongly in } (L^1(\Omega))^N. \end{cases}$$

This result further specifies the structure of the solution of (3.9a). The matrix $P^{\varepsilon'}$ is called the *corrector matrix associated to* $A^{\varepsilon'}$.

From the matrix $P^{\varepsilon'}$ we can now define γ^0 by extracting a subsequence (still indexed by ε') such that

(3.11) $${}^tP^{\varepsilon'}\,\gamma^{\varepsilon'} \rightharpoonup \gamma^0 \text{ weakly in } (L^2(\Omega))^N.$$

Note that ${}^tP^{\varepsilon'}\,\gamma^{\varepsilon'}$ is bounded in $(L^2(\Omega))^N$ in view of (3.10a) and (1.10). It can be proved ([11],[17] ; in a similar spirit, see the proof of (3.15) below) that γ^0 inherits from the L^∞ bound on γ^ε the regularity property

(3.12) $$\gamma^0 \in (L^\infty(\Omega))^N.$$

Let $y^{\varepsilon'}$ be any function such that

(3.13) $$\begin{cases} -\,div(A^{\varepsilon'}\, grad\, y^{\varepsilon'} - (\gamma^{\varepsilon'} - \gamma^0)) = 0 \text{ in } \Omega \\ y^{\varepsilon'} \rightharpoonup 0 \text{ weakly in } H_0^1(\Omega) \end{cases}$$

(we will prove at the end of this Section that such a sequence exists) by extracting a new subsequence (still indexed by ε') such that

(3.14) $$\gamma^{\varepsilon'}\, grad\, y^{\varepsilon'} \rightharpoonup \kappa \text{ weakly in } L^2(\Omega).$$

It will be proved at the end of this Section that

(3.15) $$\kappa \in L^\infty(\Omega) \quad \text{and} \quad \kappa \ge 0 \text{ a.e. in } \Omega.$$

The initial conditions a^0, b^0, and c^0 have yet to be defined. In we can extract a subsequence (still indexed by ε') such that

(3.16) $$a^{\varepsilon'} \rightharpoonup a^0 \text{ weakly in } H_0^1(\Omega)$$

which defines a^0 ; we can also extract a subsequence such that

(3.17) $$\rho^{\varepsilon'}\, b^{\varepsilon'} \rightharpoonup \overline{\rho b} \text{ weakly in } L^2(\Omega).$$

From $\overline{\rho b}$ and $\overline{\rho}$ defined in (3.8) we define b^0 by

(3.18) $$b^0 = \overline{\rho b}\,/\,\overline{\rho} \in L^2(\Omega)$$

(note that $1/\overline{\rho} \in L^\infty(\Omega)$).

The initial value c^0 of the temperature field τ is obtained with t initial value $s^\varepsilon(x,0)$ of the entropy (see (2.11)). Defining d^ε as

(3.19a) $$d^\varepsilon = \beta^\varepsilon\, c^\varepsilon + \gamma^\varepsilon\, grad\, a^\varepsilon$$

we extract a subsequence (still indexed by ε') such that

(3.19b) $$d^{\varepsilon'} \rightharpoonup d^0 \text{ weakly in } L^2(\Omega).$$

Then c^0 is defined by the formula

(3.20) $$(\overline{\beta} + \kappa)\, c^0 + \gamma^0\, grad\, a^0 = d^0$$

which in view of the previous results (and particularly of (3.15) $(\overline{\beta} + \kappa)^{-1} \in L^\infty(\Omega)$) implies that

(3.21) $$c^0 \in L^2(\Omega).$$

All quantities which appear in the homogenized system (3.1) defined.

REMARK 3.1. In a periodic setting where ρ^ε, β^ε, A^ε, K^ε and γ^ε are given by formulas like $\rho^\varepsilon(x) = \rho(x/\varepsilon)$ with ρ periodic, the quantities $\bar\rho$, $\bar\beta$, A^0, K^0, γ^0 and κ are constants (scalars, matrices or vectors) and are given by "explicit" formulas involving the solutions of partial differential equations with periodic solutions : see [4],[6]. □

REMARK 3.2. In the special case where the initial conditions of the system to be homogenized are fixed, namely

$$a^\varepsilon = a\,;\ b^\varepsilon = b\,,\ c^\varepsilon = c \text{ with } a \in H_0^1(\Omega)\,,\ b \in L^2(\Omega)\,,\ c \in L^2(\Omega)$$

it is easily seen that

$$a^0 = a \quad , \quad b^0 = b.$$

Thus there is no change in the initial conditions for u and $\partial u/\partial t$.

A completely different phenomenon occurs for the initial condition on τ : indeed, if for some subsequence still indexed by ε' one has

$$\gamma^{\varepsilon'} \rightharpoonup \bar\gamma \text{ weakly in } (L^2(\Omega))^N,$$

it is easily seen from (3.19), (3.20) that the initial condition c^0 on τ is given by

$$c^0 = \frac{1}{\bar\beta + \kappa}\,[\bar\beta\, c + (\bar\gamma - \gamma^0)\, grad\, a] \tag{3.22}$$

which is in general different from c. □

REMARK 3.3. The homogenized system (3.1)-(3.7) ressembles the initial system (1.1)-(1.7). Two differences have to be pointed out :

(i) the presence of κ in the coefficient of $\partial\tau/\partial t$; we will see (Remark 4.1 below) that the presence of such a coefficient is a consequence of the hidden presence of τ in the term $\gamma^\varepsilon\, grad\, u^\varepsilon$.

(ii) the change in the initial condition for τ (see Remark 3.2 above) ; we will see in Section 4 that, as far as the initial conditions are concerned, the entropy s^ε (and not the temperature τ^ε) is the quantity to investigate.

The most surprising feature of the homogenization result does not reside however in the above described phenomena but rather in the very fact that the homogenized system is so close to the original one. Such is not generally the case when dealing with systems with time-dependent effects, see e.g. [7], [10], [14], [19]. □

PROOF OF (3.13). We prove that there exists a sequence $y^{\varepsilon'}$ which satisfies (3.13). Assume that $y^{\varepsilon'}$ satisfies

$$-div(A^{\varepsilon'}\, grad\, y^{\varepsilon'} - (\gamma^{\varepsilon'} - \gamma^0)) = 0 \tag{3.23}$$

and a convenient boundary condition implying that $y^{\varepsilon'}$ is bounded in $H^1(\Omega)$. Define

$$\zeta^{\varepsilon'} = A^{\varepsilon'}\, grad\, y^{\varepsilon'} - (\gamma^{\varepsilon'} - \gamma^0). \tag{3.24}$$

Extracting a subsequence (indexed by ε") we may assume that

$$\begin{cases} grad\, y^{\varepsilon''} \rightharpoonup grad\, y \text{ weakly in } (L^2(\Omega))^N \\ \zeta^{\varepsilon''} \rightharpoonup \zeta \text{ weakly in } (L^2(\Omega))^N. \end{cases} \tag{3.25}$$

Since $\zeta^{\varepsilon''}$ is divergence free, application of the div-curl lemma (
$(\zeta^{\varepsilon''}, grad\, w_i^{\varepsilon''})$ (see (3.10)) yields

$$(A^{\varepsilon''}\, grad\, y^{\varepsilon''} - (\gamma^{\varepsilon''} - \gamma^0))\, grad\, w_i^{\varepsilon''} \rightharpoonup \zeta e_i \text{ in } \mathcal{D}'(\Omega$$

which in view of the definition (3.11) of γ^0 implies that

$$A^{\varepsilon''}\, grad\, y^{\varepsilon''}\, grad\, w_i^{\varepsilon''} \rightharpoonup \zeta e_i \text{ in } \mathcal{D}'(\Omega).$$

A new application of the div-curl lemma to $(A^{\varepsilon''}\, grad\, w_i^{\varepsilon''}$,
$A^0 e_i\, grad\, y = \zeta\, e_i$, i.e. $\zeta = A^0\, grad\, y$. Since $-div\, \zeta = 0$ we obtain
0, which implies that $y = 0$, thus $\zeta = 0$ provided that the bound
$y^{\varepsilon'}$ are chosen in a manner such that the trace of y is zero on $\partial\Omega$.
whenever $y^{\varepsilon'} = 0$ on $\partial\Omega$, but other choices could be more conven
$y^{\varepsilon'}$ periodic when $A^\varepsilon(x) = A(x/\varepsilon)$ with A periodic.

Since, under these assumptions, $y = 0$ and $\zeta = 0$, the converg
place for the whole sequence ε' ; the existence of $y^{\varepsilon'}$ satisfying (3.1

PROOF OF (3.15). A new application of the div-curl lemma to $(\zeta^{\varepsilon'}$
$\zeta^{\varepsilon'}$ is defined by (3.24) yields

$$(A^{\varepsilon'}\, grad\, y^{\varepsilon'} - (\gamma^{\varepsilon'} - \gamma^0))\, grad\, y^{\varepsilon'} \rightharpoonup 0 \text{ in } \mathcal{D}'(\Omega)$$

which in view of the definition (3.14) of κ implies that

$$A^{\varepsilon'}\, grad\, y^{\varepsilon'}\, grad\, y^{\varepsilon'} \rightharpoonup \kappa \text{ in } \mathcal{D}'(\Omega) \tag{3.26}$$

since $grad\, y^{\varepsilon'} \rightharpoonup 0$.

This immediately implies that $\kappa \geq 0$ in the sense of measures
everywhere on Ω because κ belongs to $L^2(\Omega)$.

Let us finally prove that κ belongs to $L^\infty(\Omega)$. For any $\varphi \in \mathcal{I}$

$$\Big|\int_\Omega \gamma^{\varepsilon'}\, grad\, y^{\varepsilon'}\, \varphi^2\, dx\Big| \leq \|\gamma^{\varepsilon'}\varphi\|_{(L^2(\Omega))^N}\, \|grad\, y^{\varepsilon'}\, \varphi\|_{(L^2($$

$$\leq \|\gamma^{\varepsilon'}\|_{(L^\infty(\Omega))^N}\, \|\varphi\|_{L^2(\Omega)}\, \frac{1}{\lambda_1}\Big(\int_\Omega A^{\varepsilon'}\, grad\, y^{\varepsilon'}\, grad\, y^{\varepsilon'}\, \varphi$$

Using the bound (1.10) on γ^ε and passing to the limit with the he
yields

$$\Big|\int_\Omega \kappa\varphi^2\, dx\Big| \leq \frac{\lambda_2}{\lambda_1}\|\varphi\|_{L^2(\Omega)}\Big(\int_\Omega \kappa\varphi^2\, dx\Big)^{1/2},$$

i.e.

$$\int_\Omega \kappa \varphi^2\,dx \le \left(\frac{\lambda_2}{\lambda_1}\right)^2 \int_\Omega \varphi^2\,dx \qquad \forall \varphi \in \mathcal{D}(\Omega).$$

Since κ is non negative this implies that κ belongs to $L^\infty(\Omega)$ with $\|\sigma\|_{L^\infty(\Omega)} \le (\lambda_2/\lambda_1)^2$. Thus (3.15) is proved. □

4. PROOF OF THE HOMOGENIZATION RESULT.

We have defined in Section 3 all the quantities which appear in the homogenized system (3.1)-(3.7). Let ε' denote a subsequence such that $A^{\varepsilon'}$ and $K^{\varepsilon'}$ H-converge (see (3.9)) to A^0 and K^0 and such that all the convergences (3.8), (3.11), (3.14), (3.16), (3.17) and (3.19) take place. The statement of Theorem 3.1 is now proved to hold true for this subsequence. The proof consists in extracting a new subsequence $(u^{\varepsilon''}, \tau^{\varepsilon''})$ which converges weakly to some (u, τ) in the topology given by (3.0) which will be shown to be a solution of (3.1)-(3.7). Thus (u, τ) is uniquely determined, and the whole sequence $(u^{\varepsilon'}, \tau^{\varepsilon'})$ converges to (u, τ).

