A TWO-DIMENSIONAL LABILE AETHER THROUGH HOMOGENIZATION

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ABSTRACT. Homogenization in linear elliptic problems usually assumes coercivity of the accompanying Dirichlet form. In linear elasticity, coercivity is not ensured through mere (strong) ellipticity so that the usual estimates that render homogenization meaningful break down unless stronger assumptions, like very strong ellipticity, are put into place. Here, we demonstrate that a L^2-type homogenization process can still be performed, very strong ellipticity notwithstanding, for a specific two-phase two dimensional problem whose significance derives from prior work establishing that one can lose strong ellipticity in such a setting, *provided* that homogenization turns out to be meaningful.

A striking consequence is that, in an elasto-dynamic setting, some two-phase homogenized laminate may support plane wave propagation in the direction of lamination on a bounded domain with Dirichlet boundary conditions, a possibility which does not exist for the associated two-phase microstructure at a fixed scale. Also, that material blocks longitudinal waves in the direction of lamination, thereby acting as a two-dimensional aether in the sense of *e.g.* Cauchy.

1. INTRODUCTION

This paper may be viewed as a sequel to both [2] and [6]. Those, in turn, were a two-dimensional revisiting of [7] in the light of [9]. The issue at stake was whether one could lose strict strong ellipticity when performing a homogenization process on a periodic mixture of two isotropic elastic materials, one being (strictly) very strongly elliptic while the other is only (strictly) strongly elliptic. We start this introduction with a brief overview of the problem that had been addressed in those papers, restricting all considerations to the two-dimensional case.

We consider throughout an elasticity tensor (Hooke's law) of the form

$$\mathbb{L} \in L^\infty\big(\mathscr{T}_2; \mathscr{L}_s(\mathbb{R}_s^{2\times 2})\big),$$

where $\mathscr{T}_2$ is the 2-dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$ and $\mathscr{L}_s(\mathbb{R}_s^{2\times 2})$ denotes the set of symmetric mappings from the set of 2×2 symmetric matrices onto itself. Note that there is a canonical identification $\mathcal{I}$ between $\mathscr{T}_2$ and the unit cell $Y := [0,1)^2$; for simplicity, we will denote by y both an element of $\mathscr{T}_2$ and its image under the mapping $\mathcal{I}$.

The tensor-valued function $\mathbb{L}$ defined in $\mathscr{T}_2$ is extended by Y-periodicity to $\mathbb{R}^2$ as

$$\mathbb{L}(y + \kappa) = \mathbb{L}(y), \quad \text{a.e. in } \mathbb{R}^2, \ \forall \kappa \in \mathbb{Z}^2,$$

so that the rescaled function $\mathbb{L}(x/\varepsilon)$ is εY-periodic.

We then consider the Dirichlet boundary value problem on a bounded open domain $\Omega \subset \mathbb{R}^2$

$$(1.1) \qquad \begin{cases} -\operatorname{div}\left(\mathbb{L}(x/\varepsilon)\nabla u^\varepsilon\right) &= \ f \text{ in } \Omega \\ u^\varepsilon &= \ 0 \text{ on } \partial\Omega, \end{cases}$$

with $f \in H^{-1}(\Omega; \mathbb{R}^2)$. We could impose a very strong ellipticity condition on $\mathbb{L}$, namely

$$(1.2) \qquad \alpha_{\text{vse}}(\mathbb{L}) := \operatorname*{ess-inf}_{y \in \mathscr{T}_2} \left(\min \left\{ \mathbb{L}(y)M \cdot M : M \in \mathbb{R}^{2\times 2}_s, |M| = 1 \right\} \right) > 0.$$

In such a setting, homogenization is straightforward; see *e.g.* the remarks in [12, Ch. 6, Sec. 11].

Instead, we will merely impose (strict) strong ellipticity, that is

$$(1.3) \quad \alpha_{\text{se}}(\mathbb{L}) := \operatorname*{ess-inf}_{y \in \mathscr{T}_2} \left(\min \left\{ \mathbb{L}(y)(a \otimes b) \cdot (a \otimes b) : a, b \in \mathbb{R}^2, |a| = |b| = 1 \right\} \right) > 0,$$

and this throughout.

Remark 1.1 (Ellipticity and isotropy). Whenever $\mathbb{L}$ is isotropic, that is

$$\mathbb{L}(y)M = \lambda(y)\operatorname{tr}(M)\, I_2 + 2\mu(y)\, M, \quad \text{for } y \in \mathscr{T}_2, \ M \in \mathbb{R}^{2\times 2}_s,$$

then (1.2) reads as

$$\operatorname*{ess-inf}_{y \in \mathscr{T}_2} \left(\min \left\{ \mu(y), \lambda(y) + \mu(y) \right\} \right) > 0$$

while (1.3) reads as

$$\operatorname*{ess-inf}_{y \in \mathscr{T}_2} \left(\min \left\{ \mu(y), \lambda(y) + 2\mu(y) \right\} \right) > 0. \qquad\qquad \P$$

The strong ellipticity condition (1.3) is the starting point of the study of homogenization performed in [7] from a variational standpoint, that of Γ-convergence. Under that condition, the authors investigate the Γ-convergence, for the weak topology of $H^1_0(\Omega; \mathbb{R}^2)$ on bounded sets (a metrizable topology), of the Dirichlet integral

$$\int_\Omega \mathbb{L}(x/\varepsilon)\nabla v \cdot \nabla v \, dx.$$

Then, under certain conditions that will be recalled in Section 2, the Γ-limit is given through the expected homogenization formula

$$(1.4) \qquad \mathbb{L}^0 M \cdot M := \min \left\{ \int_Y \mathbb{L}(y)(M + \nabla v) \cdot (M + \nabla v) \, dy : v \in H^1_{\text{per}}(Y; \mathbb{R}^2) \right\}$$

(see the definition of $H^1_{\text{per}}(Y; \mathbb{R}^2)$ in the notation at the end of this section), this in spite of the lack of very strong ellipticity.

In [9, 10], the viewpoint is somewhat different. The author, S. Gutiérrez, looks at a two-phase layering of a very strongly elliptic isotropic material with a strongly elliptic isotropic material. Assuming that the homogenization process makes sense, he shows that strict strong ellipticity can be lost through that process for a very specific combination of Lamé coefficients (see (2.3) below) and for a volume fraction $1/2$ of each phase.

Our goal in the previous study [2] was to reconcile those two sets of results, or more precisely, to demonstrate that Gutiérrez' viewpoint expounded in [9, 10] fit within the variational framework set forth in [7] and that the example produced in those papers is the only possible one within the class of laminate-like microstructures. Then, it is shown in [6] (see also further improvements in [8]) that the Gutiérrez pathology is in essence canonical, that is that inclusion-type microstructures never give rise to such a pathology.

The concatenation of those results may be seen as an indictment of linear elasticity, especially when confronted with its scalar analogue where ellipticity cannot be weakened through a homogenization process. However, our results, hence those of Gutiérrez, had to be tempered by the realization that Γ-convergence *a priori* assumes convergence of the relevant sequences in the *ad hoc* topology (here the weak-topology on bounded sets of H_0^1). The derivation of a bound that allows for such an assumption not to be vacuous is not part of the Γ-convergence process, yet it is essential lest that process become a gratuitous mathematical exercise.

This is the primary task that we propose to undertake in this study. To this end we add to the Dirichlet integral a zeroth-order term of the form $\int_\Omega |v|^2 \, dx$ which will immediately provide compactness in the weak topology of $L^2(\Omega; \mathbb{R}^2)$. We are then led to an investigation, for the *weak topology on bounded sets of $L^2(\Omega; \mathbb{R}^2)$*, of the Γ-limit of the Dirichlet integral

$$\int_\Omega \mathbb{L}(x/\varepsilon)\nabla v \cdot \nabla v \, dx.$$

Our "elliptic" results, detailed in Theorems 3.3, 3.4, essentially state that, at least for periodic mixtures of two isotropic materials that satisfy the constraints imposed in [9], the ensuing Γ-limit is in essence identical to that which had been previously obtained for the *weak topology on bounded sets of $H_0^1(\Omega; \mathbb{R}^2)$*. An immediate consequence is that Gutiérrez' example does provide a *bona fide* loss of strict strong ellipticity in two-phase two-dimensional periodic homogenization, and not only one that would be conditioned upon some otherwordly bound on minimizing sequences; see Proposition 4.2.

We then move on to the hyperbolic setting and demonstrate that the results of Theorems 3.3, 3.4 imply a weak homogenization result for the equations of elastodynamics which leads, in the Gutiérrez example, to the striking and, to the best of our knowledge, new realization that homogenization may lead to a plane wave propagation for the homogenized system on a bounded domain with Dirichlet boundary conditions, although, at a fixed scale, the microstructure would of course prevent such a propagation, precisely because of the Dirichlet boundary condition. This is roughly because a degeneracy of strong ellipticity in some direction relaxes the boundary condition on a certain part of the boundary.

Further, the Gutiérrez material is unique in its anisotropy class (2D orthorhombic) in blocking longitudinal waves – those for which propagation and oscillation are in the same direction – in a some preset direction. This feature motivates the title of our contribution because such a property was precisely the focus of pre-Maxwellian investigations by, among others, Cauchy, Green, Thomson (Lord Kelvin). There, an elastic substance called *labile aether* was meant to carry light throughout space, thereby spatially co-existing with the various materials it permeated [13, Chapter 5]. In order to conform to the various available observations for the propagation of light, it was deemed imperative that aether, as an elastic material, should allow

for transverse plane waves while inhibiting longitudinal waves. According to [13], Green's 1837 theory of wave reflection for elastic solids that assumed, in Fresnel's footstep, that aether should be much stiffer in compression than in shear prompted Cauchy's 1839 publication of his third theory of reflection in a material for which the Lamé coefficients λ, μ satisfy

$$(1.5) \qquad\qquad \lambda + 2\mu = 0.$$

This is precisely what the Gutiérrez material achieves, at least in a crystalline way, by forbidding longitudinal waves in the direction of lamination.

In Section 2, we provide a quick review of the results that are relevant to our investigation. Then Section 3 details the precise assumptions under which we obtain Theorems 3.3, 3.4 and present the proofs of those theorems. Section 4 details the impact of our results on the actual minimization of the above mentioned Dirichlet integrals augmented by a linear (force) term. Such a minimization process provides in turn a homogenization result for elasto-dynamics (Theorem 4.4) in a setting where strict strong ellipticity is lost in the limit. We conclude with a discussion of the propagation properties of the Gutiérrez material.

