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STUDY OF A DOUBLY NONLINEAR HEAT EQUATION WITH NO GROWTH ASSUMPTIONS ON THE PARABOLIC TERM*

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Abstract. A doubly nonlinear equation with no growth assumptions on the parabolic term or on the heat flux is studied. Two existence and comparison results are established under different assumptions on the data. The technique uses truncation-penalization of the energy and energy estimates through convex conjugate functions.

Key words. heat equation, fast growing energy, nonlinear flux, energy estimates, convexity

AMS(MOS) subject classification. 35K55

Introduction. Doubly nonlinear evolution equations of the form

$$\frac{\partial b(u)}{\partial t} - \operatorname{div} A(\nabla u) = f \quad \text{on } \Omega \times (0, T),$$

$$u = 0 \quad \text{on } \partial\Omega \times (0, T),$$

$$b(u)|_{t=0} = b(u_0),$$

$$\Omega \text{ bounded domain of } \mathbb{R}^N$$

were first studied, to our knowledge, by Lions [8], Raviart [10], and Bamberger [2] in the case where

$$b(u) = |u|^{\alpha-2}u, \qquad A(w) = |w|^{p-2}w.$$

Grange and Mignot [7] address this problem in an abstract setting, namely

$$\frac{d}{dt}(Bu) + Au = f, \qquad Bu|_{t=0} = Bu_0,$$

where A and B denote the subdifferentials of the convex functions Φ and Ψ. The analysis developed in [7] is based on the essential restriction that Φ must be continuous on a Banach space V_1, and Ψ on a Banach space V_2, where V_1 is *densely and compactly embedded* in V_2. Power type nonlinearities are then restricted to satisfy

$$\frac{1}{\alpha} > \frac{1}{p} - \frac{1}{N}.$$

Furthermore A and B are assumed to be bounded on the bounded sets of V_1 and V_2 and Φ is assumed to be coercive.

Similar equations are also investigated with the help of semigroup techniques in L_1 (cf., e.g., Benilan [3]).

In this paper existence of a solution of semi-abstract equations of the form

$$\frac{\partial}{\partial t} b(u) - \operatorname{div} D\Phi(\nabla u) = f \quad \text{in } \Omega \times (0, T),$$

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$$u=0 \quad \text{on } \partial\Omega \times (0, T),$$

$$b(u)|_{t=0} = b(u_0),$$

is established in the following framework:

—Φ is a C^1 convex functional on $[L_q(\Omega)]^N$, $q>1$, with $\Phi(w) \geqq \beta(\int_\Omega |w|^q \, dx)^{r/q}$, $r>1$, $2N/(N+2)<q$.

—b is a locally Lipschitz monotone real-valued function.

Loosely speaking, there need not exist a Banach space V_2 on which b is the subdifferential of a convex *continuous* function Ψ, and, if it exists, V_1 need not be embedded in V_2. We are, for instance, in a position to solve the doubly nonlinear evolution equation with power type nonlinearities for any values of α and p (greater than one). Furthermore the function b may grow faster than any power function at infinity ($b(u)=e^{e^u}$, for example). In contrast, it need not be strictly increasing on any part of $\mathbb{R}$. Thus the evolution equation may become stationary in subdomain of $\Omega \times (0, T)$.

Similar results are given by Alt and Luckhaus [1] in a setting that includes equations of the form

$$\frac{db(u)}{\partial t} - \operatorname{div} A(\nabla u) = f,$$

where A is a monotone strongly elliptic operator on $\mathbb{R}^N$, i.e.,

$$(A(z)-A(z'), z-z')_{\mathbb{R}^N} \geqq \alpha |z-z'|^p.$$

Note that in the case when $A(w)=|w|^{p-2}w$, p is then restricted to be greater than or equal to two.

In Alt and Luckhaus [1], as well as in Grange and Mignot [7], the proof of the existence of a solution is based on a backward time difference scheme. Our method uses penalization through addition of a term of the form $\varepsilon(\partial u/\partial t)$ together with a truncation of the function b.

The detailed hypotheses on b, Φ, the initial condition $b(u_0)$, and the forcing term f are given in § 1, together with the existence results. The first result (Theorem 1) is concerned with forcing terms f in $W^{1,1}(0, T; L_2(\Omega))$ and initial conditions $b(u_0)$ in $L_2(\Omega)$ with u_0 in $W_0^{1,q}(\Omega)$. It states the existence of a solution u that also satisfies a maximum principle if f has a distinguished sign. The second result (Theorem 2) addresses the case of a forcing term f in $W^{1,1}(0, T; W^{-1,q'}(\Omega))(1/q+1/q'=1)$ and an initial condition $b(u_0)$ in $L_1^{\text{loc}}(\Omega) \cap W^{-1,q'}(\Omega)$ with u_0 in $W_0^{1,q}(\Omega)$.

Section 2 is devoted to the proof of Theorem 1 while § 3 addresses the proof of Theorem 2. The details of the different steps are briefly described at the end of § 1. It should be noted however that our proof of Theorem 2 (§ 3) is inspired by the Lemmas 1.8 and 1.9 of Alt and Luckhaus [1].

Throughout the paper, the notation $\|u\|_{m,s}$ denotes the usual Sobolev norm of u on $W^{m,s}(\Omega)$, where $W^{m,s}(\Omega)$ is the space of all $L_s(\Omega)$-functions with derivatives up to order m in $L_s(\Omega)$. Unless otherwise specified, the product $\langle \, , \rangle$ stands for the duality product between $W_0^{1,q}(\Omega)$ and $W^{-1,q'}(\Omega)$.

1. Assumptions and statement of the existence results. Let Ω be a bounded domain of $\mathbb{R}^N$ ($N \geqq 1$) with Lipschitz boundary $\partial\Omega$. Let q, r, α, and T be four real numbers satisfying

$$1 < q < +\infty,$$
$$q > \frac{2N}{N+2}, \qquad r > 1, \tag{1}$$
$$\alpha > 0, \qquad T > 0.$$

Inequalities (1) result in the following *compact* imbeddings:

$$W_0^{1,q}(\Omega) \hookrightarrow L_2(\Omega) \hookrightarrow W^{-1,q'}(\Omega). \tag{2}$$

In (2) the space $W_0^{1,q}(\Omega)$ is the subspace of all $W^{1,q}(\Omega)$-functions with null traces, whereas q' is the conjugate of q, i.e., $1/q + 1/q' = 1$.

Let b be defined as a real-valued function of the real variable with the following properties:

$$b \text{ is locally Lipschitz,}$$
$$b \text{ is monotone increasing,} \tag{3}$$
$$b(0) = 0.$$

Remark 1. The function b is *not* restricted by any growth assumption at infinity, *nor* is it assumed to be strictly increasing.

If Ψ denotes the primitive of b, i.e.,

$$\Psi(t) = \int_0^t b(s)\, ds,$$

Ψ is a positive C^1 convex function, and its convex conjugate function Ψ^*, defined as

$$\Psi^*(t) = \sup_{s \in \mathbb{R}} \{ts - \Psi(s)\},$$

satisfies, for every t of $\mathbb{R}$,

$$\Psi^*(t) \geqq 0, \qquad \Psi^*(b(t)) = b(t)t - \Psi(t). \tag{4}$$

Let Φ be defined as a real-valued functional on $[L_q(\Omega)]^N$ with the following properties:

$$\Phi \text{ is } C^1,$$
$$\Phi \text{ is convex,}$$
$$D\Phi \text{ is bounded on the bounded sets of } [L_q(\Omega)]^N, \tag{5}$$
$$\Phi(0) = 0,$$
$$\Phi(w) \geqq \alpha \|w\|_{0,q}^r \quad \text{for any } w \text{ in } [L_q(\Omega)]^N.$$

The remainder of this paper is devoted to the proof of the following theorems.

THEOREM 1. *Under the assumptions* (1), (3), *and* (5), *and if*

$$u_0 \in W_0^{1,q}(\Omega), \qquad b(u_0) \in L_2(\Omega), \tag{6}$$

$$f \in W^{1,1}(0, T; L_2(\Omega)), \tag{7}$$

the problem

$$(8)\qquad \begin{aligned} &\frac{\partial b(u)}{\partial t} - \operatorname{div} D\Phi(\nabla u) = f \quad \textit{in } \Omega \times (0, T),\\ &u = 0 \quad \textit{on } \partial\Omega \times (0, T),\\ &b(u)|_{t=0} = b(u_0), \end{aligned}$$

admits a solution u such that

$$(9)\qquad u \in L_\infty(0, T;\ W_0^{1,q}(\Omega)),$$

$$(10)\qquad b(u) \in L_\infty(0, T;\ L_2(\Omega)) \cap W^{1,\infty}(0, T;\ W^{-1,q'}(\Omega)).$$

The norm u in $L_\infty(0, T;\ W_0^{1,q}(\Omega))$ *is bounded above by a continuous function of* $\Phi(\nabla u_0)$ *and of the norm* $|||f|||$ *of f in* $W^{1,1}(0, T;\ W^{-1,q'}(\Omega))$.

Furthermore, if u_{01} *and* u_{02} *satisfy* (6); *while* f_1 *and* f_2 *satisfy* (7), *and if* $b(u_{01}) - b(u_{02})$ *is almost everywhere positive on* Ω *while* $f_1 - f_2$ *is almost everywhere positive on* $\Omega \times (0, T)$, *there exist a solution* u_1 *associated to* u_{01}, f_1 *and a solution* u_2 *associated to* u_{02}, f_2 *such that* $b(u_1) - b(u_2)$ *is almost everywhere positive on* $\Omega \times (0, T)$. □

THEOREM 2. *Under the assumption* (1), (3), *and* (5), *and if*

$$(11)\qquad u_0 \in W_0^{1,q}(\Omega), \qquad b(u_0) \in L_1^{\text{loc}}(\Omega) \cap W^{-1,q'}(\Omega),$$

$$(12)\qquad f \in W^{1,1}(0, T;\ W^{-1,q'}(\Omega)),$$

the problem

$$(13)\qquad \begin{aligned} &\frac{\partial b(u)}{\partial t} - \operatorname{div} D\Phi(\nabla u) = f \quad \textit{in } \Omega \times (0, T),\\ &u = 0 \quad \textit{on } \partial\Omega \times (0, T),\\ &b(u)|_{t=0} = b(u_0), \end{aligned}$$

admits a solution u such that

$$(14)\qquad u \in L_\infty(0, T;\ W_0^{1,q}(\Omega)),$$

$$(15)\qquad b(u) \in \mathscr{C}^0(0, T;\ L_1(\Omega)) \cap W^{1,\infty}(0, T;\ W^{-1,q'}(\Omega)).$$

Furthermore, if u_{01} *and* u_{02} *satisfy* (11) *while* f_1 *and* f_2 *satisfy* (12) *and if* $b(u_{01}) - b(u_{02})$ *is almost everywhere positive on* Ω, *while* $\langle (f_1 - f_2)(t), \varphi \rangle$ *is almost everywhere positive on* $(0, T)$ *for any* φ *in* $W_0^{1,q}(\Omega)$, *there exist a solution* u_1 *associated to* u_{01}, f_1 *and a solution* u_2 *associated to* u_{02}, f_2 *such that* $b(u_1) - b(u_2)$ *is almost everywhere positive on* $\Omega \times (0, T)$. □

Remark 2. In the settings of both theorems,

$$b(u_0)u_0 \in L_1(\Omega), \qquad b(u(t))u(t) \in L_\infty(0, T;\ L_1(\Omega)).$$

This property is trivially checked in the case of Theorem 1. It results from a theorem of Brézis and Browder [5, Thm. 1] in the case of Theorem 2. The positivity of Ψ and Ψ^* then implies that

$$\Psi(u_0), \Psi^*(b(u_0)) \in L_1(\Omega) \quad \text{and} \quad \Psi(u), \Psi^*(b(u)) \in L_\infty(0, T;\ L_1(\Omega)).$$

Furthermore in the setting of Theorem 2 the initial condition $b(u_0)$ will be shown in Remark 10 to lie in $L_1(\Omega)$, which is consistent with the continuity property of $b(u)$ with respect to time.