For the sake of notational simplicity the subsequence ε' is from now onward denoted by ε.

4.1. Passing to the limit in the equations.

In order to pass to the limit in the equations (1.1) and (1.2) we transform these evolution equations into elliptic equations, which could be performed with the help of the Laplace transform (see [4], [6]). Our preference goes here to the introduction of the following auxiliary functions defined for a given $\varphi \in \mathcal{D}(0,T)$ by :

$$(4.1)\qquad \begin{cases} \hat{u}^\varepsilon(x) = \displaystyle\int_0^T u^\varepsilon(x,t)\,\varphi(t)\,dt \\ \hat{\tau}^\varepsilon(x) = \displaystyle\int_0^T \tau^\varepsilon(x,t)\,\varphi(t)\,dt. \end{cases}$$

In view of (3.0) we obtain

$$(4.2)\qquad \begin{cases} \hat{u}^\varepsilon \rightharpoonup \hat{u} = \displaystyle\int_0^T u(\cdot,t)\,\varphi(t)\,dt \text{ weakly in } H_0^1(\Omega) \text{ and strongly in } L^2(\Omega) \\ \hat{\tau}^\varepsilon \rightharpoonup \hat{\tau} = \displaystyle\int_0^T \tau(\cdot,t)\,\varphi(t)\,dt \text{ weakly in } H_0^1(\Omega) \text{ and strongly in } L^2(\Omega). \end{cases}$$

On the other hand it is deduced from (1.1) that

$$(4.3)\qquad -div(A^\varepsilon\, grad\, \hat{u}^\varepsilon - \gamma^\varepsilon \hat{\tau}^\varepsilon) = \int_0^T f\,\varphi\,dt - \rho^\varepsilon \int_0^T u^\varepsilon \frac{\partial^2 \varphi}{\partial t^2}\,dt.$$

Define $\xi^\varepsilon = A^\varepsilon\, grad\hat{u}^\varepsilon - \gamma^\varepsilon \hat{\tau}^\varepsilon$ and extract a subsequence such that

$$\xi^\varepsilon \rightharpoonup \xi \text{ weakly in } (L^2(\Omega))^N.$$

Since the right hand side of (4.3) is compact in $H^{-1}(\Omega)$, application of the div-curl lemma ([12], [18]) to $(\xi^\varepsilon, grad\, w_i^\varepsilon)$ (see (3.10) for the definition of w_i^ε) yields

$$(4.4a)\qquad (A^\varepsilon\, grad\,\hat{u}^\varepsilon - \gamma^\varepsilon \hat{\tau}^\varepsilon) grad\, w_i^\varepsilon \rightharpoonup \xi\, e_i \text{ in } \mathcal{D}'(\Omega).$$

Another application of the div-curl lemma to $(A\, grad\, w_i^\varepsilon, grad\,\hat{u}^\varepsilon)$ yields

$$(4.4b)\qquad A^\varepsilon\, grad\,\hat{u}^\varepsilon\, grad\, w_i^\varepsilon = A^\varepsilon\, grad\, w_i^\varepsilon\, grad\,\hat{u}^\varepsilon \rightharpoonup A^0\, e_i\, grad\,\hat{u} \text{ in } \mathcal{D}'(\Omega)$$

while the definition (3.11) of γ^0 and the strong $L^2(\Omega)$ convergence of $\hat{\tau}^\varepsilon$ imply that

$$(4.4c)\qquad \gamma^\varepsilon \hat{\tau}^\varepsilon\, grad\, w_i^\varepsilon = {}^tP^\varepsilon \gamma^\varepsilon e_i \hat{\tau}^\varepsilon \rightharpoonup \gamma^0 e_i \hat{\tau} \text{ in } \mathcal{D}'(\Omega).$$

Thus $\xi = A^0\, grad\,\hat{u} - \gamma^0 \hat{\tau}$. Passing to the limit in (4.3) allows to recover (3.1) in the distributional sense since $\varphi \in \mathcal{D}(0,T)$ is arbitrary.

Passing to the limit in (1.2) is more delicate. Through the above described procedure (1.2) becomes

$$(4.5)\qquad \begin{aligned} -div(K^\varepsilon\, grad\,\hat{\tau}^\varepsilon) = &\int_0^T g\,\varphi\,dt + \beta^\varepsilon \int_0^T \tau^\varepsilon \frac{\partial \varphi}{\partial t}\,dt \\ &+ \gamma^\varepsilon\, grad\Big(\int_0^T u^\varepsilon \frac{\partial \varphi}{\partial t}\,dt\Big). \end{aligned}$$

Let us first study the last term of the right hand side of (4.5). Set

$$\check{u}^\varepsilon = \int_0^T u^\varepsilon \frac{\partial \varphi}{\partial t}\,dt \quad , \quad \check{\tau}^\varepsilon = \int_0^T \tau^\varepsilon \frac{\partial \varphi}{\partial t}\,dt.$$

Then $(\check{u}^\varepsilon, \check{\tau}^\varepsilon)$ converges weakly in $(H_0^1(\Omega))^2$ to $(\check{u}, \check{\tau})$ and satisfies

$$(4.6)\qquad -div(A^\varepsilon\, grad\,\check{u}^\varepsilon - \gamma^\varepsilon \check{\tau}^\varepsilon) = \int_0^T f \frac{\partial \varphi}{\partial t}\,dt - \rho^\varepsilon \int_0^T u^\varepsilon \frac{\partial^3 \varphi}{\partial t^3}\,dt.$$

Further

$$(4.7)\qquad \begin{aligned} \gamma^\varepsilon\, grad\,\check{u}^\varepsilon = &(\gamma^\varepsilon - \gamma^0 - A^\varepsilon\, grad\, y^\varepsilon)\, grad\,\check{u}^\varepsilon \\ &+ (A^\varepsilon\, grad\,\check{u}^\varepsilon - \gamma^\varepsilon \check{\tau}^\varepsilon)\, grad\, y^\varepsilon \\ &+ \gamma^0\, grad\,\check{u}^\varepsilon + \gamma^\varepsilon\, grad\, y^\varepsilon \check{\tau}^\varepsilon. \end{aligned}$$

Application of the div-curl lemma implies that the first two terms converge to 0 in $\mathcal{D}'(\Omega)$; definition (3.14) of κ together with the boundedness of $\gamma^\varepsilon\, grad\,\check{u}^\varepsilon$ in $L^2(\Omega)$ yield

$$(4.8)\qquad \gamma^\varepsilon\, grad\,\check{u}^\varepsilon \rightharpoonup \gamma^0\, grad\,\check{u} + \kappa\,\check{\tau} \text{ weakly in } L^2(\Omega).$$

It is now easy to pass to the limit in (4.5) using the definition (3.9) of H-convergence. Since φ is arbitrary in $\mathcal{D}(0,T)$, (3.2) is recovered.

REMARK 4.1. The convergence (4.8) of $\gamma^\varepsilon \, grad\, \check{u}^\varepsilon$ to $\gamma^0 \, grad\, \check{u} + \kappa \check{\tau}$ is the source of the presence of κ in the coefficient of $\partial\tau/\partial t$ in (3.2). This phenomenon is better explained by application to equation (4.6) of the following corrector result, which proves that the displacement $grad\, \check{u}^\varepsilon$ is the sum of two parts, carrying $grad\, \check{u}$ and $\check{\tau}$ respectively. □

PROPOSITION 4.1. *Let v^ε be the solution of*

$$\begin{cases} - div(A^\varepsilon \, grad\, v^\varepsilon - \gamma^\varepsilon \theta^\varepsilon) = h^\varepsilon \text{ in } \Omega \\ v^\varepsilon \in H_0^1(\Omega) \end{cases} \tag{4.9}$$

where the sequence A^ε satisfies (1.9) and H-converges (see (3.9)) to A^0, where γ^ε, γ^0 and κ satisfy (1.10), (3.11), (3.12) and (3.15), and where

$$\begin{aligned} \theta^\varepsilon &\to \theta \text{ strongly in } L^2(\Omega) \\ h^\varepsilon &\to h \text{ strongly in } H^{-1}(\Omega). \end{aligned}$$

Then

$$\begin{cases} grad\, v^\varepsilon = P^\varepsilon \, grad\, v + \theta \, grad\, y^\varepsilon + r^\varepsilon \\ \text{with } r^\varepsilon \to 0 \text{ strongly in } (L^1(\Omega))^N \end{cases} \tag{4.10}$$

where v is the solution of

$$-div(A^0 \, grad\, v - \gamma^0 \theta) = h \text{ in } \Omega, \; v \in H_0^1(\Omega). \tag{4.11}$$

□

REMARK 4.2. When additional regularity holds for v and θ, the strong convergence of r^ε in (4.10) takes place in a better space : for example when v belongs to $C^1(\bar{\Omega})$ and θ to $C^0(\bar{\Omega})$, one gets

$$r^\varepsilon \to 0 \text{ strongly in } (L^2(\Omega))^N.$$

The same result holds true if P^ε and $grad\, y^\varepsilon$ are bounded in $(L^\infty(\Omega))^{N^2}$ and $(L^\infty(\Omega))^N$ while v and θ merely belongs to $H_0^1(\Omega)$ and $L^2(\Omega)$ respectively. □

PROOF OF PROPOSITION 4.1. Application of the div-curl lemma as in (4.4) proves that the sequence v^ε tends to v in $H_0^1(\Omega)$ where v is the solution of (4.11). The analogue of decomposition (4.7) then proves that

$$\gamma^\varepsilon \, grad\, v^\varepsilon \rightharpoonup \gamma^0 \, grad\, v + \kappa \theta \text{ weakly in } (L^2(\Omega))^N. \tag{4.12}$$

Consider now for any given ϕ in $\mathcal{D}(\Omega)$ and ψ in $\mathcal{D}(\Omega)$

(4.13)

$$\begin{aligned} X^\varepsilon &= \int_\Omega A^\varepsilon [grad\, v^\varepsilon - P^\varepsilon \, grad\, \phi - \psi \, grad\, y^\varepsilon][grad\, v^\varepsilon - P^\varepsilon \, grad\, \phi - \psi \, grad\, y^\varepsilon] \, dx \\ &= I^\varepsilon + II^\varepsilon \end{aligned}$$

where

$$I^\varepsilon = \int_\Omega [A^\varepsilon \, grad\, v^\varepsilon - \gamma^\varepsilon \theta^\varepsilon - A^\varepsilon P^\varepsilon \, grad\, \phi - \psi \{A^\varepsilon \, grad\, y^\varepsilon - (\gamma^\varepsilon - \cdots$$
$$[grad\, v^\varepsilon - P^\varepsilon \, grad\, \phi - \psi \, grad\, y^\varepsilon] \, dx$$

$$II^\varepsilon = \int_\Omega [\gamma^\varepsilon (\theta^\varepsilon - \psi) + \gamma^0 \psi][grad\, v^\varepsilon - P^\varepsilon \, grad\, \phi - \psi \, grad\, y^\varepsilon] \, dx.$$