Throughout the paper, the following remark, which finds its origin in [9], will play a decisive role. Since, for $v \in H_0^1(\Omega; \mathbb{R}^2)$, the mapping $v \mapsto \det(\nabla v)$ is a null Lagrangian, we are at liberty to replace the Dirichlet integral under investigation by

$$\int_\Omega \left\{ \mathbb{L}(x/\varepsilon) \nabla v \cdot \nabla v + c \det(\nabla v) \right\} dx,$$

for any $c \in \mathbb{R}$, thereby replacing

$$M \mapsto \mathbb{L}(y)M, \ M \in \mathbb{R}^{2\times 2},$$

by

$$M \mapsto \mathbb{L}(y)M + \frac{c}{2} \operatorname{cof}(M), \ M \in \mathbb{R}^{2\times 2}.$$

Notationwise,

- I_2 is the unit matrix of $\mathbb{R}^{2\times 2}$; $R^\perp$ is the $\pi/2$-rotation matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$;
- $A \cdot B$ is the Frobenius inner product between two elements of $A, B \in \mathbb{R}^{2\times 2}$, that is $A \cdot B := \operatorname{tr}(A^T B)$;
- If $A := \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathbb{R}^{2\times 2}$, the cofactor matrix of A is $\operatorname{cof}(A) := \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$;
- If $\mathbb{K} : \mathbb{R}^p \to \mathbb{R}^p$ is a linear mapping, the *pseudo-inverse* of $\mathbb{K}$, denoted by $\mathbb{K}^{-1}$, is defined on its range $\operatorname{Im}(\mathbb{K})$ as follows: for any $\xi \in \mathbb{R}^p$, $\mathbb{K}^{-1}(\mathbb{K}\xi)$ is the orthogonal projection of ξ onto the orthogonal space $[\operatorname{Ker}(\mathbb{K})]^\perp$, so that $\mathbb{K}\big(\mathbb{K}^{-1}(\mathbb{K}\xi)\big) = \mathbb{K}\xi$;
- If u is a distribution (an element of $\mathscr{D}'(\mathbb{R}^2; \mathbb{R}^2)$), then

$$\operatorname{curl} u := \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}$$

  while

$$\mathbf{E}(u) = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} \end{pmatrix};$$

- $H_{\operatorname{per}}^1(Y_2; \mathbb{R}^p)$ (resp. $L_{\operatorname{per}}^2(Y_2; \mathbb{R}^p), L_{\operatorname{per}}^\infty(Y_2; \mathbb{R}^p), C_{\operatorname{per}}^p(Y_2; \mathbb{R}^p)$) is the space of those functions in $H_{\operatorname{loc}}^1(\mathbb{R}^2; \mathbb{R}^p)$ (resp. $L_{\operatorname{loc}}^2(\mathbb{R}^2; \mathbb{R}^p), L^\infty(\mathbb{R}^2; \mathbb{R}^p), C^p(\mathbb{R}^2; \mathbb{R}^p)$) that are Y_2-periodic;

- For any subset $\mathscr{Z} \in \mathscr{T}_2$, we agree to denote by Z its representative in Y through the canonical representation $\mathcal{I}$ introduced earlier, and by $Z^\#$ its representative in $\mathbb{R}^2$, that is the open "periodic" set

$$Z^\# := \overset{\circ}{\overbrace{\bigcup_{k \in \mathbb{Z}^2} (k + \overline{Z})}}.$$

- Throughout, the variable x will refer to a running point in a *bounded* open domain $\Omega \subset \mathbb{R}^2$, while the variable y will refer to a running point in Y (or $\mathscr{T}_2$, or $k + \overline{Y}$, $k \in \mathbb{Z}^2$);
- If $\mathscr{I}^\varepsilon$ is an ε-indexed sequence of functionals with

$$\mathscr{I}^\varepsilon : X \to \mathbb{R},$$

(X reflexive Banach space), we will write that $\mathscr{I}^\varepsilon \overset{\Gamma(X)}{\rightharpoonup} \mathscr{I}^0$, with

$$\mathscr{I}^0 : X \to \mathbb{R},$$

if $\mathscr{I}^\varepsilon$ Γ-converges to $\mathscr{I}^0$ for the weak topology on bounded sets of X (see *e.g.* [4] for the appropriate definition); and

- $u^\varepsilon \overset{2}{\rightrightarrows} u^0$ where $u^\varepsilon \in L^2(\Omega; \mathbb{R}^2)$ and $u^0 \in L^2(\Omega \times \mathscr{T}_2; \mathbb{R}^2)$ iff u^ε two-scale converges to u^0 in the sense of Nguetseng; see *e.g.* [11, 1].

2. KNOWN RESULTS

As previously announced, this short section recalls the relevant results obtained in [7], [3]. For vector-valued (linear) problems, a successful application of Lax-Milgram's lemma to a Dirichlet problem of the type (1.1) hinges on the positivity of the following functional coercivity constant:

$$\Lambda(\mathbb{L}) := \inf \left\{ \int_{\mathbb{R}^2} \mathbb{L}(y)\nabla v \cdot \nabla v \, dy : v \in C_c^\infty(\Omega; \mathbb{R}^2), \int_{\mathbb{R}^2} |\nabla v|^2 \, dy = 1 \right\}.$$

As long as $\Lambda(\mathbb{L}) > 0$, existence and uniqueness of the solution to (1.1) is guaranteed by Lax-Milgram's lemma.

Further, according to classical results in the theory of homogenization, under condition (1.2) the solution $u^\varepsilon \in H_0^1(\Omega; \mathbb{R}^2)$ of (1.1) satisfies

$$\begin{cases} u^\varepsilon \rightharpoonup u, & \text{weakly in } H_0^1(\Omega; \mathbb{R}^2) \\ \mathbb{L}(x/\varepsilon)\nabla u^\varepsilon \rightharpoonup \mathbb{L}^0 \nabla u, & \text{weakly in } L^2(\Omega; \mathbb{R}^{2 \times 2}) \\ -\operatorname{div}\left(\mathbb{L}^0 \nabla u\right) = f, \end{cases}$$

with $\mathbb{L}^0$ given by (1.4). The same result holds true when (1.2) is replaced by the condition that $\Lambda(\mathbb{L}) > 0$; see [5].

When $\Lambda(\mathbb{L}) = 0$, the situation is more intricate. Define

$$\Lambda_{\mathrm{per}}(\mathbb{L}) := \inf \left\{ \int_Y \mathbb{L}(y)\nabla v \cdot \nabla v \, dy : v \in H_{\mathrm{per}}^1(Y; \mathbb{R}^2), \int_Y |\nabla v|^2 dy = 1 \right\}$$

and set

$$\mathscr{I}^\varepsilon(v) := \int_\Omega \mathbb{L}(x/\varepsilon)\nabla v \cdot \nabla v \, dx \quad \text{for } v \in H_0^1(\Omega; \mathbb{R}^2).$$

A first result was obtained in [7, Theorem 3.4(i)], namely

Theorem 2.1. *If $\Lambda(\mathbb{L}) \geq 0$ and $\Lambda_{\mathrm{per}}(\mathbb{L}) > 0$, $\mathscr{J}^{\varepsilon} \overset{\Gamma(H_0^1)}{\longrightarrow} \mathscr{J}^0$, with*

$$\mathscr{J}^0(v) := \int_\Omega \mathbb{L}^0 \nabla v \cdot \nabla v \, dx \quad \text{for } v \in H_0^1(\Omega; \mathbb{R}^2),$$

and $\mathbb{L}^0$ given by (1.4).

This was very recently improved by A. Braides and M. Briane as reported in [3, Theorem 2.4]. The result is as follows:

Theorem 2.2. *If $\Lambda(\mathbb{L}) \geq 0$, then, $\mathscr{J}^{\varepsilon} \overset{\Gamma(H_0^1)}{\longrightarrow} \mathscr{J}^0$, with $\mathbb{L}^0$ given by*

$$(2.1) \qquad \mathbb{L}^0 M \cdot M := \inf \left\{ \int_Y \mathbb{L}(y)(M + \nabla v) \cdot (M + \nabla v) \, dy : v \in H_{\mathrm{per}}^1(Y; \mathbb{R}^2) \right\}.$$

Note that dropping the restriction that $\Lambda_{\mathrm{per}}(\mathbb{L})$ (which is always above $\Lambda(\mathbb{L})$) be positive changes the minimum in (1.4) into an infimum in (2.1).

As announced in the introduction, we are only interested in the kind of two-phase mixture that can lead, in the layering case, to the degeneracy first observed in [9]. Specifically, we assume the existence of 2 isotropic phases $\mathscr{Z}_1, \mathscr{Z}_2$ of $\mathscr{T}_2$ – and of the associated subsets Z_1 and Z_2 of Y, or still $Z_1^{\#}$ and $Z_2^{\#}$ of $\mathbb{R}^2$ (see notation) – such that

$$(2.2) \quad \begin{cases} \mathscr{Z}_1, \mathscr{Z}_2 \text{ are open, } C^2 \text{ subsets of } \mathscr{T}_2; \\[4pt] \mathscr{Z}_1 \cap \mathscr{Z}_2 = \varnothing \quad \text{and} \quad \bar{\mathscr{Z}}_1 \cup \bar{\mathscr{Z}}_2 = \mathscr{T}_2; \\[4pt] Z_2^{\#} \text{ has an unbounded connected component in } \mathbb{R}^2, \text{ denoted by } X_2^{\#}, \\ \text{and } X_2^{\#} \cap Y = Z_2; \\[4pt] \mathscr{Z}_1 \text{ has a finite number of connected components in } \mathscr{T}_2. \end{cases}$$

We denote henceforth by $\theta \in (0,1)$ the volume fraction of $\mathscr{Z}_1$ in $\mathscr{T}_2$.
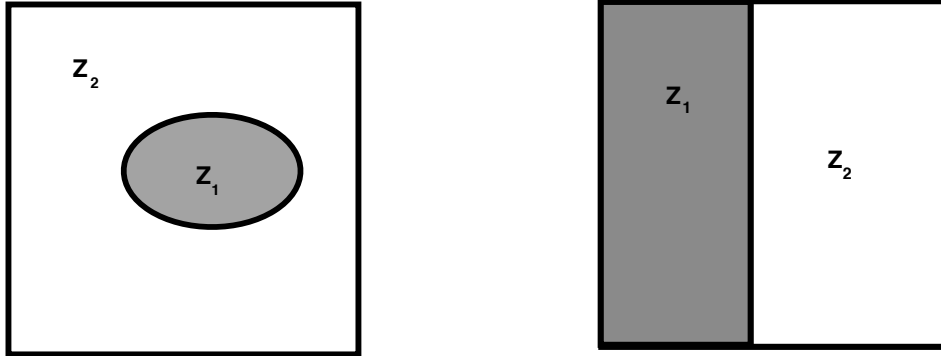

FIGURE 1. Typical allowed micro-geometries: inclusion of the good material or layering.