Remark 3. Theorem 2 provides an existence result for a class of nonlinear parabolic homogenization problems. A family A^ε of symmetric bounded measurable matrices is considered. It satisfies, for almost every x of $\mathbb{R}^N$,

$$\alpha|\xi|^2 \leqq (A^\varepsilon(x)\xi, \xi)_{\mathbb{R}^N} \leqq \beta|\xi|^2,$$

where α and β are two strictly positive real numbers.

If u_0^ε is an element of $H_0^1(\Omega)$ such that

$$b(u_0^\varepsilon) \in L_1^{\text{loc}}(\Omega) \cap H^{-1}(\Omega)$$

and if f is an arbitrary element of $W^{1,1}(0, T; H^{-1}(\Omega))$, the problem

$$\frac{\partial b(u^\varepsilon)}{\partial t} - \operatorname{div} A^\varepsilon \nabla u^\varepsilon = f \quad \text{in } \Omega,$$

$$b(u^\varepsilon)|_{t=0} = b(u_0^\varepsilon),$$

$$u^\varepsilon = 0 \quad \text{on } \partial\Omega$$

admits a solution u^ε in $L_\infty(0, T; H_0^1(\Omega))$ with $b(u^\varepsilon)$ in $L_\infty(0, T; L_1(\Omega)) \cap W^{1,\infty}(0, T; H^{-1}(\Omega))$. We assume that, as ε tends to zero,

$$u_0^\varepsilon \text{ converges weakly to } u_0^0 \text{ in } H_0^1(\Omega),$$

$$\langle b(u_0^\varepsilon), u_0^\varepsilon\rangle \text{ remains bounded independently of } \varepsilon.$$

The theory of H-convergence (Tartar [13]) ensures the existence of a subsequence $A^{\varepsilon'}$ of A^ε and of a symmetric bounded measurable matrix A^0 with

$$\alpha|\xi|^2 \leqq (A^0(x)\xi, \xi)_{\mathbb{R}^N} \leqq \beta|\xi|^2,$$

almost everywhere in $\mathbb{R}^N$, such that, as ε tends to zero,

$$A^{\varepsilon'} \text{ } H\text{-converges to } A^0.$$

It is then fairly straightforward to prove the following homogenization result: there exists a subsequence $u^{\varepsilon''}$ of u^ε such that, as ε'' tends to zero,

$$u^{\varepsilon''} \rightharpoonup u^0 \quad \text{weakly in } H_0^1(\Omega),$$

$$A^{\varepsilon''}\nabla u^{\varepsilon''} \rightharpoonup A^0\nabla u^0 \quad \text{weakly in } [L_2(\Omega)]^N,$$

where u^0 is a solution of

$$\frac{\partial b(u^0)}{\partial t} - \operatorname{div} A^0\nabla u^0 = f \quad \text{in } \Omega \times (0, T),$$

$$b(u^0)|_{t=0} = b(u_0^0),$$

$$u^0 = 0 \quad \text{on } \partial\Omega \times (0, T).$$

The proof of this result will not be presented here for the sake of brevity.

The proof of Theorem 1 is presented in § 2. It is divided into five steps to which correspond five sections. Section 2.1 is devoted to the formulation of a Galerkin approximation. To this effect, the function b is truncated and a small linear perturbation is added; $b(t)$ becomes $b^\eta(t) + \varepsilon t$, where b^η is the function resulting from the truncation of b at height $1/\eta$. In § 2.2 the limit process is performed in the Galerkin approximation. In § 2.3 the truncation height $1/\eta$ is increased to infinity. The coercivity parameter ε tends to zero in § 2.4. Finally the comparison result is derived in § 2.5.

The proof of Theorem 2 is presented in § 3. The initial condition u_0 is truncated at the height n while f is approximated by a sequence f^n in $W^{1,1}(0, T; L_2(\Omega))$. Theorem 1 is applied with the truncated u_0 as initial condition and with f^n as forcing term. The parameter n is then increased to infinity.

2. Proof of Theorem 1.

2.1. The Galerkin approximation. As previously mentioned, we introduce $b^\eta(t)$, $\Psi^\eta(t)$ to be

$$b^\eta(t)=\begin{cases} b(t) & \text{if } |b(t)|\leqq \dfrac{1}{\eta},\\ \dfrac{1}{\eta}\operatorname{sg}(t) & \text{if } |b(t)|>\dfrac{1}{\eta},\end{cases}$$

$$\Psi^\eta(t)=\int_0^t b^\eta(s)\,ds.$$

The function Ψ^η is C^1 convex and the function b^η is monotone. We propose to solve

$$(16)\qquad \begin{aligned} &\frac{\partial}{\partial t}(b^\eta(u_\varepsilon^\eta)+\varepsilon u_\varepsilon^\eta)-\operatorname{div} D\Phi(\nabla u_\varepsilon^\eta)=f \quad \text{in } \Omega\times(0,T),\\ &u_\varepsilon^\eta=0 \quad \text{on } \partial\Omega\times(0,T),\\ &b^\eta(u_\varepsilon^\eta)|_{t=0}=b^\eta(u_0),\end{aligned}$$

using a Galerkin approximation.

Let $\varphi_1,\cdots,\varphi_m,\cdots$ be a basis of $W_0^{1,q}(\Omega)$ consisting of $\mathscr{C}_0^\infty(\Omega)$-functions. If v is an arbitrary element of $W_0^{1,q}(\Omega)$, there exists a sequence V_1^m of $\mathbb{R}$ such that

$$\sum_{i=1}^m V_i^m\varphi_i \xrightarrow{m\to+\infty} v \quad \text{strongly in } W_0^{1,q}(\Omega).$$

Let m be an arbitrary but fixed integer. To any element V^m of $\mathbb{R}^m$ corresponds the element v^m of $\mathscr{C}_0^\infty(\Omega)$ defined as

$$v^m=\sum_{i=1}^m V_i^m\varphi_i.$$

The mapping is one-to-one since the φ_i are a basis of $W_0^{1,q}(\Omega)$.

Let G^m be the mapping from $\mathbb{R}^m$ into itself whose ith component is

$$[G^m(V^m)]_i=\int_\Omega (b^\eta(v^m)+\varepsilon v^m)\varphi_i\,dx.$$

If V^m and W^m are two arbitrary elements of $\mathbb{R}^m$,

$$(17)\qquad \begin{aligned}(G^m(V^m)-G^m(W^m),\,V^m-W^m)_{\mathbb{R}^m}&=\int_\Omega (b^\eta(v^m)-b^\eta(w^m))(v^m-w^m)\,dx\\ &\quad+\varepsilon\int_\Omega |v^m-w^m|^2\,dx\geqq \varepsilon\mathscr{C}_m|V^m-W^m|_{\mathbb{R}^m}^2,\end{aligned}$$

where $\mathscr{C}_m$ is a constant such that for any V^m in $\mathbb{R}^m$

$$\mathscr{C}_m|V^m|_{\mathbb{R}^m}^2\leqq\int_\Omega |v^m|^2\,dx.$$

Hence G^m is monotone coercive; it is also clearly continuous. We thus conclude with the help of Brouwer's fixed point theorem that G^m is onto (cf., for example, [8, Lemma 4.3, p. 53]). In view of (17),

$$(G^m)^{-1} \text{ is Lipschitz with Lipschitz ratio } 1/\varepsilon. \tag{18}$$

Let Φ^m be the mapping from $\mathbb{R}^m$ into itself whose ith component is

$$[\Phi^m(V^m)]_i = \int_\Omega (D\Phi(\nabla v^m), \nabla \varphi_i)_{\mathbb{R}^N}\, dx.$$

Since $D\Phi$ is continuous on $[L_q(\Omega)]^m$, Φ^m is continuous.

Finally define $F^m(t)$ to be the vector of $\mathbb{R}^m$ whose ith component is

$$[F^m(t)]_i = \int_\Omega f(t)\varphi_i\, dx.$$

By virtue of (7), $F^m(t)$ is a continuous function of t.

The continuity properties of $(G^m)^{-1}$, Φ^m, and F^m imply that the ordinary differential equation

$$\frac{dW^m}{dt}(t) + \Phi^m((G^m)^{-1}(W^m))(t) = F^m(t), \qquad W^m(0) = W_0^m, \tag{19}$$

where W_0^m is an arbitrary element of $\mathbb{R}^m$, has a local C^1 solution $W^m(t)$ on a time interval $[0, T(W_0^m))$; $T(W_0^m)$ is a strictly positive real number which depends on W_0^m.

The existence of a global solution on $[0, T]$ is ensured if $|W^m(t)|$ is proved not to blow up whenever t tends to $T(W_0^m)$ with $T(W_0^m) \leqq T$. In view of the continuity properties of G^m, it suffices to show that $V^m(t)$, defined as

$$V^m(t) = (G^m)^{-1}(W^m(t)),$$

has a bounded norm as t tends to $T(W_0^m)$.

If we set

$$V_0^m = (G^m)^{-1}(W_0^m),$$

the system (19) reads as follows:

$$\frac{d}{dt} G^m(V^m(t)) + \Phi^m(V^m(t)) = F^m(t), \qquad V^m(0) = V_0^m. \tag{20}$$

Multiplication of the first equality of (20) by $V^m(t)$ and integration over the time interval $(0, t)$ of the resulting expression leads to

$$\begin{aligned} &\int_0^t \int_\Omega \left(\frac{\partial b^\eta(v^m)}{\partial t} v^m + \varepsilon \frac{\partial v^m}{\partial t} v^m \right)(s)\, dx\, ds + \int_0^t \int_\Omega (D\Phi(\nabla v^m(s)), \nabla v^m(s))_{\mathbb{R}^N}\, dx\, ds \\ &\quad = \int_0^t \int_\Omega f(s) v^m(s)\, dx\, ds. \end{aligned} \tag{21}$$

We now appeal to the following result first established by Alt and Luckhaus [1, Lemma 1.5, p. 315].

LEMMA 1. *Let Ω be a bounded domain of $\mathbb{R}^n$. Let $\tilde{\Psi}$ be a C^1 convex function on $\mathbb{R}$, with $\tilde{b}$ as derivative ($\tilde{\Psi}(0) = 0$). Let $\tilde{\Psi}^*$ be its convex conjugate. Assume that*

$$\begin{aligned} &u \in L_\infty(0, T; W_0^{1,s}(\Omega)), \qquad 1 < s < +\infty, \\ &\tilde{b}(u) \in L_\infty(0, T; L_1(\Omega)), \\ &\frac{\partial}{\partial t}\tilde{b}(u) \in L_\infty(0, T; W^{-1,s'}(\Omega)), \qquad \frac{1}{s} + \frac{1}{s'} = 1. \end{aligned} \tag{22}$$

Assume further that there exists an element u_0 *in* $W_0^{1,s}(\Omega)$ *such that*

$$\tilde{b}(u)|_{t=0}=\tilde{b}(u_0), \tag{23}$$

and that

$$\tilde{b}(u_0)\in L_1^{\text{loc}}(\Omega)\cap W^{-1,s'}(\Omega). \tag{24}$$

Then

$$\tilde{\Psi}^*(\tilde{b}(u))\in L^\infty(0,T;L_1(\Omega)),\qquad \tilde{\Psi}^*(\tilde{b}(u_0))\in L_1(\Omega) \tag{25}$$

and, for almost any t *in* $(0, T)$,

$$\int_\Omega \tilde{\Psi}^*(b(u(t)))\,dx-\int_\Omega \tilde{\Psi}^*(\tilde{b}(u_0))\,dx=\int_0^t\left\langle\frac{\partial\tilde{b}(u(s))}{\partial t},u(s)\right\rangle ds, \tag{26}$$

where $\langle\,,\rangle$ *stands for the duality bracket between* $W_0^{1,s}(\Omega)$ *and* $W^{-1,s'}(\Omega)$. □

Lemma 1 is applied in our context with $s=q$, $\tilde{\Psi}(t)=\Psi^\eta(t)+\varepsilon(t^2/2)$, $T=t<T(W_0^m)$, $u_0=v_0^m$ and it yields

$$\begin{aligned}&\int_0^t\int_\Omega\frac{\partial}{\partial t}(b^\eta(v^m(s))+\varepsilon v^m(s))v^m(s)\,dx\,ds\\&\quad=\int_\Omega(\Psi^\eta)^*(b^\eta(v^m(t)))\,dx+\frac{\varepsilon}{2}\|v^m(t)\|_{0,2}^2-\frac{\varepsilon}{2}\|v_0^m\|_0^2-\int_\Omega(\psi^\eta)^*(b^\eta(v_0^m))\,dx.\end{aligned} \tag{27}$$

Remark 4. In view of the regularity of the function v^m relation (27) can be established independently of Lemma 1. At a later stage of this study however, Lemma 1 will become an essential ingredient and it will be repeatedly applied with $s=q$.