Since ϕ and ψ are smooth, application of the div-curl lemma in I^ε is li (4.12), (3.11) and (3.14) give the value of the limit of II^ε ; we obtain

$$\begin{aligned} X^\varepsilon \to & \int_\Omega [A^0 \, grad\, v - \gamma^0 \theta - A^0 \, grad\, \phi - \psi \, 0][grad\, v - grad\, \phi - \psi \, 0] \\ & + \int_\Omega (\theta - \psi)[\gamma^0 \, grad\, v + \kappa \theta - \gamma^0 \, grad\, \phi - \kappa \psi] \, dx \\ & + \int_\Omega \psi \gamma^0 [grad\, v - grad\, \phi - \psi \, 0] \, dx \\ & = \int_\Omega A^0 \, grad(v - \phi) \, grad(v - \phi) \, dx + \int_\Omega \kappa (\theta - \psi)^2 \, dx \end{aligned}$$

The definition (4.13) of X^ε implies that

$$\begin{aligned} & \limsup_{\varepsilon \to 0} \|grad\, v^\varepsilon - P^\varepsilon \, grad\, \phi - \psi \, grad\, y^\varepsilon\|^2_{(L^2(\Omega))^N} \\ & \leq \frac{\lambda_2}{\lambda_1} \|grad(v - \phi)\|^2_{(L^2(\Omega))^N} + \frac{\|\kappa\|_{L^\infty(\Omega)}}{\lambda_1} \|\theta - \psi\|^2_{L^2(\cdots} \end{aligned} \tag{4.14}$$

The result (4.10) is then deduced from (4.14) upon setting

$$\begin{aligned} r^\varepsilon &= grad\, v^\varepsilon - P^\varepsilon \, grad\, v - \theta \, grad\, y^\varepsilon \\ &= grad\, v^\varepsilon - P^\varepsilon \, grad\, \phi - \psi \, grad\, y^\varepsilon - P^\varepsilon \, grad(v - \phi) - (\theta - \psi) \end{aligned} \tag{4.15}$$

and recalling that P^ε and $grad\, y^\varepsilon$ are bounded in $(L^2(\Omega))^{N^2}$ and $(L^2(\Omega$ tions (4.14) and (4.15) also imply the results stated in Remark 4.2.

4.2. PASSING TO THE LIMIT IN THE INITIAL CONDITIONS.

Recall that for X and Y Banach spaces with X compactly imbedded set $V = \{v$ bounded in $L^\infty(0,T;X)$ with $\partial v/\partial t$ bounded in $L^2(0,T;Y)\}$ is compact in $C^0(0,T;Y)$ (see e.g. [15] Corollary 4 p. 85).

In view of (3.0), u^ε, which is bounded in $L^\infty(0,T;H_0^1(\Omega))$ with $\partial u^\varepsilon/\partial$ in $L^\infty(0,T;L^2(\Omega))$, converges strongly to u in $C^0(0,T;L^2(\Omega))$. It is th pass to the limit in (1.5) and to obtain

$$u(0) = a^0 \text{ in } \Omega. \tag{4.16}$$

Similarly, $\rho^\varepsilon \partial u^\varepsilon/\partial t$ is bounded in $L^\infty(0,T;L^2(\Omega))$ while $\partial/\partial t(\rho^\varepsilon (\cdots$ bounded in $L^\infty(0,T;H^{-1}(\Omega))$, because of (2.4) and equation (1.1). Tl that $\rho^\varepsilon\, \partial u^\varepsilon/\partial t$ converges strongly in $C^0(0,T;H^{-1}(\Omega))$ to $\bar\rho\, \partial u/\partial t$. It is now possible to pass to the limit in (2.10). We obtain in view of (3.17)

$$\bar\rho\, \frac{\partial u}{\partial t}(x,0) = \overline{\rho\, b}. \tag{4.17}$$

But we proved in Proposition 2.1 that any solution $(u\,,\,\tau)$ of (3.1), (3.2) which belongs to the class "L^∞ in time" defined by (2.4) actually belongs to the class "C^0_s in time" defined by (2.6), and that (4.17) is equivalent to the initial condition $(\partial u/\partial t)\,(x,0) = \overline{\rho\, b}/\bar\rho$ which coincides with (3.6) in view of the definition (3.18) of b^0.

Finally consider the entropy defined by (2.7), namely

$$s^\varepsilon = \beta^\varepsilon\, \tau^\varepsilon + \gamma^\varepsilon\, grad\, u^\varepsilon \tag{4.18}$$

and the entropy s corresponding to the solution $(u\,,\,\tau)$ of (3.1)-(3.7), defined by

$$s = (\bar\beta + \kappa)\tau + \gamma^0\, grad\, u. \tag{4.19}$$

Note that s^ε is bounded in $L^\infty(0,T;L^2(\Omega))$. Multiplying (4.18) by φ in $\mathcal{D}(0,T)$ integrating on (0,T) and using an analogue of (4.8) allows to deduce that

$$s^\varepsilon \rightharpoonup s \text{ weak} * \text{ in } L^\infty(0,T;L^2(\Omega)). \tag{4.20}$$

On the other hand s^ε is bounded in $L^\infty(0,T;L^2(\Omega))$ with $\partial s^\varepsilon/\partial t$ bounded in $L^2(0,T;H^{-1}(\Omega))$ by virtue of (2.4) and equation (1.2). Thus

$$s^\varepsilon \rightarrow s \text{ strongly in } C^0(0,T;H^{-1}(\Omega)). \tag{4.21}$$

In view of the definitions (3.19), (3.20), this permits to pass to the limit in

$$s^\varepsilon(0) = d^\varepsilon \text{ in } \Omega \tag{4.22}$$

and to obtain

$$s(0) = d^0 = (\bar\beta + \kappa)\, c^0 + \gamma^0\, grad\, a^0 \text{ in } \Omega. \tag{4.23}$$

Since $(u\,,\,\tau)$ is a solution of (3.1), (3.2) which belongs to the class "L^∞ in time" defined in (2.4), Proposition 2.1 proves that the initial conditions (4.16), (4.23) are equivalent to (3.5), (3.7).

The uniqueness result of Theorem 2.1 completes the proof of Theorem 3.1. □

5. STATEMENT OF THE CORRECTOR RESULT.

This Section is devoted to the study of the structure of the solution $(u^\varepsilon\,,\,\tau^\varepsilon)$ when ε tends to zero. It follows closely the method used in the study of the wave and heat equations [3], which traces back to the elliptic case (see [11], [17]). We split $(u^\varepsilon\,,\,\tau^\varepsilon)$ into two parts. The first has a "correct energy" and a corrector result is obtained. The second tends weakly but not strongly to zero. It dissipates the energy gap in its effort to reconcile the energy of the initial conditions to the energy of the initial conditions of the limit $(u\,,\,\tau)$.

Throughout this Section we assume that the convergences neede Theorem 3.1 take place for the whole sequence ε (see Section 3 and the i of Section 4).

Let $(u^\varepsilon\,,\,\tau^\varepsilon)$ be the solution of the system (1.1)-(1.7). Define $e^\varepsilon \in$

$$\left\{\begin{aligned} e^\varepsilon(t) &= \frac{1}{2}\int_\Omega [\rho^\varepsilon\, |\frac{\partial u^\varepsilon}{\partial t}|^2 + A^\varepsilon\, grad\, u^\varepsilon\, grad\, u^\varepsilon + \beta^\varepsilon(\tau^\varepsilon)^2](x,t \\ &+ \int_0^t\int_\Omega (K^\varepsilon\, grad\,\tau^\varepsilon\, grad\,\tau^\varepsilon)(x,s)\, dx\, ds; \end{aligned}\right. \tag{5.1}$$

we will improperly refer to e^ε as an "energy".

Recalling that equality (2.3) holds true we have

$$e^\varepsilon(t) = E^\varepsilon + \int_0^t\int_\Omega f\,\frac{\partial u^\varepsilon}{\partial t}\, dx\, ds + \int_0^t < g\,,\,\tau^\varepsilon > ds \quad \text{in } (0,T \tag{5.2}$$

where E^ε is the real number (the "energy of the initial conditions") def

$$E^\varepsilon = \frac{1}{2}\int_\Omega [\rho^\varepsilon\, |b^\varepsilon|^2 + A^\varepsilon\, grad\, a^\varepsilon\, grad\, a^\varepsilon + \beta^\varepsilon\, |c^\varepsilon|^2]\, dx. \tag{5.3}$$

The solution $(u\,,\,\tau)$ of the homogenized system (3.1)-(3.7) satisfies similar to (2.3). Defining $e^0 \in L^\infty(0,T)$ and $E^0 \in \mathbb{R}$ by

$$\left\{\begin{aligned} e^0(t) &= \frac{1}{2}\int_\Omega [\bar\rho\, |\frac{\partial u}{\partial t}|^2 + A^0\, grad\, u\, grad\, u + (\bar\beta + \kappa)\, |\tau|^2](x,t \\ &+ \int_0^t\int_\Omega (K^0\, grad\,\tau\, grad\,\tau)(x,s)\, dx\, ds \end{aligned}\right. \tag{5.4}$$

$$E^0 = \frac{1}{2}\int_\Omega [\bar\rho\, |b^0|^2 + A^0\, grad\, a^0\, grad\, a^0 + (\bar\beta + \kappa)\, |c^0|^2] dx \tag{5.5}$$

we obtain

$$e^0(t) = E^0 + \int_0^t\int_\Omega f\,\frac{\partial u}{\partial t} dx\, ds + \int_0^t < g\,,\,\tau > ds \quad \text{in } (0,T). \tag{5.6}$$

In view of the boundedness (2.4) and (1.12) of $(u^\varepsilon\,,\,\tau^\varepsilon)$ and of the tions the sequences e^ε and E^ε are bounded in $L^\infty(0,T)$ and $\mathbb{R}$ respe can thus extract a subsequence such that

$$\left\{\begin{aligned} & e^{\varepsilon'} \rightharpoonup e \text{ weak} * \text{ in } L^\infty(0,T) \\ & E^{\varepsilon'} \rightarrow E \text{ in } \mathbb{R} \end{aligned}\right. \tag{5.7}$$

(the first convergence actually takes places in the strong topology of $C^0(0,T)$ using Ascoli-Arzela's theorem and the analogue of (6.7) below), and of course we deduce from (5.2) that

$$e(t) = E + \int_0^t \int_\Omega f \frac{\partial u}{\partial t}\, dx\, ds + \int_0^t < g\,,\tau > ds \text{ in } (0,T). \tag{5.8}$$

The main difficulty resides in the lack of convergence of e^ε and E^ε to e^0 and E^0 ; in general

$$e(t) \neq e^0(t) \quad \text{and} \quad E \neq E^0; \tag{5.9}$$

see Remark 5.3 below for a detailed analysis of the behaviour of e^ε and E^ε.

This observation is at the root of the introduction of a solution $(\tilde{u}^\varepsilon\,,\tilde{\tau}^\varepsilon)$ corresponding to initial conditions $(\tilde{a}^\varepsilon\,,\tilde{b}^\varepsilon\,,\tilde{c}^\varepsilon)$ with a "correct energy".

We consider the initial conditions $(\tilde{a}^\varepsilon\,,\tilde{b}^\varepsilon\,,\tilde{c}^\varepsilon)$ defined by

$$\begin{cases} \tilde{c}^\varepsilon = c^0 \\ \tilde{b}^\varepsilon = b^0 \end{cases} \tag{5.10}$$

(see (3.17)-(3.20) for the definitions of b^0 and c^0)

$$\begin{cases} -\,div(A^\varepsilon\, grad\, \tilde{a}^\varepsilon - c^0(\gamma^\varepsilon - \gamma^0)) = -div(A^0\, grad\, a^0) \text{ in } \Omega \\ \tilde{a}^\varepsilon \in H_0^1(\Omega) \end{cases} \tag{5.11}$$

where a^0 is the weak limit of a^ε (see (3.16)). Note that $\tilde{b}^\varepsilon$ and $\tilde{c}^\varepsilon$ actually do not depend on ε and that they are exactly the initial conditions $\partial u/\partial t\,(0)$ and $\tau(0)$ of the limit $(u\,,\tau)$ of $(u^\varepsilon\,,\tau^\varepsilon)$ (see Theorem 3.1).