We then define

$$(2.3) \quad \begin{cases} \mathbb{L}(y)M = \lambda(y)\operatorname{tr}(M)\,I_2 + 2\mu(y)M, & y \in \mathscr{T}_2,\ M \in \mathbb{R}^{2\times 2} \\ \lambda(y) = \lambda_i,\ \mu(y) = \mu_i, & \text{in } \mathscr{Z}_i,\ i = 1,2 \\ 0 < -\lambda_2 - \mu_2 = \mu_1 < \mu_2,\ \lambda_1 + \mu_1 > 0. \end{cases}$$

which implies in particular that

$$\lambda_2 + 2\mu_2 > 0,$$

that is that phase 2 is only strongly elliptic ($\lambda_2 + \mu_2 < 0$) while phase 1 is very strongly elliptic ($\lambda_1 + \mu_1 > 0$).

Then the following result, which brings together [2, Theorem 2.2] and [6, Theorem 2.1], holds true:

Theorem 2.3. *Under assumptions,* (2.2), (2.3), $\Lambda(\mathbb{L}) \geq 0$ *and* $\Lambda_{\mathrm{per}}(\mathbb{L}) > 0$.

Consequently, Theorem 2.1 can be applied to the setting at hand and we obtain the following

Corollary 2.4. *Under assumptions* (2.2), (2.3), $\mathscr{J}^{\varepsilon} \xrightarrow{\ \Gamma(H_0^1)\ } \mathscr{J}^0$.

Remark 2.5. In Theorems 2.1, 2.2, 2.3 $\Lambda(\mathbb{L})$ may be 0, so that the standard coercivity assumption is lost.

As already remarked in the introduction the Γ-convergence results above say nothing about the convergence of putative minimizers for want of a uniform bound on minimizing sequences. In a Γ-convergence analysis with as topology the weak topology of $H_0^1(\Omega;\mathbb{R}^2)$ sequences of gradients are *a priori* bounded in $L^2(\Omega;\mathbb{R}^2)$.

Our goal in the next section is to prove that the Corollary remains true when adding to $\mathscr{J}^{\varepsilon}$ a zeroth order term of the form

$$\int_{\Omega} |v|^2 \, dx$$

and replacing the weak topology on bounded sets of $H_0^1(\Omega;\mathbb{R}^2)$ by that on bounded sets of $L^2(\Omega;\mathbb{R}^2)$.

3. The elliptic results

Consider $\mathbb{L}(y)$ given by (2.3) and $\mathbb{L}^0$ given by (1.4). Set, for $v \in L^2(\Omega;\mathbb{R}^2)$,

$$\mathscr{I}^{\varepsilon}(v) := \begin{cases} \displaystyle\int_{\Omega} \left\{ \mathbb{L}(x/\varepsilon)\nabla v \cdot \nabla v + |v|^2 \right\} dx, & v \in H_0^1(\Omega;\mathbb{R}^2) \\ \infty, & \text{else.} \end{cases}$$

Also define the following two functionals:

$$(3.1) \quad \mathscr{I}^0(v) := \begin{cases} \displaystyle\int_{\Omega} \left\{ \mathbb{L}^0 \nabla v \cdot \nabla v + |v|^2 \right\} dx, & v \in H_0^1(\Omega;\mathbb{R}^2) \\ \infty, & \text{else;} \end{cases}$$

and, under the additional assumption that

$$\mathbb{L}^0_{2222} = 0,$$

$$(3.2) \quad \mathscr{I}^0_{1/2}(v) := \begin{cases} \displaystyle\int_{\Omega} \left\{ \mathbb{L}^0 \nabla v \cdot \nabla v + |v|^2 \right\} dx, & v \in \mathbf{X} \\ \infty, & \text{else,} \end{cases}$$

where, if ν is the exterior normal on $\partial\Omega$,

$$(3.3) \quad \mathbf{X} := \Big\{ v \in L^2(\Omega; \mathbb{R}^2) : v_1 \in H_0^1(\Omega), \ v_2 \in L^2(\Omega),$$

$$\frac{\partial v_2}{\partial x_1} \in L^2(\Omega) \ \text{and} \ v_2\,\nu_1 = 0 \ \text{on} \ \partial\Omega \Big\}.$$

Remark 3.1. In (3.2) the cross terms

$$\int_\Omega \frac{\partial v_1}{\partial x_1}\frac{\partial v_2}{\partial x_2} \ dx$$

must be replaced by

$$\int_\Omega \frac{\partial v_1}{\partial x_2}\frac{\partial v_2}{\partial x_1} \ dx$$

so that, provided that $\mathbb{L}_{2222}^0 = 0$, which is the case in the specific setting at hand, the expression $\int_\Omega \mathbb{L}^0 \nabla v \cdot \nabla v \ dx$ has a meaning for $v \in \mathbf{X}$ and boils down to the classical one when $v \in H_0^1(\Omega; \mathbb{R}^2)$.

Remark 3.2. It is immediately checked that $\mathbf{X}$ is a Hilbert space when endowed with the following inner product:

$$\langle u, v \rangle_{\mathbf{X}} := \int_\Omega u \cdot v \ dx + \int_\Omega \nabla u_1 \cdot \nabla v_1 \ dx + \int_\Omega \frac{\partial u_2}{\partial x_1}\frac{\partial v_2}{\partial x_1} \ dx.$$

Furthermore, $C_c^\infty(\Omega; \mathbb{R}^2)$ is a dense subspace of $\mathbf{X}$, provided that Ω is C^1. Indeed, take $u \in \mathbf{X}$. The first component u_1 is in $H_0^1(\Omega; \mathbb{R}^2)$. Defining

$$\check{u}_2(x) := \begin{cases} u_2(x), & x \in \Omega \\ 0, & \text{else} \end{cases}$$

we have, thanks to the boundary condition in the definition (3.3) of $\mathbf{X}$,

$$\int_{\mathbb{R}^2} \check{u}_2 \frac{\partial \varphi}{\partial x_1} \ dx + \int_{\mathbb{R}^2} \frac{\partial u_2}{\partial x_1} \varphi \ dx = 0.$$

for any $\varphi \in C_c^\infty(\mathbb{R}^2)$, that is

$$\check{u}_2, \frac{\partial \check{u}_2}{\partial x_1} \in L^2(\mathbb{R}^2) \quad \text{with} \quad \frac{\partial \check{u}_2}{\partial x_1}(x) = \begin{cases} \dfrac{\partial u_2}{\partial x_1}(x), & x \in \Omega \\ \\ 0, & \text{else.} \end{cases}$$

Because Ω has a C^1-boundary, we can always assume, thanks to the implicit function theorem, that, at each point $x^0 \in \partial\Omega$, there exists a ball $B(x^0, r_{x^0})$ and a C^1-function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\Omega \cap B(x^0, r_{x^0}) = \{(x_1, x_2) \in B(x^0, r_{x^0}) : x_2 > f(x_1)\}$$

or

$$\Omega \cap B(x^0, r_{x^0}) = \{(x_1, x_2) \in B(x^0, r_{x^0}) : x_1 > f(x_2)\}.$$

In the first case, we translate $\check{u}$ in the direction x_2, thereby setting $\check{u}_2^t(x_1, x_2) := \check{u}_2(x_1, x_2 - t)$, $t > 0$, while, in the second case, we translate $\check{u}_2$ in the direction x_1, thereby setting $\check{u}_2^t(x_1, x_2) := \check{u}_2(x_1 - t, x_2)$, $t > 0$. This has the effect of creating a new function $\check{u}_2$ which is identically null near $\partial\Omega \cap B(x^0, r_{x^0})$. We then mollify

this function with a mollifier φ^t, with support depending on t, thereby creating yet a new function $\tilde{u}_2^t = \varphi^t * \check{u}_2^t \in C_c^\infty(\Omega \cap B(x^0, r_{x^0}))$ which will be such that

$$\lim_{t \to 0^+} \left\{ \|\tilde{u}_2^t - u\|_{L^2(\Omega \cap B(x^0, r_{x^0}))} + \big\| \frac{\partial \tilde{u}_2^t}{\partial x_1} - \frac{\partial u_2}{\partial x_1} \big\|_{L^2(\Omega \cap B(x^0, r_{x^0}))} \right\} = 0.$$

A partition of unity of the boundary and a diagonalization argument then allow one to construct a sequence of $C_c^\infty(\Omega)$-functions such that the same convergences take place over $L^2(\Omega)$. ¶

We propose to investigate the (sequential) Γ-convergence properties of $\mathscr{I}^\varepsilon$ to $\mathscr{I}^0$, or $\mathscr{I}^0_{1/2}$, for the weak topology on bounded sets of $L^2(\Omega; \mathbb{R}^2)$.

We will prove the following theorems which address both the case of a laminate and that of a matrix-inclusion type mixture. The first theorem does not completely characterize the Γ-limit to the extent that it is assumed *a priori* that the target field u lies in $H_0^1(\Omega; \mathbb{R}^2)$. By contrast, the second theorem is a complete characterization of the Γ-limit but it does restrict the geometry of laminate-like mixtures to be that made of *bona fide* layers, *i.e.,* straight strips of material.

Theorem 3.3 ("Smooth targets"). *Under assumptions* (2.2), (2.3), *there exists a subsequence of* $\{\varepsilon\}$ *(not relabeled) such that*

$$\mathscr{I}^\varepsilon \stackrel{\Gamma(L^2)}{\longrightarrow} \mathscr{I},$$

where, for $u \in H_0^1(\Omega; \mathbb{R}^2)$, $\mathscr{I}(u) = \mathscr{I}^0(u)$ *given by* (3.1) *and* $\mathbb{L}^0$ *given by* (1.4).