Inserting (27) into (21) leads to

$$\begin{aligned}&\int_\Omega(\Psi^\eta)^*(b^\eta(v^m(t)))\,dx+\frac{\varepsilon}{2}\|v^m(t)\|_{0,2}^2+\int_0^t\int_\Omega(D\Phi(\nabla v^m(s)),\nabla v^m(s))_{\mathbb{R}^N}\,dx\,ds\\&\quad\leqq\int_\Omega(\Psi^\eta)^*(b^\eta(v_0^m))\,dx+\frac{\varepsilon}{2}\|v_0^m\|_{0,2}^2+|||f|||\int_0^t\|v^m(s)\|_{1,q}\,ds,\end{aligned} \tag{28}$$

where from now on $|||f|||$ stands for the norm of f in $W^{1,1}(0,T;W^{-1,q'}(\Omega))$.

Since

$$\Phi(0)=0,$$

the coercivity property of Φ (cf. (5)) together with Poincaré's inequality imply that

$$\int_0^t\int_\Omega(D\Phi(\nabla v^m(s)),\nabla v^m(s))_{\mathbb{R}^N}\,dx\,ds\geqq\int_0^t\Phi(\nabla v^m(s))\,ds\geqq\alpha\int_0^t\|v^m(s)\|_{1,q}^r\,ds. \tag{29}$$

Because $(\Psi^\eta)^*$ is always positive, insertion of (25) and (29) into (28) yields

$$\frac{\varepsilon}{2}\|v^m(t)\|_{0,2}^2+\alpha\int_0^t\|v^m(s)\|_{1,q}^r\,ds\leqq\mathscr{C}+|||f|||\int_0^t\|v^m(s)\|_{1,q}\,ds, \tag{30}$$

where $\mathscr{C}$ denotes a generic constant.

Since r is strictly greater than one, (30) implies that $\|v^m(t)\|_{0,2}$ remains bounded on $[0,T(W_0^m))$ and thus that $|V^m(t)|$ remains bounded as t tends to $T(W_0^m)$, which was the result sought.

Recalling (6), we denote by U_0^m the vector of $\mathbb{R}^m$ associated with the projection u_0^m of u_0 on the span of $\varphi_1, \cdots, \varphi_m$, i.e.,

$$u_0^m = \sum_{i=1}^m U_{0i}^m \varphi_i \xrightarrow{m \to +\infty} u_0 \quad \text{strongly in } W_0^{1,q}(\Omega).$$

According to the previous analysis, the equation

$$\frac{d}{dt} G^m(U^m(t)) + \Phi^m(U^m(t)) = F^m(t), \qquad U^m(0) = U_0^m, \tag{31}$$

admits a global Lipschitz solution on $[0, T]$.

A priori estimates. Multiplication of the first equality of (31) by dU^m/dt and integration over the time interval $(0, t)$ of the resulting expression leads to

$$\begin{aligned} &\int_0^t \int_\Omega \frac{\partial b^\eta(u^m(s))}{\partial t} \frac{\partial u^m(s)}{\partial t} \, dx \, ds + \varepsilon \int_0^t \int_\Omega \left| \frac{\partial u^m(s)}{\partial t} \right|^2 dx \, ds + \Phi(\nabla u^m(t)) \\ &\quad = \Phi(\nabla u_0^m) + \int_0^t \int_\Omega f(s) \frac{\partial u^m(s)}{\partial t} \, dx \, ds, \end{aligned} \tag{32}$$

where

$$u^m(t) = \sum_{i=1}^m U_i^m(t) \varphi_i.$$

The function b^η is Lipschitz and u^m is in $\mathrm{Lip}(\Omega \times (0, T))$; thus, by virtue of the monotonicity of b^η,

$$\int_0^t \int_\Omega \frac{\partial b^\eta(u^m(s))}{\partial t} \frac{\partial u^m(s)}{\partial t} \, dx \, ds = \int_0^t \int_\Omega (b^\eta)'(u^m(s)) \left(\frac{\partial u^m(s)}{\partial t} \right)^2 dx \, ds \geqq 0. \tag{33}$$

The coercivity property of Φ (cf. (5)), (in)equalities (32), (33), and Poincaré's inequality imply that

$$\int_0^t \left\| \sqrt{\varepsilon} \frac{\partial u^m}{\partial t} \right\|_{0,2}^2 ds + \alpha \| u^m(t) \|_{1,q}^r \leqq \Phi(\nabla u_0^m) + \int_0^t \int_\Omega f(s) \frac{\partial u^m(s)}{\partial t} \, dx \, ds. \tag{34}$$

The last term of the right-hand side of inequality (34) is integrated by parts with the help of (7). We obtain

$$\int_0^t \left\| \sqrt{\varepsilon} \frac{\partial u^m}{\partial t} \right\|_{0,2}^2 ds + \alpha \| u^m(t) \|_{1,q}^r \leqq \Phi(\nabla u_0^m) + 3 |||f||| \sup_{s \in [0,t]} \| u^m(s) \|_{1,q}. \tag{35}$$

The time t can be chosen arbitrarily in $[0, T]$; thus,

$$\begin{aligned} &\int_0^T \left\| \sqrt{\varepsilon} \frac{\partial u^m(s)}{\partial t} \right\|_{0,2}^2 ds + \alpha \sup_{t \in [0,T]} \| u^m(t) \|_{1,q}^r \\ &\quad \leqq \Phi(\nabla u_0^m) + 3 |||f||| \sup_{t \in [0,T]} \| u^m(t) \|_{1,q}. \end{aligned} \tag{36}$$

Finally ∇u_0^m converges to ∇u_0 in $[L_q(\Omega)]^N$ as m goes to infinity. The continuity of Φ on $[L_q(\Omega)]^N$ implies that

$$\Phi(\nabla u_0^m) \xrightarrow{m \to +\infty} \Phi(\nabla u^0).$$

If m is taken to be large enough, (36) reads as follows:

$$\int_0^T \left\|\sqrt{\varepsilon}\,\frac{\partial u^m}{\partial t}\right\|_{0,2}^2 ds + \alpha \sup_{t\in[0,T]} \|u^m(t)\|_{1,q}^r \leqq \Phi(\nabla u_0) + 1 + 3|||f||| \sup_{t\in[0,T]} \|u^m(t)\|_{1,q}. \tag{37}$$

The function $\|u^m(t)\|_{1,q}$ is continuous on $[0, T]$; it reaches its supremum on $[0, T]$. It is then easily deduced from (37) that

(38) $\sqrt{\varepsilon}\,\partial u^m/\partial t$ is bounded in $L_2(0, T; L_2(\Omega))$ independently of m, η, or ε,

(39) $u^m(t)$ is bounded in $L_\infty(0, T; W_0^{1,q}(\Omega))$ independently of m, η, or ε.

Because $D\Phi$ is bounded on the bounded sets of $[L_q(\Omega)]^N$, (39) implies that, if $1/q' + 1/q = 1$,

(40) $D\Phi(\nabla u^m)$ is bounded in $L_\infty(0, T; [L_{q'}, (\Omega)]^N)$ independently of m, η, or ε.

Remark 5. The bounds in (38)-(40) continuously depend on $\Phi(\nabla u_0)$ and $|||f|||$ only (recall that $|||f|||$ is the $W^{1,1}(0, T; W^{-1,q'}(\Omega))$-norm of f).

Finally, since b^η is bounded on $\mathbb{R}$,

(41) $b^\eta(u^m)$ is bounded in $L_\infty((0, T)\times\Omega)$,

but the bound depends on η.

With the help of (38)-(41) we conclude that there exists a suitably extracted subsequence of u^m (still denoted u^m) such that, as m tends to infinity,

$$\begin{aligned} &u^m \rightharpoonup u_\varepsilon^\eta \quad \text{weak-* in } L_\infty(0, T; W_0^{1,q}(\Omega)),\\ &\sqrt{\varepsilon}\,\frac{\partial u^m}{\partial t} \rightharpoonup \sqrt{\varepsilon}\,\frac{\partial u_\varepsilon^\eta}{\partial t} \quad \text{weakly in } L_2(0, T; L_2(\Omega)),\\ &D\Phi(\nabla u^m) \rightharpoonup Y_\varepsilon^\eta \quad \text{weak-* in } L_\infty(0, T; [L_{q'}(\Omega)]^N),\\ &b^\eta(u^m) \rightharpoonup \chi_\varepsilon^\eta \quad \text{weak-* in } L_\infty((0, T)\times\Omega). \end{aligned} \tag{42}$$

In the following section we propose to use (42) to pass to the limit in (31) as m tends to infinity, and to identify the quantities Y_ε^η and χ_ε^η.

2.2. Passing to the limit in the Galerkin approximation. Let ζ be an arbitrary element of $\mathscr{C}_0^\infty((0, T))$. Equation (31) together with the convergence (42) imply that, for any integer i,

$$\left\langle\!\!\left\langle \frac{\partial \chi_\varepsilon^\eta}{\partial t} + \varepsilon\frac{\partial u_\varepsilon^\eta}{\partial t} - \operatorname{div} Y_\varepsilon^\eta - f, \varphi_i\zeta \right\rangle\!\!\right\rangle = 0,$$

where the duality bracket refers to the duality between $\mathscr{C}_0^\infty((0, T)\times\Omega)$ and $\mathscr{D}'((0, T)\times\Omega)$ (the basis vectors φ_i lie in $\mathscr{C}_0^\infty(\Omega)$).