We then define $(\tilde{u}^\varepsilon\,,\tilde{\tau}^\varepsilon)$ and $(v^\varepsilon\,,\theta^\varepsilon)$ as the solutions of the following systems.

$$\begin{cases} \rho^\varepsilon \dfrac{\partial^2 \tilde{u}^\varepsilon}{\partial t^2} - div(A^\varepsilon\, grad\, \tilde{u}^\varepsilon - \gamma^\varepsilon\, \tilde{\tau}^\varepsilon) = f \quad \text{in } \Omega \times (0,T) \\ \beta^\varepsilon \dfrac{\partial \tilde{\tau}^\varepsilon}{\partial t} - div(K^\varepsilon\, grad\, \tilde{\tau}^\varepsilon) + \gamma^\varepsilon\, grad\, \dfrac{\partial \tilde{u}^\varepsilon}{\partial t} = g \quad \text{in } \Omega \times (0,T) \\ \tilde{u}^\varepsilon(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T) \\ \tilde{\tau}^\varepsilon(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T) \\ \tilde{u}^\varepsilon(x,0) = \tilde{a}^\varepsilon(x) \quad \text{in } \Omega \\ \dfrac{\partial \tilde{u}^\varepsilon}{\partial t}(x,0) = \tilde{b}^\varepsilon(x) = b^0(x) \quad \text{in } \Omega \\ \tilde{\tau}^\varepsilon(x,0) = \tilde{c}^\varepsilon(x) = c^0(x) \quad \text{in } \Omega \end{cases} \tag{5.12}$$

$$\begin{cases} \rho^\varepsilon \dfrac{\partial^2 v^\varepsilon}{\partial t^2} - div(A^\varepsilon\, grad\, v^\varepsilon - \gamma^\varepsilon\, \theta^\varepsilon) = 0 \quad \text{in } \Omega \times (0,T) \\ \beta^\varepsilon \dfrac{\partial \theta^\varepsilon}{\partial t} - div(K^\varepsilon\, grad\, \theta^\varepsilon) + \gamma^\varepsilon\, grad\, \dfrac{\partial v^\varepsilon}{\partial t} = 0 \quad \text{in } \Omega \times (0, \\ v^\varepsilon(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T) \\ \tau^\varepsilon(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T) \\ v^\varepsilon(x,0) = a^\varepsilon(x) - \tilde{a}^\varepsilon(x) \quad \text{in } \Omega \\ \dfrac{\partial v^\varepsilon}{\partial t}(x,0) = b^\varepsilon(x) - \tilde{b}^\varepsilon(x) = b^\varepsilon(x) - b^0(x) \quad \text{in } \Omega \\ \theta^\varepsilon(x,0) = c^\varepsilon(x) - \tilde{c}^\varepsilon(x) = c^\varepsilon(x) - c^0(x) \quad \text{in } \Omega. \end{cases} \tag{5.13}$$

Denote by $\tilde{e}^\varepsilon(t)$ and $\tilde{E}^\varepsilon$ the "energy" associated to $(\tilde{u}^\varepsilon\,,\tilde{\tau}^\varepsilon)$ and tl associated to the initial conditions" $(\tilde{a}^\varepsilon\,,\tilde{b}^\varepsilon\,,\tilde{c}^\varepsilon)$ respectively and by $\eta^\varepsilon($ the corresponding "energies" associated to $(v^\varepsilon\,,\theta^\varepsilon)$, namely

$$\begin{cases} \tilde{e}^\varepsilon(t) = \dfrac{1}{2} \displaystyle\int_\Omega [\rho^\varepsilon\, |\frac{\partial \tilde{u}^\varepsilon}{\partial t}|^2 + A^\varepsilon\, grad\, \tilde{u}^\varepsilon\, grad\, \tilde{u}^\varepsilon + \beta^\varepsilon\, |\tilde{\tau}^\varepsilon|^2](x, \\ \quad + \displaystyle\int_0^t \int_\Omega (K^\varepsilon\, grad\, \tilde{\tau}^\varepsilon\, grad\, \tilde{\tau}^\varepsilon)(x,s)\, dx\, ds \\ \tilde{E}^\varepsilon = \dfrac{1}{2} \displaystyle\int_\Omega [\rho^\varepsilon\, |b^0|^2 + A^\varepsilon\, grad\, \tilde{a}^\varepsilon\, grad\, \tilde{a}^\varepsilon + \beta^\varepsilon\, |c^0|^2]\, dx \end{cases} \tag{5.14}$$

$$\begin{cases} \eta^\varepsilon(t) = \dfrac{1}{2} \displaystyle\int_\Omega [\rho^\varepsilon\, |\frac{\partial v^\varepsilon}{\partial t}|^2 + A^\varepsilon\, grad\, v^\varepsilon\, grad\, v^\varepsilon + \beta^\varepsilon\, |\theta^\varepsilon|^2](x, \\ \quad + \displaystyle\int_0^t \int_\Omega (K^\varepsilon\, grad\, \theta^\varepsilon\, grad\, \theta^\varepsilon)(x,s)\, dx\, ds \\ H^\varepsilon = \dfrac{1}{2} \displaystyle\int_\Omega [\rho^\varepsilon\, |b^\varepsilon - b^0|^2 + A^\varepsilon\, grad(a^\varepsilon - \tilde{a}^\varepsilon)\, grad(a^\varepsilon - \tilde{a}^\varepsilon \\ \qquad\qquad + \beta^\varepsilon\, |c^\varepsilon - c^0|^2](x)\, dx \end{cases} \tag{5.15}$$

Equalities similar to (5.2) (or (2.3)) hold for $(\tilde{u}^\varepsilon\,,\tilde{\tau}^\varepsilon)$ and $(v^\varepsilon\,,\theta^\varepsilon)$ and ;

$$\tilde{e}^\varepsilon(t) = \tilde{E}^\varepsilon + \int_0^t \int_\Omega f \frac{\partial \tilde{u}^\varepsilon}{\partial t}\, dx\, ds + \int_0^t < g\,, \tilde{\tau}^\varepsilon > ds \quad \text{in } (0 \tag{5.16a}$$

$$\eta^\varepsilon(t) = H^\varepsilon \quad \text{in } (0,T). \tag{5.16b}$$

Obviously

$$\begin{cases} u^\varepsilon = \tilde{u}^\varepsilon + v^\varepsilon \\ \tau^\varepsilon = \tilde{\tau}^\varepsilon + \tau^\varepsilon. \end{cases} \tag{5.17}$$

We will successively study $(\tilde{u}^\varepsilon, \tilde{\tau}^\varepsilon)$ and $(v^\varepsilon, \theta^\varepsilon)$. The obtained results will be summed up in Theorem 5.3 below, which describes in detail the structure of $(u^\varepsilon, \tau^\varepsilon)$.

Denote by $\tilde{s}^\varepsilon$ and σ^ε the entropies associated to $(\tilde{u}^\varepsilon, \tilde{\tau}^\varepsilon)$ and $(v^\varepsilon, \theta^\varepsilon)$, i.e.

$$\begin{cases} \tilde{s}^\varepsilon = \beta^\varepsilon \tilde{\tau}^\varepsilon + \gamma^\varepsilon \, grad\, \tilde{u}^\varepsilon \\ \sigma^\varepsilon = \beta^\varepsilon \theta^\varepsilon + \gamma^\varepsilon \, grad\, v^\varepsilon \end{cases} \tag{5.18a}$$

and recall that the entropy associated to (u, τ) is

$$s = (\bar{\beta} + \kappa)\tau + \gamma^0 \, grad\, u \tag{5.18b}$$

(see (4.19)). The initial conditions on $\tilde{s}^\varepsilon$, σ^ε and s are given by (see Proposition 2.1)

$$\begin{cases} \tilde{s}^\varepsilon(0) = \tilde{d}^\varepsilon \\ \sigma^\varepsilon(0) = d^\varepsilon - \tilde{d}^\varepsilon \\ s(0) = d^0 \end{cases} \tag{5.19}$$

where $\tilde{d}^\varepsilon$ is defined by

$$\tilde{d}^\varepsilon = \beta^\varepsilon \tilde{c}^\varepsilon + \gamma^\varepsilon \, grad\, \tilde{a}^\varepsilon = \beta^\varepsilon c^0 + \gamma^\varepsilon \, grad\, \tilde{a}^\varepsilon \tag{5.20}$$

and d^ε, d^0 are defined by (3.19a), (3.20).

THEOREM 5.1. *When ε tends to zero the following convergences hold true :*

$$\begin{cases} \tilde{a}^\varepsilon \rightharpoonup a^0 \text{ weakly in } H_0^1(\Omega) \\ \tilde{d}^\varepsilon \rightharpoonup d^0 \text{ weakly in } L^2(\Omega) \end{cases} \tag{5.21}$$

$$\begin{cases} \tilde{e}^\varepsilon \to e^0 \text{ strongly in } C^0(0,T) \\ \tilde{E}^\varepsilon \to E^0 \text{ in } \mathbb{R} \end{cases} \tag{5.22}$$

$$\begin{cases} \tilde{u}^\varepsilon \rightharpoonup u \text{ weak} * \text{ in } L^\infty(0,T; H_0^1(\Omega)) \\ \dfrac{\partial \tilde{u}^\varepsilon}{\partial t} \rightharpoonup \dfrac{\partial u}{\partial t} \text{ weak} * \text{ in } L^\infty(0,T; L^2(\Omega)) \\ \tilde{\tau}^\varepsilon \rightharpoonup \tau \text{ weak} * \text{ in } L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)) \end{cases} \tag{5.23}$$

where (u, τ) is the solution of the homogenized system (3.1)-(3.7).

Moreover the following corrector result hold true :

$$\begin{cases} \dfrac{\partial \tilde{u}^\varepsilon}{\partial t} \to \dfrac{\partial u}{\partial t} \text{ strongly in } C^0(0,T; L^2(\Omega)) \\ \tilde{\tau}^\varepsilon \to \tau \text{ strongly in } C^0(0,T; L^2(\Omega)) \end{cases} \tag{5.24}$$

$$\begin{cases} grad\, \tilde{u}^\varepsilon - P^\varepsilon \, grad\, u - \tau \, grad\, y^\varepsilon \to 0 \text{ strongly in } C^0(0,T; \\ grad\, \tilde{\tau}^\varepsilon - Q^\varepsilon \, grad\, \tau \to 0 \text{ strongly in } L^2(0,T; (L^1(\Omega))^N) \end{cases} \tag{5.25}$$

where the corrector matrix P^ε (associated to A^ε) is defined by (: defined by (3.13) ; Q^ε is the corrector matrix associated to K^ε.