Theorem 3.4 ("General targets"). *Under assumptions* (2.2), (2.3), *then the following holds true:*

(i) the inclusion case: *If* $\bar{Z}_1 \subset \mathring{Y}$ *and if* Ω *is a bounded open Lipschitz domain in* $\mathbb{R}^2$, *then*

$$\mathscr{I}^\varepsilon \stackrel{\Gamma(L^2)}{\longrightarrow} \mathscr{I}^0$$

given by (3.1) *and* $\mathbb{L}^0$ *given by* (1.4);

(ii) the straight layer case: *If* $Z_1 = (0, \theta) \times (0, 1)$ *(or* $(0, 1) \times (0, \theta)$*) and if* Ω *is a bounded open* C^1 *domain in* $\mathbb{R}^2$, *then*

$$\mathscr{I}^\varepsilon \stackrel{\Gamma(L^2)}{\longrightarrow} \begin{cases} \mathscr{I}^0 & \text{if } \theta \neq 1/2 \\ \mathscr{I}^0_{1/2} & \text{if } \theta = 1/2 \text{ (the Gutiérrez case)} \end{cases}$$

which are given by (3.1), (3.2), *respectively, and with* $\mathbb{L}^0$ *given by* (1.4).

Remark 3.5. In strict parallel with Remark 2.6 in [2], we do not know whether the result of those Theorems still hold true when $H_0^1(\Omega; \mathbb{R}^2)$ is replaced by $H^1(\Omega; \mathbb{R}^2)$ in the Γ-convergence statement. ¶

Remark 3.6. We could generalize the inclusion condition (i) $\bar{Z}_1 \subset \mathring{Y}$ to the case of several inclusions of the strong material as follows. Consider n regular compact sets $K_1, \ldots, K_n$ of $\mathbb{R}^2$ such that all the translated $K_j + \kappa$ for $j \in \{1, \ldots, n\}$ and $\kappa \in \mathbb{Z}^2$, are pairwise disjoint. Then define the Y-periodic phase 1 by

$$Z_1^\# := \bigcup_{j=1}^n \bigcup_{\kappa \in \mathbb{Z}^2} (K_j + \kappa).$$

All subsequent results pertaining to case (i) extend to this enlarged setting. ¶

Remark 3.7. The strict strong ellipticity of $\mathbb{L}^0$ in Theorem 3.4 is known.

In case (i), $\mathbb{L}^0$ remains strictly strongly elliptic. This is explicitly stated in [6, Theorem 2.2] under a restriction of isotropy although the proof immediately extends to the fully anisotropic case as well.

In case (ii), strict strong ellipticity is preserved except in the Gutiérrez case $(\theta = 1/2)$ in which case $\mathbb{L}^0_{2222} = 0$, as first evidenced in [9]. ¶

Subsections 3.1, 3.2 are devoted to the proofs of Theorem 3.3, 3.4.

3.1. **Proof of Theorem 3.3.** First, because of the compactness of the injection mapping from $H^1_0(\Omega; \mathbb{R}^2)$ into $L^2(\Omega; \mathbb{R}^2)$ and in view of Corollary 2.4,

$$\mathscr{I}^\varepsilon \overset{\Gamma(H^1_0)}{\longrightarrow} \mathscr{I}^0$$

with $\mathbb{L}^0$ actually given by (1.4) (a min in lieu of an inf). We want to prove that the same result holds for the weak L^2-topology, at least for a subsequence of $\{\varepsilon\}$. By a classical compactness result we can assert the existence of a subsequence of $\{\varepsilon\}$ such that the Γ- lim exists. Our goal is to show that that limit, denoted by $\mathscr{I}(u)$, is precisely $\mathscr{I}^0(u)$ when $u \in H^1_0(\Omega; \mathbb{R}^2)$. Clearly, the Γ- lim sup inequality will *a fortiori* hold in that topology, *provided that the target field $u \in H^1_0(\Omega; \mathbb{R}^2)$.* It thus remains to address the proof of the Γ- lim inf inequality which is what the rest of this subsection is about.

To that end and in the spirit of [9], we add an integrated null Lagrangian to the energy so as to render the energy density pointwise nonnegative. Thus we set, for any $M \in \mathbb{R}^{2 \times 2}$,
(3.4)
$$\mathbb{K}_j M := \mathbb{L}_j M + 2\mu_1 \mathrm{cof}\,(M) = \lambda_j \mathrm{tr}\,(M)I_2 + \mu_j(M + M^T) + 2\mu_1 \mathrm{cof}\,(M), \quad j = 1, 2$$

(thereby taking c at the end of the introduction to be $4\mu_1$) so that

$$\mathbb{K}_j M \cdot M = \mathbb{L}_j M \cdot M + 4\mu_1 \det(M) \geq 0, \quad j = 1, 2$$

and define

$$\mathbb{K}(y) \equiv \mathbb{K}_j \text{ in } \mathscr{Z}_j, \quad j = 1, 2.$$

Because the determinant is a null Lagrangian, for $v \in H^1_0(\Omega; \mathbb{R}^2)$,

(3.5) $$\mathscr{I}^\varepsilon(v) = \int_\Omega \left\{ \mathbb{K}(x/\varepsilon)\nabla v \cdot \nabla v + |v|^2 \right\} dx$$

Consider a sequence $\{u^\varepsilon\}_\varepsilon$ converging weakly in $L^2(\Omega; \mathbb{R}^2)$ to $u \in L^2(\Omega; \mathbb{R}^2)$. Then, for a subsequence (still indexed by ε), we are at liberty to assume that $\liminf \mathscr{I}^\varepsilon(u^\varepsilon)$ is actually a limit. The Γ- lim inf inequality is trivial if that limit is ∞ so that we can also assume henceforth that, for some $\infty > C > 0$,

(3.6) $$\mathscr{I}^\varepsilon(u^\varepsilon) \leq C.$$

Further, according to *e.g.* [1, Theorem 1.2], a subsequence (still indexed by ε) of that sequence two-scale converges to some $u^0(x, y) \in L^2(\Omega \times \mathscr{T}_2; \mathbb{R}^2)$. In other words,

(3.7) $$u^\varepsilon \rightrightarrows u^0.$$

Also, in view of (3.6) and because

(3.8) $$\mathbb{K}(y) \text{ is a nonnegative as a quadratic form}$$

while clearly all its components are bounded, for yet another subsequence (not relabeled),

$$\mathbb{K}(x/\varepsilon)\nabla u^\varepsilon \; \rightrightarrows \; H(x,y) \quad \text{with } H \in L^2(\Omega \times Y; \mathbb{R}^{2\times 2}),$$

and also, for future use,

(3.9) $$\mathbb{K}^{\frac{1}{2}}(x/\varepsilon)\nabla u^\varepsilon \; \rightrightarrows \; S(x,y) \quad \text{with } S \in L^2(\Omega \times Y; \mathbb{R}^{2\times 2}).$$

In particular,

(3.10) $$\varepsilon\,\mathbb{K}(x/\varepsilon)\nabla u^\varepsilon \; \rightrightarrows \; 0.$$

Take $\Phi(x,y) \in C_c^\infty(\Omega \times \mathscr{T}_2; \mathbb{R}^{2\times 2})$ with compact support in $\Omega \times \mathscr{Z}_1$. From (3.10) we get, with obvious notation, that

$$0 = -\lim_{\varepsilon \to 0} \int_\Omega \varepsilon\,\mathbb{K}(x/\varepsilon)\nabla u^\varepsilon \cdot \Phi(x, x/\varepsilon)\, dx =$$

$$\lim_{\varepsilon \to 0} \sum_{ijkh} \int_\Omega (\mathbb{K})_{ijkh}(x/\varepsilon)\, u_k^\varepsilon\, \frac{\partial \Phi_{ij}}{\partial y_h}(x, x/\varepsilon)\, dx =$$

$$\sum_{ijkh} \int_{\Omega \times Z_1} (\mathbb{K}_1)_{ijkh}\, u_k^0(x,y)\, \frac{\partial \Phi_{ij}}{\partial y_h}\, dx\, dy = -\int_{\Omega \times Z_1} \left(\mathbb{K}_1 \nabla_y u^0(x,y)\right) \cdot \Phi(x,y)\, dx\, dy,$$

so that

(3.11) $$\mathbb{K}_1 \nabla_y u^0(x,y) \equiv 0 \quad \text{in } \Omega \times Z_1^{\#},$$

and similarly

(3.12) $$\mathbb{K}_2 \nabla_y u^0(x,y) \equiv 0 \quad \text{in } \Omega \times Z_2^{\#}.$$

In view of the explicit expressions (3.4) for $\mathbb{K}_j$, (3.11), (3.12) imply that

$$\lambda_j \left(\frac{\partial u_1^0}{\partial y_1} + \frac{\partial u_2^0}{\partial y_2}\right) + 2\mu_j \frac{\partial u_1^0}{\partial y_1} + 2\mu_1 \frac{\partial u_2^0}{\partial y_2} = 0$$

$$\lambda_j \left(\frac{\partial u_1^0}{\partial y_1} + \frac{\partial u_2^0}{\partial y_2}\right) + 2\mu_j \frac{\partial u_2^0}{\partial y_2} + 2\mu_1 \frac{\partial u_1^0}{\partial y_1} = 0$$

$$\mu_j \left(\frac{\partial u_1^0}{\partial y_2} + \frac{\partial u_2^0}{\partial y_1}\right) - 2\mu_1 \frac{\partial u_1^0}{\partial y_2} = 0$$

$$\mu_j \left(\frac{\partial u_1^0}{\partial y_2} + \frac{\partial u_2^0}{\partial y_1}\right) - 2\mu_1 \frac{\partial u_2^0}{\partial y_1} = 0.$$

So, in phase 1, that is on $Z_1^{\#}$, using (2.3) we get

(3.13) $$\frac{\partial u_1^0}{\partial y_1} + \frac{\partial u_2^0}{\partial y_2} = 0, \quad \frac{\partial u_2^0}{\partial y_1} - \frac{\partial u_1^0}{\partial y_2} = 0$$

while in phase 2, that is on $Z_2^{\#}$, still using (2.3) we get

(3.14) $$\frac{\partial u_1^0}{\partial y_1} = \frac{\partial u_2^0}{\partial y_2}, \quad \frac{\partial u_2^0}{\partial y_1} = \frac{\partial u_1^0}{\partial y_2} = 0.$$

From (3.13) we conclude that, in phase 1,

(3.15) $$\triangle_y u_1^0 = \triangle_y u_2^0 = 0.$$

Step 1 – u^0 does not oscillate. We now exploit the two previous set of relations under the micro-geometric assumptions of Theorem 3.3 to demonstrate that

$$(3.16) \qquad u^0(x,y) = u(x) \text{ is independent of } y,$$

where, thanks to (3.7),

$$(3.17) \qquad u^\varepsilon \rightharpoonup u, \text{ weakly in } L^2(\Omega; \mathbb{R}^2).$$

We first notice that, in view of (3.14) and because $X_2^{\#}$ is connected (see (2.2)),

$$u_1^0(x,y) = \alpha(x)\, y_1 + \beta(x), \quad u_2^0(x,y) = \alpha(x)\, y_2 + \gamma(x), \quad y \in X_2^{\#},$$

for some functions $\alpha(x), \beta(x), \gamma(x)$. By Y_2-periodicity of u_i^0 and because $X_2^{\#}$ is unbounded, $\alpha(x) = 0$. Thus $\nabla_y u^0 = 0$, or equivalently,

$$(3.18) \qquad u^0(x,y) = u(x),\ y \in Z_2^{\#} \left(= \bigcup_{\kappa \in \mathbb{Z}^2} (X_2^{\#} + \kappa) \right),$$

for some $u \in L^2(\Omega; \mathbb{R}^2)$.