The sequence $\{\varphi_i\}$ is a basis of $W_0^{1,q}(\Omega)$ which contains $\mathscr{C}_0^\infty(\Omega)$. Thus, if φ is an arbitrary element of $\mathscr{C}_0^\infty(\Omega)$,

$$\left\langle\!\!\left\langle \frac{\partial \chi_\varepsilon^\eta}{\partial t} + \varepsilon\frac{\partial u_\varepsilon^\eta}{\partial t} - \operatorname{div} Y_\varepsilon^\eta - f, \varphi\zeta \right\rangle\!\!\right\rangle = 0,$$

and, by the density of $\mathscr{C}_0^\infty((0, T))\times\mathscr{C}_0^\infty(\Omega)$ in $\mathscr{C}_0^\infty((0, T)\times\Omega)$,

$$\frac{\partial \chi_\varepsilon^\eta}{\partial t} + \varepsilon\frac{\partial u_\varepsilon^\eta}{\partial t} - \operatorname{div} Y_\varepsilon^\eta - f = 0. \tag{43}$$

In view of (43), χ_ε^η has a trace in $W^{-1,q'}(\Omega)$. A proper choice of ζ in $\mathscr{C}_0^\infty([0, T))$ and convergence (42) lead to

$$\langle\!\langle \chi_\varepsilon^\eta(0)+\varepsilon u_\varepsilon^\eta(0)\varphi_i\zeta(0)\rangle\!\rangle=\lim_{m\to+\infty}\int_\Omega (b^\eta(u_0^m)+\varepsilon u_0^m)\varphi_i\zeta(0).$$

But $\{\varphi_i\}$ is a basis of $W_0^{1,q}(\Omega)$ and u_0^m converges strongly to u_0 in $W_0^{1,q}(\Omega)$; thus

$$\chi_\varepsilon^\eta(0)+\varepsilon u_\varepsilon^\eta(0)=b^\eta(u_0)+\varepsilon u_0. \tag{44}$$

Because of the estimates on u^m and $\partial u^m/\partial t$ (cf. (42))

$$u^m(0)\rightharpoonup u_\varepsilon^\eta(0) \quad \text{weakly in } L_2(\Omega)$$

as m tends to infinity; thus

$$u_\varepsilon^\eta(0)=u_0, \tag{45}$$

and (44) yields

$$\chi_\varepsilon^\eta(0)=b^\eta(u_0). \tag{46}$$

We now seek to identify the quantities χ_ε^η and Y_ε^η. The identification of χ_ε^η is very simple. A straightforward application of Aubin's lemma (cf., e.g., [12, Cor. 4]) to u^m implies that

$$u^m\to u_\varepsilon^\eta \quad \text{strongly in } \mathscr{C}^0([0, T]; L_2(\Omega))$$

as m tends to infinity. Since b^η is Lipschitz and bounded, it follows that for any finite s

$$b^\eta(u^m)\to b^\eta(u_\varepsilon^\eta) \quad \text{strongly in } \mathscr{C}^0([0, T]; L_s(\Omega))$$

as m tends to infinity and, in view of (42), that

$$\chi_\varepsilon^\eta=b^\eta(u_\varepsilon^\eta), \qquad b^\eta(u_\varepsilon^\eta)\in\mathscr{C}^0([0, T]; L_s(\Omega)), \qquad 1\leqq s<+\infty. \tag{47}$$

The identification of Y_ε^η is performed with the help of the following simple lemma.

LEMMA 2. *Assume that* Φ *satisfies* (5) *and that* w_m *is a sequence of* $L_\infty(0, T; [L_q(\Omega)]^N)$ *such that*

$$\begin{aligned} &w_m\rightharpoonup w \quad \textit{weak-* in } L_\infty(0, T; [L_q(\Omega)]^N),\\ &D\Phi(w_m)\rightharpoonup Y \quad \textit{weak-* in } L_\infty(0, T; [L_{q'}(\Omega)]^N), \end{aligned} \tag{48}$$

as m tends to infinity (or zero). If

$$\int_0^T\int_0^t\int_\Omega (Y(s), w(s))_{\mathbb{R}^N}\,dx\,ds\,dt\geqq\overline{\lim}\int_0^T\int_0^t\int_\Omega (D\Phi(w_m(s), w_m(s)))\,dx\,ds\,dt, \tag{49}$$

then

$$Y=D\Phi(w). \tag{50}$$

The proof of this lemma is a straightforward adaptation of a classical result of Minty (cf., e.g., [8, Remark 2.1, p. 173 and Prop. 2.5, p. 179]). It will not be reproduced here.

In our setting w_m, w, and Y are to be identified with ∇u^m, $\nabla u_\varepsilon^\eta$, and Y_ε^η, respectively. In view of (42), (48) is satisfied. To show that

$$Y_\varepsilon^\eta=D\Phi(\nabla u_\varepsilon^\eta) \tag{51}$$

we only need to prove that

$$\int_0^T \int_0^t \int_\Omega (Y_\varepsilon^\eta(s), \nabla u_\varepsilon^\eta(s))_{\mathbb{R}^N}\, dx\, ds\, dt \geqq \overline{\lim_{m\to+\infty}} \int_0^T \int_0^t \int_\Omega (D\Phi(\nabla u^m(s)), \nabla u^m(s))_{\mathbb{R}^N}\, dx\, ds\, dt. \tag{52}$$

According to (21) and (27) (applied to u^m in place of v^m), we have

$$\begin{aligned}\int_0^T \int_0^t \int_\Omega &(D\Phi(\nabla u^m(s)), \nabla u^m(s))_{\mathbb{R}^N}\, dx\, ds\, dt \\ &= \int_0^T \int_0^t \int_\Omega f(s) u^m(s)\, dx\, ds\, dt + T\left[\int_\Omega (\Psi^\eta)^*(b^\eta(u_0^m))\, dx + \frac{\varepsilon}{2}\|u_0^m\|_{0,2}^2\right] \\ &\quad - \int_0^T \left[\int_\Omega (\Psi^\eta)^*(b^\eta(u^m(t)))\, dx + \frac{\varepsilon}{2}\|u^m(t)\|_{0,2}^2\right] dt.\end{aligned} \tag{53}$$

Note that Ψ^η is Lipschitz and that (4) holds for b^η, Ψ^η, and $(\Psi^\eta)^*$ in place of b, Ψ, and Ψ^*, respectively. Since u^m converges weak-* to u_ε^η in $L_\infty(0, T; W_0^{1,q}(\Omega))$ and u_0^m converges strongly in $W_0^{1,q}(\Omega)$ to u_0, the two first terms of the right-hand side of equality (53) are easily seen to converge to

$$\int_0^T \int_0^t \int_\Omega f(s) u_\varepsilon^\eta(s)\, dx\, ds\, dt + T\left[\int_\Omega (\Psi^\eta)^*(b^\eta(u_0))\, dx + \frac{\varepsilon}{2}\|u_0\|_{0,2}^2\right]$$

as m goes to infinity.

The strong convergences of the sequences $b^\eta(u^m)$ and u^m in $\mathscr{C}^0([0, T]; L_2(\Omega))$ imply that

$$\lim_{m\to+\infty} -\int_0^T \int_\Omega (\Psi^\eta)^*(b^\eta(u^m(t)))\, dx\, dt = -\int_0^T \int_\Omega (\Psi^\eta)^*(b^\eta(u_\varepsilon^\eta(t)))\, dx\, dt$$

and

$$\lim_{m\to+\infty} \int_0^T \|u^m(t)\|_{0,2}^2\, dt = \int_0^T \|u_\varepsilon^\eta(t)\|_{0,2}^2\, dt.$$

We are now in a position to pass to the limit in (53). We obtain the following:

$$\begin{aligned}\lim_{m\to+\infty} \int_0^T \int_0^t \int_\Omega &(D\Phi(\nabla u^m(s)), \nabla u^m(s))_{\mathbb{R}^N}\, dx\, ds\, dt \\ &= \int_0^T \int_0^t \int_\Omega f(s) u_\varepsilon^\eta(s)\, dx\, ds\, dt + T\left[\int_\Omega (\Psi^\eta)^*(b^\eta(u_0))\, dx + \frac{\varepsilon}{2}\|u_0\|_{0,2}^2\right] \\ &\quad - \int_0^T \left[\int_\Omega (\Psi^\eta)^*(b^\eta(u_\varepsilon^\eta(t)))\, dx + \frac{\varepsilon}{2}\|u_\varepsilon^\eta(t)\|_{0,2}^2\right] dt.\end{aligned} \tag{54}$$

Multiplication of (43) by u_ε^η and integration of the resulting expression over $(0, t)\times\Omega$, then over $(0, T)$ yields, with the help of (47),

$$\begin{aligned}\int_0^T \int_0^t \int_\Omega &(Y_\varepsilon^\eta(s), \nabla u_\varepsilon^\eta(s))_{\mathbb{R}^N}\, dx\, ds\, dt \\ &= \int_0^T \int_0^t \int_\Omega f(s) u_\varepsilon^\eta(s)\, dx\, ds\, dt \\ &\quad - \int_0^T \int_0^t \left\langle \frac{\partial}{\partial t}(b^\eta(u_\varepsilon^\eta(s)) + \varepsilon u_\varepsilon^\eta(s)), u_\varepsilon^\eta(s)\right\rangle ds\, dt.\end{aligned} \tag{55}$$

The last term in the right-hand side of equality (55) is evaluated with the help of Lemma 1 (cf. Remark 4). We obtain the following:

$$\int_0^T \int_0^t \left\langle \frac{\partial}{\partial t}(b^\eta(u_\varepsilon^\eta(s)) + \varepsilon u_\varepsilon^\eta(s)), u_\varepsilon^\eta(s) \right\rangle ds\, dt$$

$$= \int_0^T \int_\Omega (\Psi^\eta)^*(b^\eta(u_\varepsilon^\eta(t)))\, dx\, dt \tag{56}$$

$$+\frac{\varepsilon}{2}\int_0^T \|u_\varepsilon^\eta(t)\|_{0,2}^2\, dt - T\frac{\varepsilon}{2}\|u_0\|_{0,2}^2 - T\int_\Omega (\Psi^\eta)^*(b^\eta(u_0))\, dx.$$

Inserting (56) into (55) and comparing the resulting expression with the right-hand side of inequality (54) yields (52), which in turn proves (51).

At this stage of the proof we have constructed a sequence u_ε^η with the following properties:

$$u_\varepsilon^\eta \in L_\infty(0, T;\ W_0^{1,q}(\Omega)), \qquad b^\eta(u_\varepsilon^\eta) \in L_\infty((0, T)\times\Omega),$$

$$\frac{\partial b^\eta(u_\varepsilon^\eta)}{\partial t} + \varepsilon \frac{\partial u_\varepsilon^\eta}{\partial t} - \operatorname{div} D\Phi(\nabla u_\varepsilon^\eta) = f, \tag{57}$$

$$u_\varepsilon^\eta(0) = u_0, \tag{58}$$

$$b^\eta(u_\varepsilon^\eta)|_{t=0} = b^\eta(u_0). \tag{59}$$

Furthermore the weak lower semicontinuity properties of the L_2, $L_{q'}$, and L_q norms imply, by virtue of (38)-(40) and (42), that

(60) $\sqrt{\varepsilon}\, \partial u_\varepsilon^\eta/\partial t$ is bounded in $L_2(0, T; L_2(\Omega))$ independently of ε and η,

(61) u_ε^η is bounded in $L_\infty(0, T; W_0^{1,q}(\Omega))$ independently of ε and η,

(62) $D\Phi(\nabla u_\varepsilon^\eta)$ is bounded in $L_\infty(0, T; [L_{q'}(\Omega)]^N)$ independently of ε and η.

With the help of (60)-(62) we conclude that there exists a suitably extracted subsequence (still denoted u_ε^η) such that, for fixed ε,

$$u_\varepsilon^\eta \rightharpoonup u_\varepsilon \quad \text{weak-* in } L_\infty(0, T;\ W_0^{1,q}(\Omega)),$$

$$\sqrt{\varepsilon}\,\frac{\partial u_\varepsilon^\eta}{\partial t} \rightharpoonup \sqrt{\varepsilon}\,\frac{\partial u_\varepsilon}{\partial t} \quad \text{weakly in } L_2(0, T;\ L_2(\Omega)), \tag{63}$$

$$D\Phi(\nabla u_\varepsilon^\eta) \rightharpoonup Y_\varepsilon \quad \text{weak-* in } L_\infty(0, T;\ [L_{q'}(\Omega)]^N),$$

as η goes to zero.

In the following section we propose to use (63) to pass to the limit in (57)-(59) as η tends to zero, and to identify the quantity Y_ε. To this effect we need to derive an estimate on $b^\eta(u_\varepsilon^\eta)$ independent of η (and ε) and to identify its weak limit.