REMARK 5.1. When additional regularity holds for (u, τ), the stron in (5.25) take place in better spaces than $C^0(0,T; (L^1(\Omega))^N)$ and $L^2(0$ For example when u and τ belong to $C^0(0,T; C^1(\bar{\Omega}))$ and $L^2(0,T; ($ only to $C^0(0,T; H_0^1(\Omega))$ and $L^2(0,T; H_0^1(\Omega))$, one obtains the strong $C^0(0,T; (L^2(\Omega))^N)$ and $L^2(0,T; (L^2(\Omega))^N)$. The same result holds P^ε and Q^ε are bounded in $(L^\infty(\Omega))^N$ and $(L^\infty(\Omega))^{N^2}$ respectively in L^2) while (u, τ) merely belong to the "natural" space defined by

REMARK 5.2. Note that the following convergence results hold for tl tions of the "quantities with tilde" :

$$\begin{cases} \tilde{u}^\varepsilon(0) = \tilde{a}^\varepsilon \rightharpoonup a^0 = u(0) \text{ weakly in } H_0^1(\Omega) \\ \dfrac{\partial \tilde{u}^\varepsilon}{\partial t}(0) = b^0 = \dfrac{\partial u}{\partial t}(0) \\ (\rho^\varepsilon \dfrac{\partial \tilde{u}^\varepsilon}{\partial t})(0) = \rho^\varepsilon b^0 \rightharpoonup \bar{\rho} b^0 = (\bar{\rho} \dfrac{\partial u}{\partial t})(0) \text{ weakly in} \\ \tilde{\tau}^\varepsilon(0) = c^0 = \tau(0) \\ \tilde{s}^\varepsilon(0) = \tilde{d}^\varepsilon \rightharpoonup d^0 = s(0) \text{ weakly in } L^2(\Omega). \end{cases} \tag{5.26}$$

Let us now turn to the study of the solution $(v^\varepsilon, \theta^\varepsilon)$ of (5.13).

THEOREM 5.2. *When ε tends to zero the following convergences hol*

$$\begin{cases} v^\varepsilon \rightharpoonup 0 \text{ weak} * \text{ in } L^\infty(0,T; H_0^1(\Omega)) \\ \dfrac{\partial v^\varepsilon}{\partial t} \rightharpoonup 0 \text{ weak} * \text{ in } L^\infty(0,T; L^2(\Omega)) \\ \theta^\varepsilon \rightharpoonup 0 \text{ weak} * \text{ in } L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega) \end{cases} \tag{5.27}$$

The convergences in (5.27) take place in the strong topologies if initial conditions $(a^\varepsilon, b^\varepsilon, c^\varepsilon)$ satisfy

$$\begin{cases} a^\varepsilon - \tilde{a}^\varepsilon \to 0 \text{ strongly in } H_0^1(\Omega) \\ b^\varepsilon - b^0 \to 0 \text{ strongly in } L^2(\Omega) \\ c^\varepsilon - c^0 \to 0 \text{ strongly in } L^2(\Omega). \end{cases} \tag{5.28}$$

REMARK 5.3. We will also prove that (5.28) is equivalent to

$$E^\varepsilon \to E^0 \text{ in } \mathbb{R} \tag{5.29}$$

whereas in general

(5.30) $$\liminf_{\varepsilon\to 0} E^\varepsilon \geq E^0.$$

Convergences (5.28) will also be proved to be equivalent to

(5.31) $$e^\varepsilon \to e^0 \text{ strongly in } C^0(0,T)$$

as well as to

(5.32) $$H^\varepsilon \to 0.$$

This remark complements assertion (5.9). □

REMARK 5.4. Observe that in view of (5.24)

(5.33) $$\begin{cases} \tilde{\tau}^\varepsilon \to \tau \text{ strongly in } C^0(0,T;L^2(\Omega)) \\ \dfrac{\partial \tilde{u}^\varepsilon}{\partial t} \to \dfrac{\partial u}{\partial t}\ C^0(0,T;L^2(\Omega)) \end{cases}$$

but that only weak convergence results hold for θ^ε and $\partial v^\varepsilon/\partial t$: they do not converge strongly in $L^\infty(0,T;L^2(\Omega))$ except if (5.28) holds true. Actually we believe that in general

(5.34) $$\begin{cases} \theta^\varepsilon \not\to 0 \quad \text{a.e. in } \Omega\times(0,T) \\ \dfrac{\partial v^\varepsilon}{\partial t} \not\to 0 \quad \text{a.e. in } \Omega\times(0,T); \end{cases}$$

(see [4], [6] for a supporting analysis of this belief based on a formal asymptotic expansion). □

The results obtained in Theorems 5.1 and 5.2 are merged in the following theorem that provides a detailed picture of the structure of $(u^\varepsilon, \tau^\varepsilon)$ itself.

THEOREM 5.3. *When ε tends to zero, the solution $(u^\varepsilon, \tau^\varepsilon)$ of (1.1)-(1.7) has the following structure :*

(5.35) $$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} = \dfrac{\partial u}{\partial t} + \dfrac{\partial v^\varepsilon}{\partial t} + r_1^\varepsilon \\ \tau^\varepsilon = \tau + \theta^\varepsilon + r_2^\varepsilon \end{cases}$$

(5.36) $$\begin{cases} grad\, u^\varepsilon = P^\varepsilon\, grad\, u + \tau\, grad\, y^\varepsilon + grad\, v^\varepsilon + R_1^\varepsilon \\ grad\, \tau^\varepsilon = Q^\varepsilon\, grad\, \tau + grad\, \theta^\varepsilon + R_2^\varepsilon \end{cases}$$

where (u, τ) is the solution of the homogenized system (3.1)-(3.7) the corrector matrices associated to A^ε and K^ε respectively (see defined by (3.13) ; $(v^\varepsilon, \theta^\varepsilon)$ is the solution of (5.13) and satisfies

(5.37) $$\begin{cases} \dfrac{\partial v^\varepsilon}{\partial t},\ \theta^\varepsilon \rightharpoonup 0 \text{ weak} * \text{ in } L^\infty(0,T;L^2(\Omega)) \\ grad\, v^\varepsilon \rightharpoonup 0 \text{ weak} * \text{ in } L^\infty(0,T;(L^2(\Omega))^N) \\ grad\, \theta^\varepsilon \rightharpoonup 0 \text{ weakly in } L^2(0,T;(L^2(\Omega))^N) \end{cases}$$

while

(5.38) $$r_1^\varepsilon, r_2^\varepsilon \to 0 \text{ strongly in } C^0(0,T;L^2(\Omega))$$

(5.39) $$\begin{cases} R_1^\varepsilon \to 0 \text{ strongly in } C^0(0,T;(L^1(\Omega))^N) \\ R_2^\varepsilon \to 0 \text{ strongly in } L^2(0,T;(L^1(\Omega))^N). \end{cases}$$

The strong convergences in (5.39) take place in better spaces when larity assumptions hold (see Remark 5.1). Moreover the converge strong (and thus the terms $\partial v^\varepsilon/\partial t$, θ^ε, $grad\, v^\varepsilon$, $grad\, \theta^\varepsilon$ can be re (5.36)) if and only if (5.28) holds true.

REMARK 5.5. In the case of the uncoupled wave and heat equatio ponds to the case $\gamma^\varepsilon \equiv 0$ in (1.1), (1.2) the behaviour of each of the "energy" $\eta^\varepsilon(t) = H^\varepsilon$ associated to $(v^\varepsilon, \theta^\varepsilon)$ can be more preci this case there are two uncoupled energies

(5.40) $$\begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega [\rho^\varepsilon |\frac{\partial v^\varepsilon}{\partial t}|^2 + A^\varepsilon\, grad\, v^\varepsilon\, grad\, v^\varepsilon](x,t)\, dx = H_w^\varepsilon \in \mathbb{R} \\ \dfrac{1}{2}\displaystyle\int_\Omega (\beta^\varepsilon |\theta^\varepsilon|^2)(x,t)dx + \int_0^t \int_\Omega (K^\varepsilon\, grad\, \theta^\varepsilon\, grad\, \theta^\varepsilon)(x,s)dx\, c \end{cases}$$

Extracting a subsequence such that

$$H_w^{\varepsilon'} \to H_w \text{ in } \mathbb{R} \quad , \quad H_h^{\varepsilon'} \to H_h \text{ in } \mathbb{R}$$

we prove in [3] that

(5.41) $$\begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega (\rho^\varepsilon |\frac{\partial v^\varepsilon}{\partial t}|^2)(x,t)dx \rightharpoonup \frac{1}{2} H_w \text{ weak} * \text{ in } L^\infty(0,T \\ \dfrac{1}{2}\displaystyle\int_\Omega (A^\varepsilon\, grad\, v^\varepsilon\, grad\, v^\varepsilon)(x,t)dx \rightharpoonup \frac{1}{2} H_w \text{ weak} * \text{ i} \end{cases}$$

(5.42) $$\begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega (\beta^\varepsilon |\theta^\varepsilon|^2)(x,t)dx \rightharpoonup 0 \\ \qquad \text{weak} * \text{ in } L^\infty(0,T) \text{ and strongly in } C^0(\delta,T) \text{ for} \\ \displaystyle\int_0^t \int_\Omega (K^\varepsilon\, grad\, \theta^\varepsilon\, grad\, \theta^\varepsilon)(x,s)dx\, ds \rightharpoonup H_h \\ \qquad \text{weak} * \text{ in } L^\infty(0,T) \text{ and strongly in } C^0(\delta,T) \text{ for} \end{cases}$$

Thus θ^ε is concentrated in an initial layer about the time $t = 0$ while the energy of $\partial v^\varepsilon/\partial t$ and $grad\, v^\varepsilon$ permeates all times with equipartition of the energy H_w.

We were not able to carry over such a precise analysis for the case of the coupled thermoelasticity system (1.1)-(1.7). In particular we were unable to prove that the limits of the functions $t \to \frac{1}{2}\int_\Omega \rho^\varepsilon |\partial v^\varepsilon/\partial t|^2\, dx$, $t \to \frac{1}{2}\int_\Omega A^\varepsilon\, grad\, v^\varepsilon\, grad\, v^\varepsilon\, dx$ as well as of $t \to \frac{1}{2}\int_\Omega (\beta^\varepsilon |\theta^\varepsilon|^2)\, dx$ are in general different from 0 for any positive time, although we believe that such is the case. This opinion is substantiated by the formal asymptotics and the numerical computations presented in [4], [6], which suggest the presence of fast time oscillations of θ^ε and $\partial v^\varepsilon/\partial t$. □

6. PROOFS OF THE CORRECTOR RESULT.

PROOF OF THEOREM 5.1.

FIRST PART : PROOF OF (5.21)-(5.23). As in (4.3), (4.4) two applications of the div-curl lemma prove that the solution $\tilde{a}^\varepsilon$ of (5.11) satisfies

(6.1) $$\tilde{a}^\varepsilon \rightharpoonup a^0 \text{ weakly in } H_0^1(\Omega).$$

In view of equation (5.11) and of Proposition 4.1,

(6.2) $$\begin{cases} \gamma^\varepsilon\, grad\, \tilde{a}^\varepsilon = \gamma^\varepsilon[P^\varepsilon\, grad\, a^0 + c^0\, grad\, y^\varepsilon + r^\varepsilon] \\ \qquad \rightharpoonup \gamma^0\, grad\, a^0 + c^0\,\kappa \\ \qquad \text{in } \mathcal{D}'(\Omega) \text{and weakly in } L^2(\Omega) \end{cases}$$

since $\gamma^\varepsilon\, grad\, \tilde{a}^\varepsilon$ is bounded in $L^2(\Omega)$. Thus

(6.3) $$\tilde{d}^\varepsilon = \beta^\varepsilon c^0 + \gamma^\varepsilon\, grad\, \tilde{a}^\varepsilon \rightharpoonup (\bar{\beta} + \kappa)c^0 + \gamma^0\, grad\, a^0 = d^0 \text{weakly in } L^2(\Omega)$$

and (5.21) is proved.