Consider $\Phi \in C_{\text{per}}^1(Y; \mathbb{R}^{2\times2})$ with

$$(3.19) \qquad \sum_{ijh} (\mathbb{K}_1 - \mathbb{K}_2)_{ijkh} \Phi_{ij}(y)\nu_h(y) = 0 \quad \text{on } \partial Z_1^{\#},$$

that condition being necessary for $\text{div}_y(\mathbb{K}(y)\Phi(y))$ to be an admissible test function for two-scale convergence. In (3.19) $\nu(y)$ denotes the exterior normal to $Z_2^{\#}$ at y.

In view of (3.10), (3.18), (3.19), we get that, for any $\varphi \in C_c^\infty(\Omega; C_{\text{per}}^\infty(Y))$,

$$
\begin{aligned}
0 = &-\lim_{\varepsilon \to 0} \int_\Omega \varepsilon\, \mathbb{K}(x/\varepsilon)\nabla u_\varepsilon \cdot \varphi(x, x/\varepsilon)\Phi(x/\varepsilon)\, dx \\
&= \lim_{\varepsilon \to 0} \int_\Omega \left[\frac{\partial}{\partial y_h} \big\{ (\mathbb{K}(y))_{ijkh}\varphi(x,y)\Phi_{ij}(y) \big\} \right](x, x/\varepsilon)\, (u_\varepsilon)_k\, dx = \\
&\sum_{ijkh} \int_{\Omega \times Z_1} \frac{\partial}{\partial y_h} \big\{ (\mathbb{K}_1)_{ijkh}\varphi(x,y)\Phi_{ij}(y) \big\} u_k^0(x,y)\, dx\, dy\ + \\
&\sum_{ijkh} \int_{\Omega \times Z_2} \frac{\partial}{\partial y_h} \big\{ (\mathbb{K}_2)_{ijkh}\varphi(x,y)\Phi_{ij}(y) \big\} u_k(x)\, dx\, dy = \\
&\sum_{ijkh} \int_{\Omega \times Z_1} \frac{\partial}{\partial y_h} \big\{ (\mathbb{K}_1)_{ijkh}\varphi(x,y)\Phi_{ij}(y) \big\} u_k^0(x,y)\, dx\, dy\ + \\
&\qquad \sum_{ijkh} \int_{\Omega \times \partial Z_1} (\mathbb{K}_1)_{ijkh} u_k(x)\nu_h(y)\varphi(x,y)\Phi_{ij}(y)\, dx\, d\mathcal{H}_y^1.
\end{aligned}
$$

Set $v^0(x,y) := u^0(x,y) - u(x)$. Then,

$$\int_{\Omega \times Z_1} \text{div}_y \big\{ \varphi(x,y)\, \mathbb{K}_1\Phi(y) \big\} \cdot v^0(x,y)\, dx\, dy = 0.$$

Now take $\varphi \in C_c^\infty(\Omega \times \mathbb{R}^2)$. Using the periodized function

$$\varphi^{\#}(x,y) := \sum_{\kappa \in \mathbb{Z}^2} \varphi(x, y + \kappa)$$

as new test function we obtain

$$
\begin{aligned}
0 \;&=\; \int_{\Omega \times Z_1} \mathrm{div}_y\big\{\varphi^{\#}(x,y)\,\mathbb{K}_1\Phi(y)\big\} \cdot v^0(x,y)\,dx\,dy \\
(3.20) \qquad &=\; \sum_{\kappa \in \mathbb{Z}^2} \int_{\Omega \times (Z_1+\kappa)} \mathrm{div}_y\big\{\varphi(x,y)\,\mathbb{K}_1\Phi(y)\big\} \cdot v^0(x,y)\,dx\,dy \\
&=\; \int_{\Omega \times Z_1^{\#}} \mathrm{div}_y\big\{\varphi(x,y)\,\mathbb{K}_1\Phi(y)\big\} \cdot v^0(x,y)\,dx\,dy.
\end{aligned}
$$

Now simple algebra using the explicit expression for $\mathbb{K}_1, \mathbb{K}_2$ as well as (2.3) shows that, for any $\xi \in \mathbb{R}^2$ and $\nu \in S^1$, there exists a unique matrix Φ such that

$$
(3.21) \qquad\qquad (\mathbb{K}_1\Phi)\,\nu = (\mathbb{K}_2\Phi)\,\nu = \xi,
$$

so that, in particular, (3.19) can always be met, provided that each connected component of $\mathscr{Z}_1$ has a C^2 boundary because the normal $\nu(y)$ is then a C^1-function of $y \in \partial \mathscr{Z}_1$ so that one can define $\Phi(y)$ satisfying (3.21) as a C^1 function on $\partial \mathscr{Z}_1$, hence by *e.g.* Whitney's extension theorem as a C^1 function on $\mathscr{T}_2$.

Consider a connected component $Z^{\#}$ of $Z_1^{\#}$ in $\mathbb{R}^2$. Recall that $\mathbb{K}_1 \nabla_y v^0 = 0$ in $Z_1^{\#}$. In view of (3.20), (3.21), and the arbitrariness of φ, ξ, an integration by parts yields that $v^0(x,\cdot)$ has a trace on $\partial Z^{\#}$ which satisfies

$$
(3.22) \qquad\qquad v^0(x,\cdot) = 0 \text{ on } \partial Z^{\#}.
$$

Fix x. According to (3.13), there exists a potential $\zeta_x \in H^1(Z^{\#} \cap (-A,A)^2)$ for any $A > 0$, such that

$$
v^0(x,y) = R^{\perp}\nabla \zeta_x(y)
$$

and

$$
\triangle_y \zeta_x = 0 \ \text{ in } Z^{\#}.
$$

Further, in view of (3.22),

$$
R^{\perp}\nabla \zeta_x \cdot \nu = \nabla \zeta_x \cdot \nu^{\perp} = 0 \ \text{ on } \partial Z^{\#},
$$

so that ζ_x is constant on each connected component of $\partial Z^{\#}$. Thus, by elliptic regularity $\zeta_x \in H^2(Z^{\#} \cap (-A,A)^2)$ for any $A > 0$, hence $v^0 \in H^1(Z^{\#} \cap (-A,A)^2)$ for any $A > 0$. Thanks to (3.15), (3.22) and the periodicity of $v^0(x,\cdot)$, we conclude that $v^0 \equiv 0$, hence (3.16).

Step 2 – Identification of the Γ-$\liminf$. Consider $\Phi \in L^2_{\mathrm{per}}(Y;\mathbb{R}^{2\times2})$ such that

$$
(3.23) \qquad\qquad \mathrm{div}\,\big(\mathbb{K}^{\frac{1}{2}}(y)\Phi(y)\big) = 0 \ \ \text{ in } \mathbb{R}^2,
$$

or equivalently,

$$
\int_Y \mathbb{K}^{\frac{1}{2}}(y)\Phi(y) \cdot \nabla \psi(y)\,dy = 0 \ \ \ \forall\,\psi \in H^1_{\mathrm{per}}(Y;\mathbb{R}^2),
$$

and also consider $\varphi \in C^{\infty}(\bar{\Omega})$.

Then, since $u^{\varepsilon} \in H^1_0(\Omega;\mathbb{R}^2)$ and in view of (3.23),

$$
\int_{\Omega} \varphi(x)\,\mathbb{K}^{\frac{1}{2}}(x/\varepsilon)\nabla u^{\varepsilon} \cdot \Phi(x/\varepsilon)\,dx = -\sum_{ijkh} \int_{\Omega} u^{\varepsilon}_k \cdot \mathbb{K}^{\frac{1}{2}}_{ijkh}(x/\varepsilon)\Phi_{ij}(x/\varepsilon)\frac{\partial \varphi}{\partial x_h}(x)\,dx.
$$

Recalling (3.7), (3.9), we can pass to the two-scale limit in the previous expression and obtain, thanks to (3.16),
(3.24)
$$\int_{\Omega \times Y} \varphi(x) S(x,y) \cdot \Phi(y) \, dx \, dy = - \sum_{ijkh} \int_{\Omega \times Y} u_k(x) \cdot \mathbb{K}^{\frac{1}{2}}_{ijkh}(y) \Phi_{ij}(y) \frac{\partial \varphi}{\partial x_h}(x) \, dx \, dy.$$

Assume henceforth that $u \in H^1(\Omega; \mathbb{R}^2)$. Then, (3.24) implies that
$$\int_{\Omega \times Y} S(x,y) \cdot \Phi(y) \, \varphi(x) \, dx \, dy = \int_{\Omega \times Y} \mathbb{K}^{\frac{1}{2}}(y) \nabla_x u(x) \cdot \Phi(y) \, \varphi(x) \, dx \, dy.$$

By density, the result still holds with the test functions $\varphi(x)\Phi(y)$ replaced by the set of $\Psi(x,y) \in L^2(\Omega; L^2_{\text{per}}(Y; \mathbb{R}^{2 \times 2}))$ such that
$$\text{div}_y\big(\mathbb{K}^{\frac{1}{2}}(y) \Psi(x,y)\big) = 0 \quad \text{in } \mathbb{R}^2,$$

or equivalently, due to the symmetry of $\mathbb{K}(y)$,
$$\int_{\Omega \times Y} \Psi(x,y) \cdot \mathbb{K}^{\frac{1}{2}}(y) \nabla_y v(x,y) \, dy = 0 \quad \forall v \in L^2(\Omega; H^1_{\text{per}}(Y; \mathbb{R}^2)).$$

The $L^2(\Omega; L^2_{\text{per}}(Y; \mathbb{R}^{2 \times 2}))$-orthogonal to that set is the L^2-closure of
$$\mathscr{K}_\nabla := \left\{ \mathbb{K}^{\frac{1}{2}}(y) \nabla_y v(x,y) : v \in L^2(\Omega; H^1_{\text{per}}(Y; \mathbb{R}^2)) \right\}.$$

Thus,
$$S(x,y) = \mathbb{K}^{\frac{1}{2}}(y) \nabla_x u(x) + \xi(x,y)$$

for some ξ in the closure of $\mathscr{K}_\nabla$ and there exists a sequence
$$v_n \in L^2(\Omega; H^1_{\text{per}}(Y; \mathbb{R}^2))$$

such that $\mathbb{K}^{\frac{1}{2}}(y) \nabla_y v_n \to \xi$, strongly in $L^2(\Omega; L^2_{\text{per}}(Y; \mathbb{R}^{2 \times 2}))$.