2.3. Passing to the limit in the truncation. The quantities $\Psi^\eta(u_\varepsilon^\eta)$, $b^\eta(u_\varepsilon^\eta)$ lie in $L_\infty(0, T; W_0^{1,q}(\Omega)) \cap W^{1,2}(0, T; L_2(\Omega))$ because Ψ^η and b^η are Lipschitz. Thus, $b^\eta(u_\varepsilon^\eta)$ is an admissible test function in (57). Upon integration over the time interval $(0, t)$ of the result of the multiplication of (57) by $b^\eta(u_\varepsilon^\eta)$, we obtain

$$\frac{1}{2}\|b^\eta(u_\varepsilon^\eta(t))\|_{0,2}^2 + \varepsilon\int_\Omega \Psi^\eta(u_\varepsilon^\eta(t))\, dx$$

$$+\int_0^t \int_\Omega (b^\eta)'(u_\varepsilon^\eta(s))(D\Phi(\nabla u_\varepsilon^\eta(s)), \nabla u_\varepsilon^\eta(s))_{\mathbb{R}^N}\, dx\, ds \tag{64}$$

$$= \frac{1}{2}\|b^\eta(u_0)\|_{0,2}^2 + \varepsilon\int_\Omega \Psi^\eta(u_0)\, dx + \int_0^t\int_\Omega f(s) b^\eta(u_\varepsilon^\eta(s))\, dx\, ds.$$

The derivation formula for the composition of a $W^{1,q}$ function by a Lipschitz function is implicitly used in (64). It is classical if the Lipschitz function is piecewise C^1. For a proof in the case of an arbitrary Lipschitz function see, for example, [9, Cor. 1.3, p. 353] or [4, Thm. 4.3].

Since b^η is monotone and Ψ^η is positive, (64) yields

$$\begin{aligned}\frac{1}{2}\|b^\eta(u_\varepsilon^\eta(t))\|_{0,2}^2 &\leqq \frac{1}{2}\|b^\eta(u_0)\|_{0,2}^2+\varepsilon\int_\Omega \Psi^\eta(u_0)\,dx \\ &\quad + T\sup_{t\in[0,T]}\|f(t)\|_{0,2}\sup_{t\in[0,T]}\|b^\eta(u_\varepsilon^\eta(t))\|_{0,2}.\end{aligned}\tag{65}$$

Remark 6. The L_2 space regularity of f is required in estimate (65), and it motivates the regularity hypothesis (7) on f.

As η tends to zero, $b^\eta(u_0)$ and $\Psi^\eta(u_0)$ converge almost everywhere to $b(u_0)$ and $\Psi(u_0)$, respectively, while

$$|b^\eta(u_0(x))|\leqq|b(u_0(x))|,\qquad \Psi^\eta(u_0(x))\leqq\Psi(u_0(x)),$$

for almost every x of Ω. By hypothesis, $b(u_0)$ belongs to $L_2(\Omega)$ (see (6)). In view of (4) and the positivity of Ψ,

$$0\leqq\Psi(u_0(x))\leqq b(u_0(x))u_0(x)$$

for almost every x of Ω, and thus $\Psi(u_0)$ belongs to $L_1(\Omega)$.

The dominated convergence theorem permits us to conclude that

$$\begin{aligned}b^\eta(u_0)&\to b(u_0)\qquad \text{strongly in } L_2(\Omega),\\ \Psi^\eta(u_0)&\to\Psi(u_0)\qquad \text{strongly in } L_1(\Omega),\end{aligned}\tag{66}$$

and thus to give an upper bound for the right-hand side of (65). We obtain, for η small enough,

$$\begin{aligned}\frac{1}{2}\|b^\eta(u_\varepsilon^\eta(t))\|_{0,2}^2 &\leqq \frac{1}{2}\|b(u_0)\|_{0,2}^2+\varepsilon\int_\Omega \Psi(u_0)\,dx+1 \\ &\quad + T\sup_{t\in[0,T]}\|f(t)\|_{0,2}\sup_{t\in[0,T]}\|b^\eta(u_\varepsilon^\eta(t))\|_{0,2}.\end{aligned}\tag{67}$$

Since $\|b^\eta(u_\varepsilon^\eta(t))\|_{0,2}$ is continuous on $[0, T]$ (cf. (47)), an argument similar to the one that led to (38), (39), would show that

(68) $b^\eta(u_\varepsilon^\eta)$ is bounded in $L_\infty(0, T, L_2(\Omega))$ independently of ε and η.

At the possible expense of extracting one more subsequence, we are thus at liberty to assume that the sequence u_ε^η is also such that

$$b^\eta(u_\varepsilon^\eta)\rightharpoonup\chi_\varepsilon\quad\text{weak-* in } L_\infty(0, T; L_2(\Omega))\tag{69}$$

as η tends to zero.

Passing to the limit in (57)-(59) is now an immediate task if we remark that, in view of (57), (60), and (62),

(70) $\partial b^\eta(u_\varepsilon^\eta)/\partial t$ is bounded in $L_2(0, T; W^{-1,q'}(\Omega))$.

We obtain the following:

$$\frac{\partial\chi_\varepsilon}{\partial t}+\varepsilon\frac{\partial u_\varepsilon}{\partial t}-\operatorname{div} Y_\varepsilon=f,\tag{71}$$

and, with the help of (60), (61), (68), and (70),

$$u_\varepsilon(0) = u_0, \tag{72}$$

$$\chi_\varepsilon\big|_{t=0} = b(u_0). \tag{73}$$

It now remains to identify χ_ε and Y_ε. Once again, the identification of χ_ε directly results from Aubin's lemma. In view of (63),

$$u_\varepsilon^\eta \to u_\varepsilon \quad \text{strongly in } \mathscr{C}^0([0, T]; L_2(\Omega))$$

as η tends to zero. Since b^η converges pointwise to b on $\mathbb{R}$ as η tends to zero, a subsequence of $b^\eta(u_\varepsilon^\eta)$ (still denoted $b^\eta(u_\varepsilon^\eta)$) satisfies

$$b^\eta(u_\varepsilon^\eta(x, t)) \to b(u_\varepsilon(x, t)) \text{ for almost every } (x, t) \text{ of } \Omega \times (0, T)$$

as η tends to zero. Recalling (69) then implies that

$$\chi_\varepsilon = b(u_\varepsilon), \qquad b(u_\varepsilon) \in L_\infty(0, T; L_2(\Omega)). \tag{74}$$

The identification of Y_ε relies on Lemma 2. The quantities w_m, w, and Y are identified with $\nabla u_\varepsilon^\eta$, ∇u_ε, and Y_ε, respectively. In view of (63), (48) is satisfied. To show that

$$Y_\varepsilon = D\Phi(\nabla u_\varepsilon), \tag{75}$$

we only need to prove that

$$\begin{aligned} &\int_0^T \int_0^t \int_\Omega (Y_\varepsilon(s), \nabla u_\varepsilon(s))_{\mathbb{R}^N}\, dx\, ds\, dt \\ &\qquad \geqq \varlimsup_{\eta \to 0} \int_0^T \int_0^t \int_\Omega (D\Phi(\nabla u_\varepsilon^\eta(s)), \nabla u_\varepsilon^\eta(s))_{\mathbb{R}^N}\, dx\, ds\, dt. \end{aligned} \tag{76}$$

The proof of (76) is essentially the same as that of (52). It consists in passing to the limit in (55), (56) as η tends to zero (with Y_ε^η replaced by $D\Phi(\nabla u_\varepsilon^\eta)$). We obtain the following:

$$\begin{aligned} &\varlimsup_{\eta \to 0} \int_0^T \int_0^t \int_\Omega (D\Phi(\nabla u_\varepsilon^\eta(s)), \nabla u_\varepsilon^\eta(s))_{\mathbb{R}^N}\, dx\, ds\, dt \\ &\quad \leqq \int_0^T \int_0^t \int_\Omega f(s) u_\varepsilon(s)\, dx\, ds\, dt + T\frac{\varepsilon}{2}\|u_0\|_{0,2}^2 - \frac{\varepsilon}{2}\int_0^T \|u_\varepsilon(t)\|_{0,2}^2\, dt \\ &\qquad - \varliminf_{\eta \to 0} \int_0^T \int_\Omega (\Psi^\eta)^*(b^\eta(u_\varepsilon^\eta(t)))\, dx\, dt + T \lim_{\eta \to 0} \int_\Omega (\Psi^\eta)^*(b^\eta(u_0))\, dx. \end{aligned} \tag{77}$$

Since

$$(\Psi^\eta)^*(b^\eta(u_0)) = u_0 b^\eta(u_0) - \Psi^\eta(u_0),$$

convergences (66) imply that

$$\lim_{\eta \to 0} \int_\Omega (\Psi^\eta)^*(b^\eta(u_0))\, dx = \int_\Omega (u_0 b(u_0) - \Psi(u_0))\, dx = \int_\Omega \Psi^*(b(u_0))\, dx. \tag{78}$$

Similarly, since u_ε^η and $b^\eta(u_\varepsilon^\eta)$ converge almost everywhere to u_ε and $b(u_\varepsilon)$, respectively,

$$(\Psi^\eta)^*(b^\eta(u_\varepsilon^\eta(x, t))) \to \Psi^*(b(u_\varepsilon(x, t)))$$

for almost any (x, t) in $\Omega\times(0, T)$ as η tends to zero. The positivity of $(\Psi^\eta)^*$ together with Fatou's lemma yield

$$\underline{\lim}_{\eta\to 0}\int_0^T\int_\Omega (\Psi^\eta)^*(b^\eta(u_\varepsilon^\eta(t)))\,dx\,dt \geqq \int_0^T\int_\Omega \Psi^*(b(u_\varepsilon(t)))\,dx\,dt. \tag{79}$$

Insertion of (78) and (79) into (77) leads to

$$\begin{aligned}\overline{\lim}_{\eta\to 0}&\int_0^T\int_0^t\int_\Omega (D\Phi(\nabla u_\varepsilon^\eta(s)), \nabla u_\varepsilon^\eta(s))_{\mathbb{R}^N}\,dx\,ds\,dt\\ &\leqq \int_0^T\int_0^t\int_\Omega f(s)u_\varepsilon(s)\,dx\,ds\,dt + T\left[\int_\Omega \Psi^*(b(u_0))\,dx + \frac{\varepsilon}{2}\|u_0\|_{0,2}^2\right]\\ &\quad -\int_0^T\left[\int_\Omega \Psi^\eta(b(u_\varepsilon(t)))\,dx + \frac{\varepsilon}{2}\|u_\varepsilon(t)\|_{0,2}^2\right]dt.\end{aligned} \tag{80}$$

Multiplication of (71) by u_ε, integration of the resulting expression over $(0, t)\times\Omega$ then over $(0, T)$, and application of Lemma 1 readily shows that the right-hand side of inequality (80) is precisely $\int_0^T\int_0^t\int_\Omega (Y_\varepsilon(s), \nabla u_\varepsilon(s))_{\mathbb{R}^N}\,dx\,ds\,dt$, which proves (76) and thus (75).

We have constructed a sequence u_ε with the following properties:

$$u_\varepsilon \in L_\infty(0, T;\, W_0^{1,q}(\Omega)), \qquad b(u_\varepsilon)\in L_\infty(0, T;\, L_2(\Omega)),$$

$$\frac{\partial b(u_\varepsilon)}{\partial t} + \varepsilon\frac{\partial u_\varepsilon}{\partial t} - \operatorname{div} D\Phi(\nabla u_\varepsilon) = f, \tag{81}$$

$$u_\varepsilon(0) = u_0, \tag{82}$$

$$b(u_\varepsilon)|_{t=0} = b(u_0). \tag{83}$$

Once again, the weak lower semicontinuity properties of the L_2, $L_{q'}$, and L_q norms imply, by virtue of (60)-(63), (68), (69), (74), and (75) that

(84) $\sqrt{\varepsilon}\,\partial u_\varepsilon/\partial t$ is bounded in $L_2(0, T;\, L_2(\Omega))$ independently of ε,

(85) u_ε is bounded in $L_\infty(0, T;\, W_0^{1,q}(\Omega))$ independently of ε,

(86) $\Phi(\nabla u_\varepsilon)$ is bounded in $L_\infty(0, T;\, [L_{q'}(\Omega)]^N)$ independently of ε,

(87) $b(u_\varepsilon)$ is bounded in $L_\infty(0, T;\, L_2(\Omega))$ independently of ε.