In view of (5.21) and of $\tilde{b}^\varepsilon = b^0$, Theorem 3.1 immediately implies (5.23), where $(u\,,\,\tau)$ is the solution of the homogenized system (3.1)-(3.7).

Let us now prove (5.22). Obviously we have

(6.4) $$\begin{cases} \int_\Omega \rho^\varepsilon |\tilde{b}^\varepsilon|^2\, dx = \int_\Omega \rho^\varepsilon |b^0|^2\, dx \to \int_\Omega \bar{\rho}\, |b^0|^2\, dx \\ \int_\Omega \beta^\varepsilon |\tilde{c}^\varepsilon|^2\, dx = \int_\Omega \beta^\varepsilon |c^0|^2\, dx \to \int_\Omega \bar{\beta}\, |c^0|^2\, dx. \end{cases}$$

On the other hand, multiplication of (5.11) by $\tilde{a}^\varepsilon$, integration by parts and use of (6.3) and (6.1) imply that

(6.5) $$\begin{cases} \int_\Omega A^\varepsilon\, grad\, \tilde{a}^\varepsilon\, grad\, \tilde{a}^\varepsilon\, dx \\ = \int_\Omega c^0 \gamma^\varepsilon\, grad\, \tilde{a}^\varepsilon\, dx - \int_\Omega c^0 \gamma^0\, grad\, \tilde{a}^\varepsilon\, dx + \int_\Omega A^0\, grad\, a^0\, grad\, \tilde{a}^\varepsilon\, dx \\ \to \int_\Omega c^0(\gamma^0\, grad\, a^0 + c^0\,\kappa) dx - \int_\Omega c^0 \gamma^0\, grad\, a^0\, dx + \int_\Omega A^0\, grad\, a^0\, grad\, a^0\, dx \\ = \int_\Omega A^0\, grad\, a^0\, grad\, a^0\, dx + \int_\Omega \kappa\, |c^0|^2\, dx. \end{cases}$$

By virtue of the definitions (5.14) and (5.5) we proved that $\tilde{E}^\varepsilon$ tends to

In view of (5.16a) the first part of (5.22) follows from

(6.6) $$\begin{cases} \int_0^t \int_\Omega f \frac{\partial \tilde{u}^\varepsilon}{\partial t}\, dx\, ds + \int_0^t < g\,,\, \tilde{\tau}^\varepsilon > ds \\ \to \int_0^t \int_\Omega f \frac{\partial u}{\partial t}\, dx\, ds + \int_0^t < g\,,\, \tau > ds \text{ strongly in } C^0(0, \end{cases}$$

Because of (5.23) this convergence is clear in $L^\infty(0,T)$ weak star. convergence in $C^0(0,T)$ then follows from Ascoli-Arzela's theorem and f

(6.7) $$\begin{cases} |\int_t^{t+h} \int_\Omega f \frac{\partial \tilde{u}^\varepsilon}{\partial t}\, dx\, ds| \le \|\frac{\partial \tilde{u}^\varepsilon}{\partial t}\|_{L^2(0,T;L^2(\Omega))}\, \|f\|_{L^2(t,t+h;L^2} \\ |\int_t^{t+h} < g\,,\, \tilde{\tau}^\varepsilon > ds| \le \|\tilde{\tau}^\varepsilon\|_{L^2(0,T;H_0^1(\Omega))}\, \|g\|_{L^2(t,t+h;H^{-1}(} \end{cases}$$

SECOND PART : THE HEART OF THE MATTER. In a manner similar to that of case ([11][17] ; see also the proof of Proposition 4.1 above), and of the heat equations ([3]) we consider for given ϕ and ψ in $C^\infty(0,T;\mathcal{D}(\Omega))$ th $\tilde{X}^\varepsilon$ defined by

(6.8) $$\begin{cases} \tilde{X}^\varepsilon(t) = \frac{1}{2}\int_\Omega [\rho^\varepsilon |\frac{\partial \tilde{u}^\varepsilon}{\partial t} - \frac{\partial \phi}{\partial t}|^2 + A^\varepsilon(grad\, \tilde{u}^\varepsilon - P^\varepsilon\, grad\, \phi - \psi\, grad\, y \\ \qquad (grad\, \tilde{u}^\varepsilon - P^\varepsilon\, grad\, \phi - \psi\, grad\, y^\varepsilon) + \beta^\varepsilon |\tilde{\tau}^\varepsilon - \psi|^2](x, \\ + \int_0^t \int_\Omega (K^\varepsilon(grad\, \tilde{\tau}^\varepsilon - Q^\varepsilon\, grad\, \psi)(grad\, \tilde{\tau}^\varepsilon - Q^\varepsilon\, grad\, \psi))(x, \\ = I^\varepsilon(t) + II^\varepsilon(t) + III^\varepsilon(t) \end{cases}$$

where

$$I^\varepsilon(t) = \frac{1}{2}\int_\Omega [\rho^\varepsilon |\frac{\partial \tilde{u}^\varepsilon}{\partial t}|^2 + A^\varepsilon\, grad\, \tilde{u}^\varepsilon\, grad\, \tilde{u}^\varepsilon + \beta^\varepsilon |\tilde{\tau}^\varepsilon|^2]\, dx + \int_0^t \int_\Omega K^\varepsilon\, grad\, \tilde{\tau}^\varepsilon\, grad\, \tilde{\tau}^\varepsilon\, dx\, ds$$

$$II^\varepsilon(t) = -\int_\Omega [\rho^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial t}\frac{\partial \phi}{\partial t} + A^\varepsilon(P^\varepsilon\, grad\, \phi + \psi\, grad\, y^\varepsilon)\, grad\, \tilde{u}^\varepsilon + \beta^\varepsilon \tilde{\tau} \quad - 2\int_0^t \int_\Omega K^\varepsilon Q^\varepsilon\, grad\, \psi\, grad\, \tilde{\tau}^\varepsilon\, dx\, ds$$

$$III^\varepsilon(t) = \frac{1}{2}\int_\Omega [\rho^\varepsilon |\frac{\partial \phi}{\partial t}|^2 + A^\varepsilon(P^\varepsilon\, grad\, \phi + \psi\, grad\, y^\varepsilon)(P^\varepsilon\, grad\, \phi + \psi\, grad\, y^\varepsilon) + \int_0^t \int_\Omega K^\varepsilon Q^\varepsilon\, grad\, \psi\, Q^\varepsilon\, grad\, \psi\, dx\, ds.$$

We will successively pass to the limit in $I^\varepsilon, III^\varepsilon$ and II^ε. Since I^ε is precisely $\tilde{e}^\varepsilon$ defined by (5.14), (5.22) and definition (5.4) of e^0 yield

$$(6.9)\quad \begin{cases} I^\varepsilon \to \frac{1}{2}\int_\Omega [\bar\rho\,|\frac{\partial u}{\partial t}|^2 + A^0\, grad\, u\, grad\, u + (\bar\beta+\kappa)|\tau|^2](x,t)dx \\ \quad + \int_0^t \int_\Omega (K^0\, grad\,\tau\, grad\,\tau)(x,s)dx\, ds \quad \text{strongly in } C^0(0,T). \end{cases}$$

Note that III^ε and its time derivative $\partial III^\varepsilon/\partial t$ are bounded in $L^\infty(0,T)$. Thus III^ε converges strongly in $C^0(0,T)$ to some limit which can be identified in the sense of distributions using a test function $\varphi \in \mathcal{D}(0,T)$. Consider for example the term

$$\int_0^T \varphi\, dt \int_\Omega A^\varepsilon(P^\varepsilon\, grad\,\phi + \psi\, grad\, y^\varepsilon)(P^\varepsilon\, grad\,\phi + \psi\, grad\, y^\varepsilon)dx$$
$$= \int_0^T \int_\Omega \varphi(\sum_i A^\varepsilon P^\varepsilon e_i \frac{\partial\phi}{\partial x_i})(\sum_j P^\varepsilon e_j \frac{\partial\phi}{\partial x_j} + \psi\, grad\, y^\varepsilon)dx\, dt$$
$$+ \int_0^T \int_\Omega \varphi\,\psi(A^\varepsilon\, grad\, y^\varepsilon - (\gamma^\varepsilon - \gamma^0))(\sum_j P^\varepsilon e_j \frac{\partial\phi}{\partial x_j} + \psi\, grad\, y^\varepsilon)dx\, dt$$
$$+ \int_0^T \int_\Omega \varphi\,\psi(\gamma^\varepsilon - \gamma^0)(P^\varepsilon\, grad\,\phi + \psi\, grad\, y^\varepsilon)dx\, dt.$$

Because the divergences in x of $A^\varepsilon P^\varepsilon e_i$ and of $(A^\varepsilon\, grad\, y^\varepsilon - (\gamma^\varepsilon - \gamma^0))$ as well as the curl in x and the time derivative of $P^\varepsilon e_j$ and of $grad\, y^\varepsilon$ are compact in $H^{-1}(\Omega\times(0,T))$, a time dependent version of the div-curl lemma (see [12], [18], or [3] for a precise statement) together with the definitions (3.11) and (3.14) of γ^0 and κ permit to obtain in the limit

$$\int_0^T \int_\Omega \varphi(A^0\, grad\,\phi)(grad\,\phi + 0)dx\, dt$$
$$+ \int_0^T \int_\Omega \varphi\,\psi\,(0)\,(grad\,\phi + 0)dx\, dt$$
$$+ \int_0^T \int_\Omega \varphi\,\psi[(\gamma^0 - \gamma^0)grad\,\phi + \psi(\kappa - \gamma^0\, 0)]dx\, dt.$$

Thus

$$(6.10)\quad \begin{cases} III^\varepsilon \to \frac{1}{2}\int_\Omega [\bar\rho\,|\frac{\partial\phi}{\partial t}|^2 + A^0\, grad\,\phi\, grad\,\phi + (\bar\beta+\kappa)|\psi|^2](x,t)dx \\ \quad + \int_0^t \int_\Omega (K^0\, grad\,\psi\, grad\,\psi)(x,s)dx\, ds \quad \text{strongly in } C^0(0,T). \end{cases}$$

The second term II^ε is clearly bounded in $L^\infty(0,T)$. Writing II^ε as

$$(6.11)\quad \begin{cases} II^\varepsilon = -\int_\Omega [\rho^\varepsilon \frac{\partial\tilde u^\varepsilon}{\partial t}\frac{\partial\phi}{\partial t} + A^\varepsilon P^\varepsilon\, grad\,\phi\, grad\,\tilde u^\varepsilon \\ \quad + \psi\, grad\,\tilde u^\varepsilon(A^\varepsilon\, grad\, y^\varepsilon - (\gamma^\varepsilon - \gamma^0)) \\ \quad + \psi(\beta^\varepsilon\,\tilde\tau^\varepsilon + \gamma^\varepsilon\, grad\,\tilde u^\varepsilon) - \psi\,\gamma^0\, grad\,\tilde u^\varepsilon]dx \\ \quad - 2\int_0^t \int_\Omega K^\varepsilon Q^\varepsilon\, grad\,\psi\, grad\,\tilde\tau^\varepsilon\, dx\, ds \end{cases}$$

and using the time dependent version of the div-curl lemma as for III pass to the limit in the sense of distributions on $(0,T)$ in II^ε. It will be the fourth part of this proof that each term actually converges strongly i and thus that

$$(6.12)\quad \begin{cases} II^\varepsilon \to -\int_\Omega [\bar\rho\,\frac{\partial u}{\partial t}\frac{\partial\phi}{\partial t} + A^0\, grad\,\phi\, grad\, u \\ \quad + \psi\, grad\, u\, 0 + \psi((\bar\beta+\kappa)\tau + \gamma^0\, grad\, u) - \psi\,\gamma^0\, gra \\ \quad - 2\int_0^t \int_\Omega K^0\, grad\,\psi\, grad\,\tau\, dx\, ds \\ = -\int_\Omega [\bar\rho\,\frac{\partial u}{\partial t}\frac{\partial\phi}{\partial t} + A^0\, grad\,\phi\, grad\, u + (\bar\beta+\kappa)\tau\,\psi]dx \\ \quad - 2\int_0^t \int_\Omega K^0\, grad\,\psi\, grad\,\tau\, dx\, ds \quad \text{strongly in } C^0(0,T \end{cases}$$