We now appeal to [1, Proposition 1.6] which yields

$$(3.25) \quad \liminf_{\varepsilon \to 0} \|\mathbb{K}^{\frac{1}{2}}(x/\varepsilon) \nabla u^\varepsilon\|^2_{L^2(\Omega; \mathbb{R}^{2 \times 2}))} \geq \|S\|^2_{L^2(\Omega \times Y; \mathbb{R}^{2 \times 2})} =$$
$$\lim_n \|\mathbb{K}^{\frac{1}{2}}(y) \nabla_x u(x) + \mathbb{K}^{\frac{1}{2}}(y) \nabla_y v_n\|^2_{L^2(\Omega \times Y; \mathbb{R}^2)}.$$

But recall that
$$\|\mathbb{K}^{\frac{1}{2}}(x/\varepsilon) \nabla u^\varepsilon\|^2_{L^2(\Omega; \mathbb{R}^{2 \times 2}))} = \int_\Omega \mathbb{K}(x/\varepsilon) \nabla u^\varepsilon \cdot \nabla u^\varepsilon \, dx = \int_\Omega \mathbb{L}(x/\varepsilon) \nabla u^\varepsilon \cdot \nabla u^\varepsilon \, dx$$

because the determinant is a null Lagrangian.

Thus, from (3.25) and by weak L^2-lower semi-continuity of $\|u^\varepsilon\|_{L^2(\Omega; \mathbb{R}^2)}$ we conclude that

$$(3.26) \quad \liminf_{\varepsilon \to 0} \mathscr{I}^\varepsilon(u^\varepsilon) \geq$$

$$\lim_n \int_{\Omega \times Y} \mathbb{K}(y)(\nabla_x u(x) + \nabla_y v_n(x,y)) \cdot (\nabla_x u(x) + \nabla_y v_n(x,y)) \, dx \, dy + \int_\Omega |u|^2 \, dx \geq$$

$$\inf \left\{ \int_{\Omega \times Y} \mathbb{K}(y)(\nabla_x u(x) + \nabla_y v(y)) \cdot (\nabla_x u(x) + \nabla_y v(y)) \, dx \, dy : v \in H^1_{\text{per}}(Y; \mathbb{R}^2) \right\}$$

$$+ \int_\Omega |u|^2 \, dx.$$

In the light of the definition (2.1) for $\mathbb{L}^0$, we finally get

$$\liminf_{\varepsilon \to 0} \mathscr{I}^\varepsilon(u^\varepsilon) \geq \int_\Omega \left\{ \mathbb{L}^0 \nabla_x u \cdot \nabla_x u + |u|^2 \right\} dx$$

provided that $u \in H^1(\Omega; \mathbb{R}^2)$, hence, *a fortiori* provided that $u \in H_0^1(\Omega; \mathbb{R}^2)$.

The proof of Theorem 3.3 is complete.

3.2. **Proof of Theorem 3.4.** Recall that, in the proof of Theorem 3.3, we were at liberty to assume that

$$\mathscr{I}^\varepsilon(u^\varepsilon) \leq C < \infty,$$

otherwise the Γ-lim inf inequality is trivially verified. Consequently, if we can show that, under that condition, the target function u is in $H_0^1(\Omega; \mathbb{R}^2)$, then we will be done as remarked at the onset of Subsection 3.1. Such will be the case except when dealing with straight layers (case (ii)) under the condition that $\theta = 1/2$. In that case we will have to show that, for those target fields u that are not in $H_0^1(\Omega; \mathbb{R}^2)$, a recovery sequence for the Γ- lim sup (in)equality can be obtained by density.

Returning to (3.24), setting

(3.27) $$N_{kh} := \sum_{ij} \int_Y \mathbb{K}_{ijkh}^{\frac{1}{2}}(y) \Phi_{ij}(y) \, dy$$

and varying φ in $C_c^\infty(\Omega)$, we conclude that

(3.28) $$N \cdot \nabla u \in L^2(\Omega).$$

We now remark that $\mathbb{K}(y)$, a symmetric mapping on $\mathbb{R}^{2\times2}$, has for eigenvalues $2(\lambda(y) + \mu(y) + \mu_1)$, $2\mu_1$ and $2(\mu(y) - \mu_1)$ with eigenspaces respectively generated by

$$I_2, \quad R^\perp,$$

and, for the last eigenvalue, by

$$G := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Consequently, its kernel for $y \in Z_2$ is

(3.29) $$\text{Ker}\left(\mathbb{K}(y)\right) = \text{Ker}\left(\mathbb{K}_2\right) := \{\gamma I_2, \ \gamma \text{ arbitrary in } \mathbb{R}\},$$

while its kernel for $y \in Z_1$ is

(3.30) $$\text{Ker}\left(\mathbb{K}(y)\right) = \text{Ker}\left(\mathbb{K}_1\right) := \left\{ \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} : \alpha, \beta \text{ arbitrary in } \mathbb{R} \right\}.$$

Step 1 – Case (i). First assume that $\bar{Z}_1 \subset \mathring{Y}$ and that $M \in \mathbb{R}_s^{2\times2}$. We then define

$$\Psi(y) := \frac{1}{2} \{My \cdot y - \varphi(y - \kappa)M(y - \kappa) \cdot (y - \kappa)\} \quad \text{for any } y \in Y + \kappa, \ \kappa \in \mathbb{Z}^2,$$

with $\varphi \in C_c^2(\mathring{Y})$, $\varphi \equiv 1$ in Z_1. Then clearly, $\nabla \Psi - My \in H_{\text{per}}^1(Y; \mathbb{R}^2)$. Further, $\nabla \Psi = M\kappa$ in $Z_1 + \kappa$ hence $\nabla^2 \Psi \equiv 0$ in $Z_1 + \kappa$ while $\nabla^2 \Psi R^\perp \in \text{Ker}^\perp(\mathbb{K}_2)$ (because any Hessian matrix is symmetric) thus it belongs to the range of $\mathbb{K}_2$ in $Z_2 + \kappa$.

It is thus meaningful to define $\Phi(y) := \mathbb{K}^{-\frac{1}{2}}(y)\left(\nabla^2\Psi(y) R^\perp\right)$ where $\mathbb{K}^{-\frac{1}{2}}$ is the pseudo-inverse of $\mathbb{K}^{\frac{1}{2}}$ (see the notation at the close of the introduction).

We get

$$\int_Y \mathbb{K}^{\frac{1}{2}}(y)\Phi(y) \, dy = \int_Y \nabla^2\Psi(y) R^\perp \, dy = MR^\perp,$$

while Φ satisfies (3.23) since for any $v \in H^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^2)$ with periodic gradient,

$$\text{div}(\nabla v \, R^{\perp}) = 0 \quad \text{in } \mathbb{R}^2,$$

or equivalently,

$$\int_Y \nabla v(y) \, R^{\perp} \cdot \nabla \psi(y) \, dy = 0 \quad \forall \, \psi \in H^1_{\text{per}}(Y; \mathbb{R}^2).$$

We finally obtain by (3.28) that $MR^{\perp} \cdot \nabla u \in L^2(\Omega)$. Since, when M spans $\mathbb{R}^{2\times 2}_s$, $N := MR^{\perp}$ spans the set of all 2×2 trace-free matrices, we infer from (3.28) that

$$\frac{\partial u_1}{\partial x_2}, \; \frac{\partial u_2}{\partial x_1}, \; \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} \text{ are in } L^2(\Omega).$$

This is equivalent to stating that $\mathbf{E}(R^{\perp}u) \in L^2(\Omega; \mathbb{R}^{2\times 2}_s)$. Since Ω is Lipschitz, Korn's inequality allows us to conclude that $R^{\perp}u \in H^1(\Omega; \mathbb{R}^2)$, hence that

$$(3.31) \qquad\qquad u \in H^1(\Omega; \mathbb{R}^2)$$

in that case.

Since, for an arbitrary trace-free matrix N, we can choose Φ constrained by (3.23) so that (3.27) is satisfied, then actually

$$(3.32) \qquad\qquad u \in H^1_0(\Omega; \mathbb{R}^2).$$

Indeed, take $x^0 \in \partial\Omega$ to be a Lebesgue point for $u\lfloor \partial\Omega$ – which lies in particular in $L^2_{\mathscr{H}^1}(\partial\Omega; \mathbb{R}^2)$ – as well as for $\nu(x^0)$, the exterior normal to Ω at x^0. Then take an arbitrary trace-free N and the associated Φ.

By (3.31) we already know that $u \in H^1(\Omega; \mathbb{R}^2)$, so that (3.24) reads as

$$\int_{\Omega \times Y} \varphi(x) S(x,y) \cdot \Phi(y) \, dx \, dy =$$

$$\int_\Omega N \cdot \nabla u(x) \, \varphi(x) \, dx - \int_{\partial\Omega} N\nu(x) \cdot u(x) \, \varphi(x) \, d\mathscr{H}^1.$$

But, taking first $\varphi \in C^\infty_c(\Omega)$ and remarking that, in such a case, the first two integrals are equal and bounded by a constant times $\|\varphi\|_{L^2(\Omega)}$, we immediately conclude that, for any $\varphi \in C^\infty(\bar\Omega)$,

$$\int_{\partial\Omega} N\nu(x) \cdot u(x) \, \varphi(x) \, d\mathscr{H}^1 = 0.$$

Thus, $N\nu(x) \cdot u(x) = 0$ $\mathscr{H}^1$-a.e. on $\partial\Omega$, hence, since x^0 is a Lebesgue point, $N\nu(x^0) \cdot u(x^0) = 0$ from which it is immediately concluded that $u_i(x^0) = 0$, $i = 1, 2$, hence (3.32).

But, in such a case we can apply Theorem 3.3 which thus delivers the Γ-limit.

Step 2 – Case (ii). Assume now that Z_1 is a straight layer, that is that there exists $0 < \theta < 1$ such that $(0, \theta) \times (0, 1) = Z_1 \cap \mathring{Y}$.