With the help of (84)-(87), we conclude that there exists a suitably extracted subsequence (still denoted u_ε) such that

$$\begin{aligned} &u_\varepsilon \rightharpoonup u \quad \text{weak-* in } L_\infty(0, T;\, W_0^{1,q}(\Omega)),\\ &\sqrt{\varepsilon}\,\frac{\partial u_\varepsilon}{\partial t} \rightharpoonup 0 \quad \text{weakly in } L_2(0, T;\, L_2(\Omega)),\\ &D\Phi(\nabla u_\varepsilon) \rightharpoonup Y \quad \text{weak-* in } L_\infty(0, T;\, [L_{q'}(\Omega)]^N),\\ &b(u_\varepsilon) \rightharpoonup \chi \quad \text{weak-* in } L_\infty(0, T;\, L_2(\Omega))\end{aligned} \tag{88}$$

as ε tends to zero.

In the following section, (88) is used to pass to the limit in (81)-(83) as ε tends to zero, and the quantities χ and Y are identified.

2.4. Passing to the limit in the coercivity parameter. By virtue of (81), (84), and (86)

$$\frac{\partial b(u_\varepsilon)}{\partial t} \text{ is bounded in } L_2(0, T;\ W^{-1,q'}(\Omega)). \tag{89}$$

In view of (88) and (89), the limit of (81) as ε tends to zero is

$$\frac{\partial \chi}{\partial t} - \operatorname{div} Y = f, \tag{90}$$

while that of (83) is

$$\chi|_{t=0} = b(u_0). \tag{91}$$

Remark 7. The initial value of u_ε, i.e., u_0, is lost in the limiting process because of the absence of estimates on $\partial u_\varepsilon/\partial t$ that are independent of ε.

The absence of an estimate on $\partial u_\varepsilon/\partial t$ precludes the application of Aubin's lemma to u_ε. However, that lemma can be applied to $b(u_\varepsilon)$ because of (87) and (89); it implies that, as ε tends to zero,

$$b(u_\varepsilon) \to \chi \quad \text{strongly in } \mathscr{C}^0([0, T];\ W^{-1,q'}(\Omega)).$$

Since u_ε converges weak-* to u in $L_\infty(0, T;\ W_0^{1,q}(\Omega))$ we conclude that

$$\int_0^T \int_\Omega b(u_\varepsilon(t))u_\varepsilon(t)\,dx\,dt \to \int_0^T \int_\Omega \chi(t)u(t)\,dx\,dt \tag{92}$$

as ε tends to zero.

We introduce the functional J defined on $L_2(\Omega \times (0, T))$ as

$$J(v) = \begin{cases} \displaystyle\int_0^T \int_\Omega \Psi(v(t))\,dx\,dt & \text{if } \Psi(v) \text{ belongs to } L_1(\Omega \times (0, T)), \\ +\infty & \text{otherwise.} \end{cases}$$

In view of the properties of Ψ (cf. § 2.1), J is positive, convex, and lower semicontinuous. It is also proper, since $\Psi(0) = 0$.

A classical result of convex analysis [11, Thm. 2, p. 532] allows us to identify the subdifferential $\partial J(v)$ of J at any point v of $L_2(\Omega \times (0, T))$ as

$$\partial J(v) = \{w \in L_2(\Omega \times (0, T)) \mid w(x, t) = b(v(x, t)) \text{ almost everywhere in } \Omega \times (0, T)\}. \tag{93}$$

Since both u^ε and $b(u^\varepsilon)$ lie, in particular, in $L_2(\Omega \times (0, T))$, $b(u^\varepsilon)$ belongs to $\partial J(u^\varepsilon)$. Thus,

$$\int_0^T \int_\Omega b(u_\varepsilon(t))u_\varepsilon(t)\,dx\,dt + J(w) \geqq J(u_\varepsilon) + \int_0^T \int_\Omega b(u_\varepsilon(t))w(t)\,dx\,dt, \tag{94}$$

for any w in $L_2(\Omega \times (0, T))$. Because of (88), (92), and the weak lower semicontinuity of J, we are in a position to compute the limit of each term in inequality (94). We obtain that, for any w in $L_2(\Omega \times (0, T))$,

$$\int_0^T \int_\Omega \chi(t)u(t)\,dx\,dt + J(w) \geqq J(u) + \int_0^T \int_\Omega \chi(t)w(t)\,dx\,dt. \tag{95}$$

Inequality (95) implies that u belongs to the domain of J and that

$$\chi \in \partial J(u), \tag{96}$$

and the characterization (93) of ∂J enables us to conclude that

$$\chi = b(u). \tag{97}$$

Once again the identification of Y relies on Lemma 2. The quantities w_m, w, and Y are identified with ∇u_ε, ∇u, and Y, respectively, and (48) is satisfied with the help of the convergences (88). To show that

$$Y = D\Phi(\nabla u) \tag{98}$$

reduces to proving that

$$\begin{aligned} &\int_0^T \int_0^t \int_\Omega (Y(s), \nabla u(s))_{\mathbb{R}^N}\, dx\, ds\, dt \\ &\geqq \overline{\lim_{\varepsilon\to 0}} \int_0^T \int_0^t \int_\Omega (D\Phi(\nabla u_\varepsilon(s)), \nabla u_\varepsilon(s))_{\mathbb{R}^N}\, dx\, ds\, dt. \end{aligned} \tag{99}$$

As seen earlier, the right-hand side of (99) is the lim-sup of the right-hand side of (80) as ε tends to zero. We obtain the following:

$$\begin{aligned} &\overline{\lim_{\varepsilon\to 0}} \int_0^T \int_0^t \int_\Omega (D\Phi(\nabla u_\varepsilon(s)), \nabla u_\varepsilon(s))_{\mathbb{R}^N}\, dx\, ds\, dt \\ &\leqq \int_0^T \int_0^t \int_\Omega f(s)u(s)\, dx\, ds\, dt + T\int_\Omega \Psi^*(b(u_0))\, dx \\ &- \underline{\lim_{\varepsilon\to 0}} \int_0^T \int_\Omega \Psi^*(b(u_\varepsilon(t)))\, dx\, dt. \end{aligned}$$

But Ψ^* is positive lower semicontinuous and convex on $\mathbb{R}$; thus, with the help of (97),

$$0 \leqq \int_0^T \int_\Omega \Psi^*(b(u(t)))\, dx\, dt \leqq \underline{\lim_{\varepsilon\to 0}} \int_0^T \int_\Omega \Psi^*(b(u_\varepsilon(t)))\, dx\, dt,$$

which leads to

$$\begin{aligned} &\overline{\lim_{\varepsilon\to 0}} \int_0^T \int_0^t \int_\Omega (D\Phi(\nabla u_\varepsilon(s)), \nabla u_\varepsilon(s))\, dx\, ds\, dt \\ &\leqq \int_0^T \int_0^t \int_\Omega f(s)u(s)\, dx\, ds\, dt + T\int_\Omega \Psi^*(b(u_0))\, dx \\ &- \int_0^T \int_\Omega \Psi^*(b(u(t)))\, dx. \end{aligned} \tag{100}$$

The right-hand side of inequality (100) is easily seen to coincide with $\int_0^T \int_0^t \int_\Omega (Y(s), \nabla u(s))_{\mathbb{R}^N}\, dx\, ds\, dt$ after multiplication of (90) by u, integration of the resulting expression over $(0, t)\times\Omega$ then over $(0, T)$, and application of Lemma 1. Inequality (99) is proved and equality (98) is established.

Recalling (88), (90), (91), (97), and (98) we conclude that there exists an element u of $L_\infty(0, T;\ W_0^{1,q}(\Omega))$ which satisfies (8)-(10). The proof of the existence part of Theorem 1 is now complete.

The bound on the norm of u in $L_\infty(0, T;\ W_0^{1,q}(\Omega))$ is a direct consequence of Remark 5 and of the weak lower semicontinuity property of the L_q norm.

2.5. Comparison result. It is now assumed that u_{01}, u_{02}, f_1, f_2 satisfy the hypotheses of Theorem 1 and that

$$b(u_{01}) \geqq b(u_{02}) \quad \text{almost everywhere on } \Omega,$$

$$f_1 \geqq f_2 \quad \text{almost everywhere on } \Omega \times (0, T).$$

Let u_1 and u_2 denote the associated solutions of (8)-(10) whose existence was established in the previous sections.

Let $\mathrm{sg}_\alpha^-(t)$ and $\mathrm{sg}_0(t)$ denote the following real-valued functions of the real variable:

$$\mathrm{sg}_\alpha^-(t) = \begin{cases} 0 & \text{if } t \geqq 0, \\ \dfrac{1}{\alpha} t & \text{if } -\alpha \leqq t \leqq 0, \\ -1 & \text{if } t \leqq -\alpha, \end{cases}$$

$$\mathrm{sg}_0(t) = \begin{cases} 0 & \text{if } t \geqq 0, \\ -1 & \text{if } t < 0. \end{cases}$$

In view of (60) and (61), the quantity $\mathrm{sg}_\alpha^-(u_{1\varepsilon}^\eta - u_{2\varepsilon}^\eta)$ is an element of $L_\infty(0, T; W_0^{1,q}(\Omega)) \cap W^{1,2}(0, T; L_2(\Omega))$ (and $b^\eta(u_\varepsilon^\eta)$ as well). Thus it is an admissible test function in (57) (with u_ε^η, f replaced by $u_{1\varepsilon}^\eta, f_1$ or $u_{2\varepsilon}^\eta, f_2$). Upon integration over the time interval $(0, t)$ of the result of the multiplication of (57) by $\mathrm{sg}_\alpha^-(u_{1\varepsilon}^\eta - u_{2\varepsilon}^\eta)$, we obtain by difference

$$\begin{aligned} &\int_0^t \int_\Omega \frac{\partial}{\partial t}[(b^\eta(u_{1\varepsilon}^\eta) + \varepsilon u_{1\varepsilon}^\eta) - (b^\eta(u_{2\varepsilon}^\eta) + \varepsilon u_{2\varepsilon}^\eta)](s)\, \mathrm{sg}_\alpha^-(u_{1\varepsilon}^\eta - u_{2\varepsilon}^\eta)(s)\, dx\, ds \\ &\qquad + \int_0^t \int_\Omega (\mathrm{sg}_\alpha^-)'(u_{1\varepsilon}^\eta - u_{2\varepsilon}^\eta)(s)(D\Phi(\nabla u_{1\varepsilon}^\eta(s)) \\ &\qquad\qquad - D\Phi(\nabla u_{2\varepsilon}^\eta(s)), (\nabla u_{1\varepsilon}^\eta - \nabla u_{2\varepsilon}^\eta)(s))_{\mathbb{R}^N}\, dx\, ds \\ &= \int_0^t \int_\Omega (f_1 - f_2)(s)\, \mathrm{sg}_\alpha^-(u_{1\varepsilon}^\eta - u_{2\varepsilon}^\eta)(s)\, dx\, ds. \end{aligned} \tag{101}$$

Once again the derivation formula for the composition of a $W_0^{1,q}$ function by a Lipschitz function is implicitly used in (101). By virtue of the monotonicity of sg_α^- and $D\Phi$ and the positivity of $f_1 - f_2$, (101) yields

$$\int_0^t \int_\Omega \frac{\partial}{\partial t}[(b^\eta(u_{1\varepsilon}^\eta) + \varepsilon u_{1\varepsilon}^\eta) - (b^\eta(u_{2\varepsilon}^\eta) + \varepsilon u_{2\varepsilon}^\eta)](s)\, \mathrm{sg}_\alpha^-(u_{1\varepsilon}^\eta - u_{2\varepsilon}^\eta)(s)\, dx\, ds \leqq 0. \tag{102}$$