From (6.8)-(6.12) we conclude that

$$(6.13)\quad \begin{cases} \tilde X^\varepsilon \to \frac{1}{2}\int_\Omega [\bar\rho\,|\frac{\partial u}{\partial t} - \frac{\partial\phi}{\partial t}|^2 + A^0\,(grad\, u - grad\,\phi)(grad\, u - grad\,\phi) \\ \quad + (\bar\beta+\kappa)|\tau - \psi|^2]dx \\ \quad + \int_0^t \int_\Omega K^0(grad\,\tau - grad\,\psi)(grad\,\tau - grad\,\psi)dx\, ds \quad \text{in } C \end{cases}$$

which implies

$$(6.14)\quad \begin{cases} \limsup_{\varepsilon\to 0} \{\lambda_1 \|\frac{\partial\tilde u^\varepsilon}{\partial t} - \frac{\partial\phi}{\partial t}\|^2_{C^0(0,T;L^2(\Omega))} \\ \quad + \lambda_1\|grad\,\tilde u^\varepsilon - P^\varepsilon\, grad\,\phi - \psi\, grad\, y^\varepsilon\|^2_{C^0(0,T;(L^2(\Omega))^N)} \\ \quad + \lambda_1\|\tilde\tau^\varepsilon - \psi\|^2_{C^0(0,T;L^2(\Omega))} + 2\lambda_1\|grad\,\tilde\tau^\varepsilon - Q^\varepsilon\, grad\,\psi\|^2_{(L^2(0,T;L} \\ \le 2\{\lambda_2\|\frac{\partial u}{\partial t} - \frac{\partial\phi}{\partial t}\|^2_{C^0(0,T;L^2(\Omega))} + \lambda_2\|grad\, u - grad\,\phi\|^2_{C^0(0,T;(L^2(\Omega)} \\ \quad + (\lambda_2 + \|\kappa\|_{L^\infty(\Omega)})\,\|\tau - \psi\|^2_{C^0(0,T;L^2(\Omega))} \\ \quad + 2\lambda_2\|grad\,\tau - grad\,\psi\|^2_{L^2(0,T;(L^2(\Omega))^N)}\}. \end{cases}$$

THIRD PART : END OF THE PROOF OF (5.24), (5.25). The corrector results (5.24) (5.25) for $(\tilde{u}^\varepsilon, \tilde{\tau}^\varepsilon)$ are then deduced from (6.14) with the help of the following decomposition :

(6.15)
$$\begin{cases} \dfrac{\partial \tilde{u}^\varepsilon}{\partial t} - \dfrac{\partial u}{\partial t} = \dfrac{\partial \tilde{u}^\varepsilon}{\partial t} - \dfrac{\partial \phi}{\partial t} + \dfrac{\partial \phi}{\partial t} - \dfrac{\partial u}{\partial t} \\ grad\, \tilde{u}^\varepsilon - P^\varepsilon\, grad\, u - \tau\, grad\, y^\varepsilon \\ = grad\, \tilde{u}^\varepsilon - P^\varepsilon\, grad\, \phi - \psi\, grad\, y^\varepsilon + P^\varepsilon(grad\, \phi - grad\, u) + (\psi - \tau) grad\, y^\varepsilon \end{cases}$$

(and the analogous formulas for $\tilde{\tau}^\varepsilon$). We choose ϕ and ψ in $C^\infty(0,T;\mathcal{D}(\Omega))$ such that $((\partial u/\partial t) - (\partial \phi/\partial t))$ and $(grad\, u - grad\, \phi)$ are small in $C^0(0,T;L^2(\Omega))$ and $C^0(0,T;(L^2(\Omega))^N)$ respectively, while $\tau - \psi$ is small in $C^0(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$: this choice is possible because u belongs to $C^0(0,T;H_0^1(\Omega))$ with $\partial u/\partial t$ in $C^0(0,T;L^2(\Omega))$ and τ belongs to $C^0(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$; recall also that P^ε and $grad\, y^\varepsilon$ are bounded in $(L^2(\Omega))^{N^2}$ and $(L^2(\Omega))^N$ respectively. This proves (5.24), (5.25). The "convergences in better spaces" announced in Remark 5.1 are proved in the same manner.

FOURTH PART : PROOF OF THE STRONG CONVERGENCE IN $C^0(0,T)$ OF THE TERMS OF II^ε (see (6.11), (6.12)). Using equation (5.12) leads to

(6.16)
$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_\Omega \rho^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial t}\frac{\partial \phi}{\partial t}\, dx \\ = \displaystyle\int_\Omega \rho^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial t}\frac{\partial^2 \phi}{\partial t^2}\, dx + < \rho^\varepsilon \frac{\partial^2 \tilde{u}^\varepsilon}{\partial t^2}, \frac{\partial \phi}{\partial t} > \\ = \displaystyle\int_\Omega \rho^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial t}\frac{\partial^2 \phi}{\partial t^2}\, dx + \int_\Omega f \frac{\partial \phi}{\partial t}\, dx \\ \quad - \displaystyle\int_\Omega (A^\varepsilon\, grad\, \tilde{u}^\varepsilon - \gamma^\varepsilon \tilde{\tau}^\varepsilon) grad\, \frac{\partial \phi}{\partial t}\, dx. \end{cases}$$

By application of the Cauchy-Schwartz inequality this term is bounded in $L^2(0,T)$. Thus $t \to \int_\Omega \rho^\varepsilon (\partial \tilde{u}^\varepsilon/\partial t)(\partial \phi/\partial t)\, dx$ converges strongly in $C^0(0,T)$.

In a similar manner, (5.12) leads to

(6.17)
$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_\Omega \psi(\beta^\varepsilon \tilde{\tau}^\varepsilon + \gamma^\varepsilon\, grad\, \tilde{u}^\varepsilon) dx = \\ = \displaystyle\int_\Omega \frac{\partial \psi}{\partial t}(\beta^\varepsilon \tilde{\tau}^\varepsilon + \gamma^\varepsilon\, grad\, \tilde{u}^\varepsilon) dx + \int_\Omega \psi \frac{\partial}{\partial t}(\beta^\varepsilon \tilde{\tau}^\varepsilon + \gamma^\varepsilon\, grad\, \tilde{u}^\varepsilon) dx \\ = \displaystyle\int_\Omega \frac{\partial \psi}{\partial t}(\beta^\varepsilon \tilde{\tau}^\varepsilon + \gamma^\varepsilon\, grad\, \tilde{u}^\varepsilon) dx - \int_\Omega K^\varepsilon\, grad\, \tilde{\tau}^\varepsilon\, grad\, \psi\, dx + < g, \psi > . \end{cases}$$

By application of the Cauchy-Schwartz inequality this term is bounded in $L^2(0,T)$. Thus $t \to \int_\Omega \psi(\beta^\varepsilon \tilde{\tau}^\varepsilon + \gamma^\varepsilon\, grad\, \tilde{u}^\varepsilon) dx$ converges strongly in $C^0(0,T)$.

Using Ascoli-Arzela's theorem, it is easy to see that the strong con $C^0(0,T)$ of the term $\int_0^t \int_\Omega K^\varepsilon Q^\varepsilon\, grad\, \psi\, grad\, \tilde{\tau}^\varepsilon\, dx\, ds$ results from

(6.18)
$$\begin{cases} \left| \displaystyle\int_t^{t+h} \int_\Omega K^\varepsilon\, grad\, \tilde{\tau}^\varepsilon\, Q^\varepsilon\, grad\, \psi\, dx\, ds \right| \\ \leq \|K^\varepsilon\, grad\, \tilde{\tau}^\varepsilon\|^2_{L^2(0,T;(L^2(\Omega))^N)} \|Q^\varepsilon\|_{(L^2(\Omega))^{N^2}} \|grad\, \psi\|_{L^2(t,t+h;(L^\infty} \end{cases}$$

Using integration by parts and the definition (3.10) of P^ε yields

(6.19a)
$$\begin{cases} \displaystyle\int_\Omega A^\varepsilon P^\varepsilon\, grad\, \phi\, grad\, \tilde{u}^\varepsilon\, dx = \sum_i \int_\Omega A^\varepsilon\, grad\, w_i^\varepsilon \frac{\partial \phi}{\partial x_i} grad\, \tilde{u}^\varepsilon\, dx \\ = -\displaystyle\sum_i < div(A^\varepsilon\, grad\, w_i^\varepsilon) \frac{\partial \phi}{\partial x_i}, \tilde{u}^\varepsilon > - \sum_i \int_\Omega A^\varepsilon\, grad\, w_i^\varepsilon\, grad \frac{\partial}{\partial} \\ = -\displaystyle\sum_i < div(A^0\, e_i) \frac{\partial \phi}{\partial x_i}, \tilde{u}^\varepsilon > - \sum_i \int_\Omega A^\varepsilon\, grad\, w_i^\varepsilon\, grad \frac{\partial \phi}{\partial x_i} \tilde{u}^\varepsilon \end{cases}$$

and similarly in view of the definition (3.13) of y^ε

(6.19b)
$$\begin{cases} \displaystyle\int_\Omega \psi(A^\varepsilon\, grad\, y^\varepsilon - (\gamma^\varepsilon - \gamma^0)) grad\, \tilde{u}^\varepsilon\, dx \\ = -\displaystyle\int_\Omega grad\, \psi(A^\varepsilon\, grad\, y^\varepsilon - (\gamma^\varepsilon - \gamma^0)) \tilde{u}^\varepsilon\, dx. \end{cases}$$

Thus defining k^ε by

(6.20)
$$\begin{cases} k^\varepsilon = [-\displaystyle\sum_i div(A^0\, e_i) \frac{\partial \phi}{\partial x_i} + div(\psi\, \gamma^0)] \\ \quad - \displaystyle\sum_i A^\varepsilon\, grad\, w_i^\varepsilon\, grad \frac{\partial \phi}{\partial x_i} - grad\, \psi(A^\varepsilon\, grad\, y^\varepsilon - (\gamma^\varepsilon \end{cases}$$

the part $F^\varepsilon(t)$ of $II^\varepsilon(t)$ that has yet to be proved to converge strongly reads as

(6.21)
$$F^\varepsilon(t) = < k^\varepsilon, \tilde{u}^\varepsilon > (t).$$

But k^ε is the sum of a first term which is fixed in $C^0(0,T;H^{-1}($ sequence which is bounded in $C^1(0,T;L^2(\Omega))$. Thus (for a subsequenc

(6.22)
$$k^\varepsilon \to k \text{ strongly in } C^0(0,T;H^{-1}(\Omega))$$

where k belongs to $C^0(0,T;H^{-1}(\Omega))$.