For an arbitrary matrix $M \in \mathbb{R}^{2\times 2}$ define $v(y)$ as

$$v(y) := My + \left(\int_0^{y_1} [\chi(t) - \theta] \, dt \right) \xi, \; \xi \in \mathbb{R}^2,$$

where $\chi(y)$ is the characteristic function of phase 1 with volume fraction

$$\int_{Y_2} \chi(y) \, dy := \theta.$$

Then, $v(y) - My$ is Y_2-periodic and

$$\nabla v(y) = \chi(y_1)(M + (1-\theta)\xi \otimes e_1) + (1-\chi(y_1))(M - \theta\xi \otimes e_1).$$

According to (3.30), (3.29), for $\nabla v(y)R^\perp$ to be in the range of $\mathbb{K}(y)$ we must have both

$$(M + (1-\theta)\xi \otimes e_1)R^\perp \cdot G = (M + (1-\theta)\xi \otimes e_1)R^\perp \cdot H = 0$$

i.e., $(1-\theta)\xi_1 = M_{22} - M_{11}$, $(1-\theta)\xi_2 = -M_{12} - M_{21}$ and

$$(M - \theta\xi \otimes e_1)R^\perp \cdot I_2 = 0,$$

i.e., $\theta\xi_2 = -M_{12} + M_{21}$.

Since ξ can be arbitrary, this imposes as sole condition on M that

$$(2\theta - 1)M_{12} + M_{21} = 0.$$

Then,

$$\int_{Y_2} \nabla v(y)R^\perp \, dy = MR^\perp = \begin{pmatrix} M_{12} & -M_{11} \\ M_{22} & (2\theta-1)M_{12} \end{pmatrix} = N_\theta$$

where

$$N_\theta = \begin{pmatrix} a & c \\ b & (2\theta-1)a \end{pmatrix}, \quad \text{with } a, b, c \text{ arbitrary.}$$

In view of (3.28), we obtain that

$$\frac{\partial u_1}{\partial x_2}, \ \frac{\partial u_2}{\partial x_1}, \ \frac{\partial u_1}{\partial x_1} + (2\theta-1)\frac{\partial u_2}{\partial x_2} \text{ are in } L^2(\Omega),$$

or equivalently when $\theta \neq 1/2$,

$$\mathbf{E}(P_\theta u) \in L^2(\Omega; \mathbb{R}^{2 \times 2}) \quad \text{with} \quad P_\theta := \begin{pmatrix} 0 & 2\theta-1 \\ 1 & 0 \end{pmatrix}.$$

Using Korn's inequality once again, we thus conclude that $u \in H^1(\Omega; \mathbb{R}^2)$, except when $\theta = 1/2$ in which case $\dfrac{\partial u_2}{\partial x_2}$ might not be in $L^2(\Omega)$.

Remark 3.8. Actually, when $\theta = 1/2$, then all Φ's that are such that (3.23) is satisfied produce, through (3.27), a matrix M with $M_{21} = 0$, hence a matrix $N := MR^\perp$ such that $N_{22} = 0$.

Indeed, the existence of Φ is equivalent to that of $v(y) = My + w(y)$ with $w \in H^1_{\text{per}}(Y; \mathbb{R}^2)$ such that

$$\nabla v(y)R^\perp = \mathbb{K}^{\frac{1}{2}}(y_1)\Phi(y)$$

which implies that, for a.e. $y_1 \in (0,1)$,

$$\mathbb{K}^{\frac{1}{2}}(y_1)\int_0^1 \Phi(y_1, t) \, dt = \int_0^1 \nabla v(y_1, t)R^\perp \, dt.$$

In view of (3.30), (3.29), the last relation yield in particular that

$$\begin{cases} v_1(y_1, 1) - v_1(y_1, 0) = -\displaystyle\int_0^1 \frac{\partial v_2}{\partial y_1}(y_1, y_2) \, dy_2, \ 0 \leq y_1 \leq 1/2 \\[3mm] v_1(y_1, 1) - v_1(y_1, 0) = +\displaystyle\int_0^1 \frac{\partial v_2}{\partial y_1}(y_1, y_2) \, dy_2, \ 1/2 \leq y_1 \leq 1, \end{cases}$$

or still, since $v(y) = My + w(y)$ with w Y-periodic,

$$\begin{cases} M_{12} = + \displaystyle\int_0^1 \frac{\partial v_2}{\partial y_1}(y_1, y_2) \, dy_2, \ 0 \le y_1 \le 1/2 \\[2mm] M_{12} = - \displaystyle\int_0^1 \frac{\partial v_2}{\partial y_1}(y_1, y_2) \, dy_2, \ 1/2 \le y_1 \le 1, \end{cases}$$

But then

$$M_{21} = \int_Y \frac{\partial v_2}{\partial y_1}(y_1, y_2) \, dy_1 dy_2 = 0. \qquad\qquad ¶$$

Then, through an argument identical to that used in case (i), we find that, for x^0 Lebesgue point for $u_1 \lfloor \partial\Omega$ (and for $u_2 \lfloor \partial\Omega$ as well if $\theta \neq 1/2$), $u_1(x^0) = 0$ (and $u_2(x^0) = 0$ if $\theta \neq 1/2$) while, if $\theta = 1/2$, $u_2 \, \nu_1$, which is well defined as an element of $H^{-\frac{1}{2}}(\partial\Omega)$, satisfies $u_2 \, \nu_1 = 0$.

So, here again, we can apply Theorem 3.3 provided that $\theta \neq 1/2$. It thus remains to compute the Γ-limit in case (ii) when $\theta = 1/2$. This is the object of the last step below.

Step 3 – Identification of the Γ-limit – case (ii) – $\theta = 1/2$; the Gutiérrez case. As far as the Γ-lim sup inequality is concerned there is nothing to prove once again, because, as already stated at the onset of Subsection 3.1 we know the existence of a recovery sequence for any target field $u \in H_0^1(\Omega; \mathbb{R}^2)$. But, according to Remark 3.2, $H_0^1(\Omega; \mathbb{R}^2)$ is *a fortiori* dense in $\mathbf{X}$. So any element $u \in \mathbf{X}$ can be in turn viewed as the limit in the topology induced by the inner product $\langle,\rangle_{\mathbf{X}}$ of a sequence $u^p \in H_0^1(\Omega; \mathbb{R}^2)$. Since, as noted in Remark 3.1, $\partial u_2^p / \partial x_2$ does not enter the expression

$$\int_\Omega \mathbb{L}^0 \nabla u^p \cdot \nabla u^p \, dx,$$

we immediately get that

$$\lim_p \int_\Omega \mathbb{L}^0 \nabla u^p \cdot \nabla u^p \, dx = \int_\Omega \mathbb{L}^0 \nabla u \cdot \nabla u \, dx.$$

A diagonalization process concludes the argument.

Consider now, for $u \in \mathbf{X}$, a sequence $u^\varepsilon \in H_0^1(\Omega; \mathbb{R}^2)$ such that $u^\varepsilon \rightharpoonup u$ weakly in $L^2(\Omega; \mathbb{R}^2)$. We revisit Step 2 in the proof of Theorem 3.3 in Subsection 3.1, taking into account Remark 3.8. Since, because of that remark, $N_{22} = 0$, (3.24) now reads as

$$\int_{\Omega \times Y} \varphi(x) S(x, y) \cdot \Phi(y) \, dx \, dy = - \sum_{\substack{ijkh \\ (k,h) \neq (2,2)}} \int_{\Omega \times Y} u_k(x) \, \mathbb{K}_{ijkh}^{\frac{1}{2}}(y) \Phi_{ij}(y) \frac{\partial \varphi}{\partial x_h}(x) \, dx \, dy.$$

The rest of the argument goes through exactly as in Step 2, yielding, in lieu of (3.26),

$$(3.33) \quad \liminf_{\varepsilon \to 0} \mathscr{I}^\varepsilon(u^\varepsilon) \geq$$

$$\inf \left\{ \int_{\Omega \times Y} \mathbb{K}(y)(\nabla'_x u(x) + \nabla_y v(y)) \cdot (\nabla'_x u(x) + \nabla_y v(y)) \, dx \, dy : v \in H^1_{\mathrm{per}}(Y; \mathbb{R}^2) \right\}$$

$$+ \int_\Omega |u|^2 \, dx$$

$$= \inf \left\{ \int_{\Omega \times Y} \mathbb{L}(y)(\nabla'_x u(x) + \nabla_y v(y)) \cdot (\nabla'_x u(x) + \nabla_y v(y)) \, dx \, dy : v \in H^1_{\mathrm{per}}(Y; \mathbb{R}^2) \right\}$$

$$+ 4\mu_1 \int_\Omega \det \nabla'_x u \, dx + \int_\Omega |u|^2 \, dx$$

$$= \int_\Omega \mathbb{L}^0 \nabla'_x u \cdot \nabla'_x u \, dx + 4\mu_1 \int_\Omega \det \nabla'_x u \, dx + \int_\Omega |u|^2 \, dx$$

with $\nabla'_x u := \nabla_x u - \dfrac{\partial u_2}{\partial x_2} e_2 \otimes e_2$.

It now suffices to remark that, in this specific setting and because $-\lambda_2 - \mu_2 = \mu_1$, the precise expression for $\mathbb{L}^0$ in the basis (e_1, e_2) is as follows (see [9]):

$$(3.34) \qquad\qquad \mathbb{L}^0_{1111} = \frac{2}{\dfrac{1}{\lambda_1 + 2\mu_1} + \dfrac{1}{\lambda_2 + 2\mu_2}} =: \bar\lambda + 2\bar\mu_1,$$

$$\mathbb{L}^0_{1212} = \mathbb{L}^0_{1221} = \mathbb{L}^0_{2112} = \mathbb{L}^0_{2121} = \frac{2\mu_1\mu_2}{\mu_1 + \mu_2} =: \bar\mu_2,$$

$$\mathbb{L}^0_{1122} = \mathbb{L}^0_{2211} = \frac{\dfrac{\lambda_1}{\lambda_1 + 2\mu_1} + \dfrac{\lambda_2}{\lambda_2 + 2\mu_2}}{\dfrac{1}{\lambda_1 + 2\mu_1} + \dfrac{1}{\lambda_2 + 2\mu_2}} = -2\mu_1 =: \bar\lambda,$$

$$\mathbb{L}^0_{1112} = \mathbb{L}^0_{1121} = \mathbb{L}^0_{2111} = \mathbb{L}^0_{1211} = 0,$$

$$\mathbb{L}^0_{1222} = \mathbb{L}^0_{2122} = \mathbb{L}^0_{2212} = \mathbb{L}^0_{2221} = 0 \quad \text{and} \quad \mathbb{L}^0_{2222} = 0.$$

Consequently, recalling Remark 3.1,

$$\int_\Omega \mathbb{L}^0 \nabla_x u \cdot \nabla_x u \, dx = \int_\Omega \mathbb{L}^0 \nabla'_x u \cdot \nabla'_x u \, dx + 2 \, \mathbb{L}^0_{1122} \int_\Omega \frac{\partial u_1}{\partial x_2} \frac{\partial u_2}{\partial x_1} \, dx$$

$$= \int_\Omega \mathbb{L}^0 \nabla'_x u \cdot \nabla'_x u \, dx + 4\mu_1 \int_\Omega \det \nabla'_x u \, dx.$$

which, in view of (3.33), proves the Γ-$\liminf$ inequality in the Gutiérrez case.