As α tends to zero, $\mathrm{sg}_\alpha^-(u_{1\varepsilon}^\eta - u_{2\varepsilon}^\eta)$ converges weak-* in $L_\infty(\Omega \times (0, T))$ to $\mathrm{sg}_0^-(u_{1\varepsilon}^\eta - u_{2\varepsilon}^\eta)$ and (102) becomes

$$\int_0^t \int_\Omega \frac{\partial}{\partial t}[(b^\eta(u_{1\varepsilon}^\eta) + \varepsilon u_{1\varepsilon}^\eta) - (b^\eta(u_{2\varepsilon}^\eta) + \varepsilon u_{2\varepsilon}^\eta)](s)\, \mathrm{sg}_0^-(u_{1\varepsilon}^\eta - u_{2\varepsilon}^\eta)(s)\, dx\, ds \leqq 0.$$

Since $b^\eta(t) + \varepsilon t$ is monotone and takes the value zero for t equal to zero,

$$\mathrm{sg}_0^-(u_{1\varepsilon}^\eta - u_{2\varepsilon}^\eta) = \mathrm{sg}_0^-[(b^\eta(u_{1\varepsilon}^\eta) + \varepsilon u_{1\varepsilon}^\eta) - (b^\eta(u_{2\varepsilon}^\eta) + \varepsilon u_{2\varepsilon}^\eta)]$$

almost everywhere in $\Omega\times(0,T)$. The function $b^\eta(u^\eta_{i\varepsilon})+\varepsilon u^\eta_{i\varepsilon}$ $(i=1,2)$ lies in $W^{1,2}(0,T;L_2(\Omega))$. Consequently the last inequality reads as

$$\begin{aligned}\int_\Omega [(b^\eta(u^\eta_{1\varepsilon})+\varepsilon u^\eta_{1\varepsilon})-(b^\eta(u^\eta_{2\varepsilon})+\varepsilon u^\eta_{2\varepsilon})]^-(t)\,dx \\ \leqq \int_\Omega [(b^\eta(u_{01})+\varepsilon u_{01})-(b^\eta(u_{02})+\varepsilon u_{02})]^-\,dx,\end{aligned}\tag{103}$$

for any t in $[0,T]$. In (103), $[\cdot]^-$ denotes the convex Lipschitz function $-\inf(\cdot,0)$.

Since $[\ ]^-$ is convex and continuous, the convergences (63), (66), (69), and (88) together with relations (74) and (97), easily allow passing to the limit in the result of the integration of inequality (103) over the time interval $(0,T)$ as η and ε successively tend to zero. We finally conclude that

$$\int_0^T\int_\Omega [b(u_1)-b(u_2)]^-(t)\,dx\,dt \leqq T\int_\Omega [b(u_{01})-b(u_{02})]^-\,dx. \tag{104}$$

The function $[b(u_{01})-b(u_{02})]^-$ is by assumption equal to zero, which implies that for almost every (x,t) in $\Omega\times(0,T)$,

$$(b(u_1)-b(u_2))(x,t)\leqq 0.$$

The proof of Theorem 1 is now complete.

Remark 8. Note that the hypothesis on $b(u_{01})-b(u_{02})$ has not been used in the derivation of inequality (104) which thus holds true only under the assumption that f_1-f_2 is almost everywhere positive on $\Omega\times(0,T)$.

3. Proof of Theorem 2. The proof of Theorem 2 is based on the existence result given by Theorem 1. Specifically we introduce

$$u_0^n=T_n(u_0),$$

where, for any positive r, T_r is the Lipschitz function defined as

$$T_r(t)=\begin{cases} t & \text{if } |t|\leqq r,\\ r\,\mathrm{sg}\,(t) & \text{if } |t|>r.\end{cases}$$

Since T_n is a piecewise C^1 Lipschitz function and u_0 is in $W_0^{1,q}(\Omega)$, u_0^n is in $W_0^{1,q}(\Omega)$, and the derivation formula applies, from which it is easily deduced that

$$u_0^n\rightharpoonup u_0 \quad \text{weakly in } W_0^{1,q}(\Omega), \tag{105}$$

as n tends to infinity.

We also introduce a sequence f^n in $W^{1,1}(0,T;L_2(\Omega))$ such that

$$f^n\to f \quad \text{strongly in } W^{1,1}(0,T;W^{-1,q'}(\Omega)) \tag{106}$$

as n tends to infinity, and we propose to study the behavior of u^n, the solution of

$$\begin{aligned}&\frac{\partial b(u^n)}{\partial t}-\operatorname{div} D\Phi(\nabla u^n)=f^n \quad \text{in } \Omega\times(0,T),\\ &u^n=0 \quad \text{on } \partial\Omega\times(0,T),\\ &b(u^n)|_{t=0}=b(u_0^n),\end{aligned}\tag{107}$$

as n tends to infinity. Theorem 1 ensures the existence of such a u^n in $L_\infty(0,T;W_0^{1,q}(\Omega))$ with $b(u^n)$ in $L_\infty(0,T;L_2(\Omega))\cap W^{1,\infty}(0,T;W^{-1,q'}(\Omega))$.

Furthermore the norm of u^n in $L_\infty(0, T; W_0^{1,q}(\Omega))$ is bounded by a continuous function of $\Phi(\nabla u_0^n)$ and of the norm $|||f^n|||$ of f^n in $W^{1,1}(0, T; W^{-1,q'}(\Omega))$ (cf. Theorem 1). It is then immediately deduced from (105), (106), and the properties (5) of Φ that

(108) u^n is bounded in $L_\infty(0, T; W_0^{1,q}(\Omega))$ independently of n.

Once again the boundedness of $D\Phi$ on the bounded sets of $L_q(\Omega)$ implies that

(109) $D\Phi(\nabla u^n)$ is bounded in $L_\infty(0, T; [L_{q'}(\Omega)]^N)$.

With the help of (108) and (109) we conclude that there exists a suitably extracted subsequence (still denoted u^n) such that

$$u^n \rightharpoonup u \quad \text{weak-* in } L_\infty(0, T; W_0^{1,q}(\Omega)),$$
$$D\Phi(\nabla u^n) \rightharpoonup Y \quad \text{weak-* in } L_\infty(0, T; [L_{q'}(\Omega)]^N) \tag{110}$$

as n tends to infinity. Furthermore, by virtue of (106)-(109),

(111) $\dfrac{\partial b(u^n)}{\partial t}$ is bounded in $L_\infty(0, T; W^{-1,q'}(\Omega))$ independently of n.

We need to derive an estimate on $b(u^n)$ so as to be in a position to pass to the limit in (107). The function u^n is an admissible test function in the first equation of (107). Upon integration over $\Omega \times (0, t)$ of the result of the multiplication of the first equation of (107) by u^n we obtain the following estimate:

$$\int_\Omega \Psi^*(b(u^n(t)))\,dx \leqq \int_\Omega \Psi^*(b(u_0^n))\,dx + |||f^n||| \int_0^t \|u^n(s)\|_{1,q}\,ds, \tag{112}$$

where $|||f^n|||$ denotes the norm of f^n in $W^{1,1}(0, T; W^{-1,q'}(\Omega))$. Lemma 1 and the coercivity properties of Φ are implicitly used in establishing (112) (refer to (21), (27)-(29) for an identical argument).

A subsequence of $\Psi^*(b(u_0^n))$, still denoted $\Psi^*(b(u_0^n))$, converges almost everywhere to $\Psi^*(b(u_0))$. Furthermore, in view of (4) and the positivity of Ψ,

$$0 \leqq \Psi^*(b(u_0^n(x))) \leqq b(u_0^n(x))u_0^n(x) \leqq b(u_0(x))u_0(x),$$

for almost every x of Ω. Recalling Remark 2 we conclude that, as n tends to infinity

$$\int_\Omega \Psi^*(b(u_0^n))\,dx \to \int_\Omega \Psi^*(b(u_0))\,dx, \tag{113}$$

and, with the help of (112), that

(114) $\Psi^*(b(u^n))$ is bounded in $L_\infty(0, T; L_1(\Omega))$ independently of n.

In (114) we have identified $\Psi^*(b(u^n))$ with one of its subsequences.

We now make use of the following remark (cf. [1, Remark 1.2, p. 314]).

Remark 9. Let δ be an arbitrary strictly positive real number. Then

$$|b(t)| \leqq \delta\Psi^*(b(t)) + \sup_{|\sigma| \leqq 1/\delta} |b(\sigma)|$$

for every t in $\mathbb{R}$.

Remark 10. Note that Remarks 2 and 9 immediately imply that $b(u_0)$ is in fact an element of $L_1(\Omega)$.

Thus, for any strictly positive δ and almost any (x, t) in $\Omega \times (0, T)$,

$$|b(u^n(x, t))| \leqq \delta\Psi^*(b(u^n(x, t))) + \sup_{|\sigma| \leqq 1/\delta} |b(\sigma)|. \tag{115}$$

Integration of (115) over any measurable subset Q of Ω implies, in view of (114), that, for almost any t in $(0, T)$,

$$\int_Q |b(u^n(t))|\,dx \leqq \delta\mathscr{C} + \text{mes}\,(Q) \sup_{|\sigma|\leqq 1/\delta} |b(\sigma)|, \tag{116}$$

where $\mathscr{C}$ is a generic constant independent of n.

Thus,

$$\begin{aligned} &b(u^n) \text{ is bounded in } L_\infty(0, T; L_1(\Omega)) \text{ independently of } n,\\ &b(u^n) \text{ is uniformly equi-integrable in } L_1(\Omega). \end{aligned} \tag{117}$$

We are now in a position to prove the following lemma.

LEMMA 3. *The sequence* $b(u_n(t))$ *lies in* $\mathscr{C}^0([0, T]; L_1(\Omega))$ *and as n tends to infinity*

$$b(u_n(t)) \to \chi \quad \textit{strongly in } \mathscr{C}^0([0, T]; L_1(\Omega)), \tag{118}$$

where χ *is also an element of* $\mathscr{C}^0([0, T]; L_1(\Omega))$.

Proof of Lemma 3. By virtue of (108) and since the embedding of $W_0^{1,q}(\Omega)$ into $L_1(\Omega)$ is compact, we conclude that there exists a measurable set Z in $(0, T)$ of zero measure such that

$$F = \{u_n(t);\ n \in \mathbb{N},\ t \in (0, T) - Z\} \text{ is relatively compact in } L_1(\Omega). \tag{119}$$

Through application of the Dunford–Pettis Theorem (see, e.g., [6, Cor. 11, p. 294]), (117) implies that

(120) $b(F) = \{b(u_n(t));\ n \in N,\ t \in (0, T) - Z\}$ is sequentially weakly relatively compact in $L_1(\Omega)$.

The compactness properties (119) and (120) of the sets F and $b(F)$ ensure that

$$b(F) \text{ is relatively compact in } L_1(\Omega). \tag{121}$$

Indeed, if h_n is an arbitrary sequence of $b(F)$, there exists a subsequence of h_n (still denoted by h_n) and a sequence w_n of F such that, as n tends to infinity,

$$w_n \to w \quad \text{in } L_1(\Omega) \text{ and almost everywhere in } \Omega, \tag{122}$$

$$b_n = b(w_n) \rightharpoonup h \quad \text{weakly in } L_1(\Omega). \tag{123}$$

Since b is a continuous function (122) implies the almost pointwise convergence of $b(w_n)$ to $b(w)$. The weak convergence (123) then implies the strong convergence in $L_1(\Omega)$ of the sequence $b(w_n)$ to $b(w)$ (see [6, Thm. 12, p. 295], which proves (121).