Consider h in $C^0(0,T;L^2(\Omega))$ and write for any $t \in (0,T)$

$$F^\varepsilon(t) - < k, u > (t) = (< k^\varepsilon, \tilde{u}^\varepsilon > - < k, u >)(t)$$
$$= (< k^\varepsilon - k, \tilde{u}^\varepsilon > + < k - h, \tilde{u}^\varepsilon > + < h, \tilde{u}^\varepsilon - u > + < h - k,$$

which implies that

$$(6.23)\quad \begin{cases} \|F^\varepsilon(t) - <k\,,u>(t)\|_{C^0(0,T)} \\ \leq \|k^\varepsilon - k\|_{C^0(0,T;H^{-1}(\Omega))}\,\|\tilde{u}^\varepsilon\|_{C^0(0,T;H_0^1(\Omega))} \\ + \|k-h\|_{C^0(0,T;H^{-1}(\Omega))}\,(\|\tilde{u}^\varepsilon\|_{L^\infty(0,T;H_0^1(\Omega))} + \|u\|_{L^\infty(0,T;H_0^1(\Omega))}) \\ + \|h\|_{C^0(0,T;L^2(\Omega))}\,\|\tilde{u}^\varepsilon - u\|_{C^0(0,T;L^2(\Omega))}. \end{cases}$$

Fix now h in $C^0(0,T;L^2(\Omega))$ such that $\|k-h\|_{C^0(0,T;H^{-1}(\Omega))}$ is small. In view of the convergence (6.22) of k^ε, of the $L^\infty(0,T;H_0^1(\Omega))$ bound of $\tilde{u}^\varepsilon$ and of its strong convergence in $C^0(0,T;L^2(\Omega))$ (recall that $\partial\tilde{u}^\varepsilon/\partial t$ is bounded in $L^\infty(0,T;L^2(\Omega))$), $F^\varepsilon - <k\,,u>$ converges strongly to 0 in $C^0(0,T)$. This completes the proof of (6.12).

Theorem 5.1 is proved. □

PROOF OF THEOREM 5.2. Decomposition (5.17) and the convergence result (5.23) on $(\tilde{u}^\varepsilon\,,\tilde{\tau}^\varepsilon)$ immediately implies (5.27). This result can also be obtained through direct application of Theorem 3.1 since

$$(6.24)\quad \begin{cases} v^\varepsilon(0) = a^\varepsilon - \tilde{a}^\varepsilon \rightharpoonup 0 \text{ weakly in } H_0^1(\Omega) \\ \rho^\varepsilon \dfrac{\partial v^\varepsilon}{\partial t}(0) = \rho^\varepsilon(b^\varepsilon - b^0) \rightharpoonup \bar{\rho}(b^0 - b^0) = 0 \text{ weakly in } L^2(\Omega) \\ \sigma^\varepsilon(0) = d^\varepsilon - \tilde{d}^\varepsilon \rightharpoonup d^0 - d^0 = 0 \text{ weakly in } L^2(\Omega) \end{cases}$$

(see (3.17), (3.18), (3.19), (5.19), (5.21)).

The conservation of the energy (5.16b), namely

$$\eta^\varepsilon(t) = H^\varepsilon \text{ in } (0,T)$$

together with the definitions (5.15) of η^ε and H^ε, clearly implies that (5.28) is equivalent to (5.32), which in turn is equivalent to the strong convergence in (5.27).

Theorem 5.2 is proved. □

PROOF OF REMARK 5.3. We firstly prove two lower semi-continuity results.

Since

$$(6.25)\quad \begin{cases} \displaystyle\int_\Omega \rho^\varepsilon\,|b^\varepsilon|^2\,dx = \\ \displaystyle = \int_\Omega \rho^\varepsilon\,|b^0|^2\,dx + 2\int_\Omega \rho^\varepsilon\,b^0(b^\varepsilon - b^0) + \int_\Omega \rho^\varepsilon|b^\varepsilon - b^0|^2\,dx \end{cases}$$

we deduce from the definition (3.18) of b^0 that

$$(6.26)\quad \liminf_{\varepsilon\to 0} \int_\Omega \rho^\varepsilon\,|b^\varepsilon|^2\,dx \geq \int_\Omega \bar{\rho}\,|b^0|^2\,dx$$

where equality takes place if and only if

$$(b^\varepsilon - b^0) \to 0 \text{ strongly in } L^2(\Omega).$$

Further

$$(6.27)\quad \begin{cases} \displaystyle\int_\Omega [A^\varepsilon\,grad\,a^\varepsilon\,grad\,a^\varepsilon + \beta^\varepsilon|c^\varepsilon|^2]dx \\ \displaystyle = \int_\Omega [A^\varepsilon\,grad(\tilde{a}^\varepsilon + (a^\varepsilon - \tilde{a}^\varepsilon))grad(\tilde{a}^\varepsilon + (a^\varepsilon - \tilde{a}^\varepsilon)) + \beta^\varepsilon|c^0 + \\ \displaystyle = \int_\Omega [A^\varepsilon\,grad\,\tilde{a}^\varepsilon\,grad\,\tilde{a}^\varepsilon + \beta^\varepsilon|c^0|^2]dx \\ \displaystyle + 2\int_\Omega [A^\varepsilon\,grad\,\tilde{a}^\varepsilon\,grad\,(a^\varepsilon - \tilde{a}^\varepsilon) + \beta^\varepsilon\,c^0(c^\varepsilon - c^0)]dx \\ \displaystyle + \int_\Omega [A^\varepsilon\,grad(a^\varepsilon - \tilde{a}^\varepsilon)grad(a^\varepsilon - \tilde{a}^\varepsilon) + \beta^\varepsilon|c^\varepsilon - c^0|^2]dx \\ \equiv I^\varepsilon + 2\,II^\varepsilon + III^\varepsilon. \end{cases}$$

But in view of (6.5) we have

$$(6.28a)\quad I^\varepsilon \to \int_\Omega [A^0\,grad\,a^0\,grad\,a^0 + (\bar{\beta} + \kappa)|c^0|^2]dx$$

while clearly

$$(6.28b)\quad \limsup_{\varepsilon\to 0} III^\varepsilon \geq 0.$$

On the other hand

$$II^\varepsilon = \int_\Omega (A^\varepsilon\,grad\,\tilde{a}^\varepsilon - c^0\,\gamma^\varepsilon)\,grad(a^\varepsilon - \tilde{a}^\varepsilon)dx + \int_\Omega c^0(\beta^\varepsilon\,c^\varepsilon + \gamma^\varepsilon\;grad\,a^\varepsilon)dx - \int_\Omega c^0(\beta^\varepsilon\,c^0 + \gamma^\varepsilon\;grad\,\tilde{a}^\varepsilon)dx.$$

Application of the div-curl lemma implies that the first te because $A^\varepsilon\,grad\,\tilde{a}^\varepsilon - c^0\,\gamma^\varepsilon$ has a fixed divergence in H^{-1} by (3.] $grad(a^\varepsilon - \tilde{a}^\varepsilon)$ tends to 0 weakly in $(L^2(\Omega))^N$ by (6.1). From (: conclude that

$$(6.28c)\quad II^\varepsilon \to 0.$$

This yields the second lower semicontinuity result, namely

$$(6.29)\quad \begin{cases} \displaystyle\liminf_{\varepsilon\to 0} \int_\Omega [A^\varepsilon\,grad\,a^\varepsilon\,grad\,a^\varepsilon + \beta^\varepsilon|c^\varepsilon|^2]dx \\ \displaystyle \geq \int_\Omega [A^0\,grad\,a^0\,grad\,a^0 + (\bar{\beta} + \kappa)|c^0|^2\,dx \end{cases}$$

where equality takes places if and only if

$$\begin{cases} a^\varepsilon - \tilde{a}^\varepsilon \to 0 \text{ strongly in } H_0^1(\Omega) \\ c^\varepsilon - c^0 \to 0 \text{ strongly in } L^2(\Omega). \end{cases}$$

In view of (6.26), (6.29), (5.30) is proved as well as the equivalence between (5.28) and (5.29). The equivalence between (5.32) and (5.28) has already been proved in the proof of Theorem 5.2. Finally (5.31) is equivalent to (5.29) by using in (5.2) an argument similar to (6.6), (6.7).

This completes the proof of Remark 5.3. □

REFERENCES.

[1] A. BENSOUSSAN, J.-L. LIONS, G. PAPANICOLAOU. *Asymptotic Analysis for Periodic Structures*, North-Holland, Amsterdam, 1978.
[2] S. BRAHIM-OTSMANE. *Homogénéisation en thermoélasticité*, Thèse de 3ème cycle, Université Pierre et Marie Curie, Paris, 1988.
[3] S. BRAHIM-OTSMANE, G.A. FRANCFORT, F. MURAT. *Correctors for the homogenization of the wave and heat equations*, to appear.
[4] G.A. FRANCFORT. *Two variational problems in thermoelasticity*, Ph.D. Thesis, Stanford University, Stanford, 1982.
[5] G.A. FRANCFORT. *Homogenization and fast oscillations in linear thermoelasticity*, in *Numerical Methods for Transient and Coupled Problems*, R.W. Lewis, E. Hinton, P. Bettess, B.A. Schrefler eds., Pineridge Press, Swansea, 1984, pp. 382-393.
[6] G.A. FRANCFORT. *Homogenization and linear thermoelasticity*, SIAM J. Math. Anal., 14 (1984), pp. 696-708.
[7] G.A. FRANCFORT, P. SUQUET. *Homogenization and mechanical dissipation in thermoviscoelasticity*, Arch. Rat. Mech. Anal., 96 (1986), pp. 265-293.
[8] J.-L. LIONS, E. MAGENES. *Problèmes aux Limites Non Homogènes et Applications Vol. 1*, Dunod, Paris, 1968.
[9] J. MARSDEN, T.J.R. HUGHES. *Topics in the mathematical foundations of elasticity*, in *Nonlinear Analysis and Mechanics, Heriot-Watt Symposium Vol. 2*, R.J. Knops ed., Research Notes in Mathematics 27, Pitman, London, 1978, pp. 30-285.
[10] M.L. MASCARENHAS. *Homogenisation of a viscoelastic equation with non-periodic coefficients*, Proc. Royal Soc. Edinburgh, 106A (1987), pp. 143-160.
[11] F. MURAT. *H-convergence*, in *Séminaire d'Analyse Fonctionnelle et Numérique*, 1977-1978, Université d'Alger, multigraphed.
[12] F. MURAT. *Compacité par compensation.* Ann. Sc. Norma. Sup. Pisa, 5 (1978), pp. 489-507.
[13] J.L. NOWINSKI. *Theory of Thermoelasticity*, Noordhoff International, Alphen aan den Rijn, 1978.
[14] E. SANCHEZ-PALENCIA. *Non Homogeneous Media and Vibration Theory*, Lecture Notes in Physics 127, Springer-Verlag, Berlin, 1980.
[15] J. SIMON. *Compact sets in $L_p(0,T;B)$*, Ann. Mat. Pura Appl., 146 (1987), pp. 65-96.

[16] S. SPAGNOLO. *Sulla convergenza di soluzioni di equazioni paraboli liticle*, Ann. Sc. Norm. Sup. Pisa, 22 (1968), pp. 577-597.
[17] L. TARTAR. *Cours Peccot*, Collège de France, Paris, 1977.
[18] L. TARTAR. *Compensated compactness and applications to partia tial equations*, in *Nonlinear Mechanics and Analysis, Heriot-Watt S Vol. 4*, R.J. Knops ed., Research Notes in Mathematics 39, Pitman 1979, pp. 136-212.
[19] L. TARTAR. *Nonlocal effects induced by homogenization*, in *Partial I Equations and the Calculus of Variations : Essays in Honor of Ennio* Birkhaüser, Boston, to appear.