By virtue of (121), there exists a compact set K in $L_1(\Omega)$ such that

$$b(u_n(t)) - b(u_n(t')) \in K \text{ for any } n \text{ and for almost every } t \text{ and } t' \text{ in } (0, T). \tag{124}$$

In view of (111) a proper choice of s (s small enough) guarantees that

$$L^1(\Omega) \hookrightarrow W^{-1,s}(\Omega), \tag{125}$$

and

$$\frac{\partial b(u_n)}{\partial t} \text{ is bounded in } L^\infty(0, T; W^{-1,s}(\Omega)). \tag{126}$$

We now appeal to a straightforward adaptation of a classical lemma of Lions (see [8, Lemma 5.1, p. 58]), which may be proved exactly as in [12, Lemma 8, p. 84].

LEMMA 4. *Let B and Y be two Banach spaces with continuous embedding of B into Y. Let X be a compact subset of B. Then, for any strictly positive number ε, there exists a strictly positive constant C_ε such that*

$$|f|_B \leqq \varepsilon + C_\varepsilon |f|_Y \quad \text{for any } f \text{ in } X. \qquad \square$$

Lemma 4 is applied in our context with $B = L_1(\Omega)$, $Y = W^{-1,s}(\Omega)$, and $X = K$. Thus, with the help of (124) and (125), for any strictly positive number ε, there exists a strictly positive constant C_ε such that

$$\|b(u_n(t)) - b(u_n(t'))\|_{0,1} \leqq \varepsilon + C_\varepsilon \|b(u_n(t)) - b(u_n(t'))\|_{-1,s} \tag{127}$$

for any n and for almost every t and t' in $(0, T)$.

Estimate (126) implies that

$$\|b(u_n(t)) - b(u_n(t'))\|_{-1,s} \leqq \int_t^{t'} \left\| \frac{\partial b(u_n)(\sigma)}{\partial t} \right\|_{-1,s} d\sigma \leqq \mathscr{C}|t - t'|, \tag{128}$$

where $\mathscr{C}$ is a generic constant independent of n.

Inserting (128) into (127) leads to

(129) $\|b(u_n(t)) - b(u_n(t'))\|_{0,1} \leqq \varepsilon + C_\varepsilon |t - t'|$ for any n and for almost every t and t' in $(0, T)$.

Estimate (129) readily yields the existence of a sequence of continuous representatives of $b(u_n(t))$ (still denoted by $b(u_n(t))$) such that for any positive number ε, there exists a strictly positive constant C_ε with

(130) $\|b(u_n(t)) - b(u_n(t'))\|_{0,1} \leqq \varepsilon + C_\varepsilon |t - t'|$ for any n and every t and t' in $[0, T]$.

The continuity of $b(u_n(t))$ and (121) imply that $b(u_n(t))$ is continuous on $[0, T]$ with value in a compact set of $L_1(\Omega)$ (which is independent of n). The uniform equicontinuity (130) of the sequence $b(u_n(t))$ permits the application of Ascoli's theorem, which completes the proof of Lemma 3.

It remains to prove that $\chi = b(u)$. Let ω be an arbitrary function in $L_1(\Omega \times (0, T))$. The function $T_R(b(u^n) - b(\omega))$ converges almost everywhere in $\Omega \times (0, T)$ to $T_R(\chi - b(\omega))$ and its L_∞-norm is bounded above by R. Hence

$$T_R(b(u^n) - b(\omega)) \to T_R(\chi - b(\omega)) \quad \text{strongly in } L_s(\Omega \times (0, T)), \qquad 1 \leqq s < +\infty, \tag{131}$$

as n tends to infinity. If φ denotes an arbitrary positive element of $\mathscr{C}_0^\infty(\Omega \times (0, T))$ we conclude, with the help of (110) and (131), that

$$\begin{aligned} \lim_{n \to +\infty} \int_0^T \int_\Omega \varphi(t) T_R(b(u^n(t)) - b(\omega(t)))(u^n(t) - \omega(t))\, dx\, dt \\ = \int_0^T \int_\Omega \varphi(t) T_R(\chi(t) - b(\omega(t)))(u(t) - \omega(t))\, dx\, dt. \end{aligned} \tag{132}$$

The integrand in the left-hand side of equality (132) is always positive because b is monotone. Thus the limit is positive and we conclude that, for almost every (x, t) in $\Omega \times (0, T)$,

$$T_R(\chi(x, t) - b(\omega(x, t)))(u(x, t) - \omega(x, t)) \geqq 0. \tag{133}$$

Since R is arbitrary, (133) implies that, for almost any (x, t) in $\Omega \times (0, T)$,

$$(\chi(x, t) - b(\omega(x, t)))(u(x, t) - \omega(x, t)) \geqq 0. \tag{134}$$

A proper choice of ω in (134) then shows that

$$\chi = b(u) \quad \text{almost everywhere in } \Omega \times (0, T). \tag{135}$$

Passing to the limit in (107) is now an immediate task in view of (106), (110), (111), (118), and (135). We obtain the following:

$$\frac{\partial b(u)}{\partial t} - \operatorname{div} Y = f \quad \text{in } \Omega \times (0, T). \tag{136}$$

Since a subsequence of $b(u_0^n)$ (still denoted $b(u_0^n)$) converges almost everywhere and monotonically to $b(u_0)$, and with the help of Remark 10,

$$b(u_0^n) \to b(u_0) \quad \text{strongly in } L_1(\Omega), \tag{137}$$

as n tends to infinity. By virtue of (111), (118), and (135)

$$b(u)|_{t=0} = b(u_0). \tag{138}$$

It now remains to identify Y. The identification relies once again on Lemma 2. The quantities w_m, w, and Y are identified with ∇u^n, ∇u, and Y, respectively, and (48) is satisfied with the help of estimates (110). To show that

$$Y = D\Phi(\nabla u) \tag{139}$$

we only need to prove that

$$\begin{aligned} &\int_0^T \int_0^t \int_\Omega (Y(s), \nabla u(s))_{\mathbb{R}^N}\, dx\, ds\, dt \\ &\geqq \overline{\lim} \int_0^T \int_0^t \int_\Omega (D\Phi(\nabla u^n(s)), \nabla u^n(s))_{\mathbb{R}^N}\, dx\, ds\, dt. \end{aligned} \tag{140}$$

As was seen earlier (cf. § 2.4) the right-hand side of (140) is the limit superior of the right-hand side of inequality (100) with f, u_0, and u, respectively, replaced by f^n, u_0^n, and u^n.

We obtain, in view of (106), (110), and (113),

$$\begin{aligned} \overline{\lim_{\varepsilon \to 0}} &\int_0^T \int_0^t \int_\Omega (D\Phi(\nabla u^n(s)), \nabla u^n(s))_{\mathbb{R}^N}\, dx\, ds\, dt \\ &\leqq \int_0^T \int_0^t \int_\Omega f(s)u(s)\, dx\, ds\, dt + T \int_\Omega \Psi^*(b(u_0))\, dx \\ &\quad - \underline{\lim}_{n \to +\infty} \int_0^T \int_\Omega \Psi^*(b(u^n(t)))\, dx\, dt. \end{aligned}$$

But Ψ^* is positive and lower semicontinuous on $\mathbb{R}$; thus, with the help of (118), (135), and Fatou's lemma,

$$0 \leqq \int_0^T \int_\Omega \Psi^*(b(u(t)))\, dx\, dt \leqq \underline{\lim}_{m \to +\infty} \int_0^T \int_\Omega \Psi^*(b(u^n(t)))\, dx\, dt,$$

which leads to

$$\begin{aligned} \overline{\lim_{\varepsilon \to 0}} &\int_0^T \int_0^t \int_\Omega (D\Phi(\nabla u^n(s)), \nabla u^n(s))_{\mathbb{R}^N}\, dx\, ds\, dt \\ &\leqq \int_0^T \int_0^t \int_\Omega f(s)u(s)\, dx\, ds\, dt + T \int_\Omega \Psi^*(b(u_0))\, dx \\ &\quad - \int_0^T \int_\Omega \Psi^*(b(u(t)))\, dx\, dt. \end{aligned} \tag{141}$$

The right-hand side of inequality (141) is easily seen to coincide with $\int_0^T \int_0^t \int_\Omega (Y(s), \nabla u(s))_{\mathbb{R}^N} \, dx \, ds \, dt$ after multiplication of (136) by u, integration of the resulting expression over $(0, t) \times \Omega$ then over $(0, T)$, and application of Lemma 1. Inequality (140) is proved and equality (139) follows.

Recalling (110), (118), and (135)-(139), we conclude that there exists an element u of $L_\infty(0, T; W_0^{1,q}(\Omega))$ which satisfies (13)-(15). The proof of the existence part of Theorem 2 is complete.

If f_1 and f_2 satisfy (12), and $f_1 - f_2$ is positive in the sense of Theorem 2, the sequences f_1^n, f_2^n introduced in (106) can be chosen such that $f_1^n - f_2^n$ is positive almost everywhere on $\Omega \times (0, T)$. According to Remark 8, inequality (104) applies to u_1^n and u_2^n. We obtain the following:

$$\int_0^T \int_\Omega [b(u_1^n) - b(u_2^n)]^-(t) \, dx \, dt \leqq T \int_\Omega [b(u_{01}^n) - b(u_{02}^n)]^- \, dx, \tag{142}$$

where u_{01}^n and u_{02}^n are the sequences associated to u_{01} and u_{02} through (105). In view of (118), (135), and (138), inequality (142) is easily seen to yield

$$\int_0^T \int_\Omega [b(u_1) - b(u_2)]^-(t) \, dx \, dt \leqq T \int_\Omega [b(u_{01}) - b(u_{02})]^- \, dx,$$

as n tends to infinity and the hypothesis on $b(u_{01}) - b(u_{02})$ permits us to conclude.

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REFERENCES

[1] H. W. ALT AND S. LUCKHAUS, *Quaslinear elliptic-parabolic differential equations*, Math. Z., 183 (1983), pp. 311-341.

[2] A. BAMBERGER, *Etude d'une équation doublement non linéaire*, J. Funct. Anal., 24 (1977), pp. 148-155.

[3] PH. BENILAN, *Sur un problème d'évolution non monotone dans $L_2(\Omega)$*, Internal Report, Publications mathématiques de l'Université de Besançon, France, 1975.

[4] L. BOCCARDO AND F. MURAT, *Remarques sur l'homogénéisation de certains problèmes quasi-linéaires*, Portugal. Math., 41 (1982), pp. 535-562.

[5] H. BRÉZIS AND F. BROWDER, *A property of Sobolev spaces*, Comm. Partial Differential Equations, 4 (1979), pp. 1077-1083.

[6] N. DUNFORD AND J. T. SCHWARTZ, *Linear Operators*, Interscience, New York, 1967.

[7] O. GRANGE AND E. MIGNOT, *Sur la résolution d'une équation et d'une inéquation paraboliques non linéaires*, J. Funct. Anal., 11 (1972), pp. 77-92.

[8] J. L. LIONS, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Gauthier-Villars, Paris, 1969.

[9] M. MARCUS AND V. J. MIZEL, *Nemitsky operators on Sobolev spaces*, Arch. Rational Mech. Anal., 51 (1973), pp. 347-370.

[10] P. A. RAVIART, *Sur la résolution de certaines équations paraboliques nonlinéaires*, J. Funct. Anal., 5 (1970), pp. 299-328.

[11] R. T. ROCKAFELLAR, *Integrals which are convex functionals*, Pacific J. Math., 24 (1968), pp. 525-539.

[12] J. SIMON, *Compact sets in $L_p(0, T; B)$*, Ann. Mat. Pura Appl. (4), 146 (1987), pp. 65-96.

[13] L. TARTAR, *Cours Peccot, Collège de France*, 1977 (partially written in H-Convergence, F. Murat Séminaire d'Analyse Fonctionnelle et Numérique, Alger, 1977/1978).