Remark 3.9. For future reference, we name the material obtained in (3.34) the Gutiérrez material and observe that it can be labeled 2D orthorhombic since it is invariant under symmetry about the two lines $x_1 = 0$ and $x_2 = 0$. ¶

3.3. **A corollary of Theorem 3.4.** We conclude this section with a corollary of Theorem 3.4 that will play an essential role in Section 4 below.

Corollary 3.10. *In the setting of Theorem 3.4, consider the following sequence of functionals*

$$\mathscr{G}^\varepsilon(v) := \begin{cases} \displaystyle\int_\Omega \left\{ \mathbb{L}(x/\varepsilon)\nabla v \cdot \nabla v + a(x/\varepsilon)|v|^2 - b(x/\varepsilon)f \cdot v \right\} dx, & v \in H_0^1(\Omega;\mathbb{R}^2) \\ \infty, & else. \end{cases}$$

where $f \in L^2(\Omega;\mathbb{R}^2)$, $a, b \in L^\infty_{\mathrm{per}}(Y_2;\mathbb{R})$ and $a(y) \geq \alpha$ a.e. in $\mathbb{R}^2$ for some $\alpha > 0$.

Then the results of those theorems still hold true upon replacing $\mathscr{I}^\varepsilon$ by $\mathscr{G}^\varepsilon$ and $\mathscr{I}^0$, resp. $\mathscr{I}^0_{1/2}$, by $\mathscr{G}^0$, resp. $\mathscr{G}^0_{1/2}$, defined by

$$\mathscr{G}^0(v) := \begin{cases} \displaystyle\int_\Omega \left\{ \mathbb{L}^0\nabla v \cdot \nabla v + \bar{a}|v|^2 - \bar{b}f \cdot v \right\} dx, & v \in H_0^1(\Omega;\mathbb{R}^2) \\ \infty, & else, \end{cases}$$

and, under the additional assumption that $\mathbb{L}^0_{2222} = 0$,

$$\mathscr{G}^0_{1/2}(v) := \begin{cases} \displaystyle\int_\Omega \left\{ \mathbb{L}^0\nabla v \cdot \nabla v + \bar{a}|v|^2 - \bar{b}f \cdot v \right\} dx, & v \in \mathbf{X} \\ \infty, & else, \end{cases}$$

where $\bar{a} := \int_Y a(y)\, dy$ (idem for $\bar{b}$).

Proof. Once again the Γ-$\limsup$ inequality is straightforward since, for any $u \in H_0^1(\Omega;\mathbb{R}^2)$, or in $\mathbf{X}$ depending on the case considered, the recovery sequences $\{u^\varepsilon\}_\varepsilon$ are bounded in $H_0^1(\Omega;\mathbb{R}^2)$, so that Rellich's theorem permits one to pass to the limit in the zeroth order term. We obtain

$$\lim_\varepsilon \int_\Omega a(x/\varepsilon)|u^\varepsilon|^2\, dx = \bar{a}\int_\Omega |u|^2\, dx, \quad \lim_\varepsilon \int_\Omega b(x/\varepsilon)f \cdot u^\varepsilon\, dx = \bar{b}\int_\Omega f \cdot v\, dx.$$

As far as the Γ-$\liminf$ inequality is concerned, we still have (3.16), that is that, if a sequence $\{u^\varepsilon\}_\varepsilon$ converges weakly in $L^2(\Omega;\mathbb{R}^2)$ to $u \in L^2(\Omega;\mathbb{R}^2)$ and two-scale converges to $u^0(x,y)$, then u^0 does not depend on y. In other words, $u^0(x,y) = u(x)$. But then

$$\begin{cases} a(x/\varepsilon)u^\varepsilon \; \overset{\scriptscriptstyle 2}{\rightharpoonup} \; a(y)u(x) \\ b(x/\varepsilon)u^\varepsilon \; \overset{\scriptscriptstyle 2}{\rightharpoonup} \; b(y)u(x) \end{cases}$$

which implies in turn that

$$\begin{cases} a(x/\varepsilon)u^\varepsilon \rightharpoonup \bar{a}\,u \\ b(x/\varepsilon)u^\varepsilon \rightharpoonup \bar{b}\,u \end{cases} \qquad \text{weakly in } L^2(\Omega;\mathbb{R}^2).$$

Consequently passing to the limit in the linear term is immediate while, as far as the quadratic zeroth order term is concerned it suffices to remark that

$$0 \leq \liminf_\varepsilon \int_\Omega a(x/\varepsilon)|u^\varepsilon - u|^2 \, dx = \liminf_\varepsilon \int_\Omega a(x/\varepsilon)|u^\varepsilon|^2 \, dx -$$

$$2 \lim_\varepsilon \int_\Omega a(x/\varepsilon)u^\varepsilon \cdot u \, dx + \lim_\varepsilon \int_\Omega a(x/\varepsilon)|u|^2 \, dx =$$

$$\liminf_\varepsilon \int_\Omega a(x/\varepsilon)|u^\varepsilon|^2 \, dx - \int_\Omega \bar{a}|u|^2 \, dx.$$

With the above inequality at hand, the rest of the proof remains unchanged. $\square$

4. Elasto-dynamics

In this last section our goal is to investigate the impact of Theorem 3.4 (or rather of Corollary 3.10) on wave propagation with a particular emphasis on the Gutiérrez setting (case (ii) of Theorem 3.4 with $\theta = 1/2$) because of the loss of (strict) strong ellipticity ($\mathbb{L}^0_{2222} = 0$) demonstrated there.

4.1. **Convergence of minimizers.** The main purpose of Γ-convergence is to ensure the convergence of minimizers. In this respect, consider the ε-indexed sequence of functionals

$$\mathscr{I}^\varepsilon(v) - 2 \int_\Omega f \cdot v \, dx$$

where *e.g.* $f \in L^2(\Omega; \mathbb{R}^2)$ is a given force load. For a fixed ε, we first note in Lemma 4.1 below that H_0^1-coercivity holds true.

Lemma 4.1. *Under assumptions (2.2), (2.3), there exists a a sequence $\{\alpha_\varepsilon > 0\}_\varepsilon$ such that*

$$\mathscr{I}^\varepsilon(v) \geq \alpha_\varepsilon \int_\Omega \left\{|v|^2 + |\nabla v|^2\right\} dx, \ v \in H_0^1(\Omega; \mathbb{R}^2).$$

Proof. Assume that the conclusion does not hold. Then there exists a sequence $v_n \in H_0^1(\Omega; \mathbb{R}^2)$ such that

$$(4.1) \qquad \int_\Omega \left\{|v_n|^2 + |\nabla v_n|^2\right\} dx = 1,$$

while

$$(4.2) \qquad \lim_n \mathscr{I}^\varepsilon(v_n) = 0.$$

We can always extend the sequence v_n to $H^1(\mathbb{R}^2; \mathbb{R}^2)$ by setting $v_n \equiv 0$ outside Ω. In view of (3.5), (3.8), convergence (4.2) implies in particular that

$$(4.3) \qquad v_n \to 0 \text{ in } L^2(\mathbb{R}^2; \mathbb{R}^2).$$

Further, the explicit expressions for $\mathbb{K}_i$ imply that

$$(4.4) \qquad \int_{\mathbb{R}^\varepsilon_2} \left[\left(\frac{\partial v_1^n}{\partial x_2}\right)^2 + \left(\frac{\partial v_2^n}{\partial x_1}\right)^2 + \left(\frac{\partial v_1^n}{\partial x_1} - \frac{\partial v_2^n}{\partial x_2}\right)^2\right] dx \to 0,$$

where $\mathbb{R}_2^\varepsilon := \mathbb{R}^2 \cap \varepsilon Z_2^\#$, while

$$(4.5) \qquad \int_{\mathbb{R}_1^\varepsilon} \left[\left(\frac{\partial v_1^n}{\partial x_1} + \frac{\partial v_2^n}{\partial x_2} \right)^2 + \left(\frac{\partial v_1^n}{\partial x_2} - \frac{\partial v_2^n}{\partial x_1} \right)^2 \right] dx \to 0.$$

where $\mathbb{R}_1^\varepsilon := \Omega \cap \varepsilon Z_1^\#$.

But, as remarked before in the proof of Step 1 - Case (i) in Subsection 3.2, (4.4) is equivalent to stating that $\mathbf{E}(R^\perp v_n) \to 0$ strongly in $L^2(\mathbb{R}_2^\varepsilon; \mathbb{R}_s^{2\times 2})$. Because of assumptions (2.2), Korn's inequality applies to $\mathbb{R}_2^\varepsilon$ and thus we conclude, with the additional help of (4.3), that $R^\perp v_n \to 0$ strongly in $H^1(\mathbb{R}_2^\varepsilon; \mathbb{R}^2)$, hence that

$$(4.6) \qquad\qquad v_n \to 0, \text{ strongly in } H^1(\mathbb{R}_2^\varepsilon; \mathbb{R}^2).$$

Now the determinant is a null Lagrangian, so

$$\int_{\mathbb{R}^2} \det \nabla v_n \, dx = 0,$$

hence, in view of (4.6),

$$\lim_n \int_{\mathbb{R}_1^\varepsilon} \det \nabla v_n \, dx = 0.$$

Subtracting twice that quantity from (4.5), we obtain

$$\lim_n \left\{ \sum_{i,j=1,2} \int_{\mathbb{R}_1^\varepsilon} \left| \frac{\partial v_i^n}{\partial x_j} \right|^2 \, dx \right\} = 0,$$

which, together with (4.6), (4.3), contradicts (4.1). $\qquad\qquad\square$

Thanks to Lemma 4.1, we can now consider the setting of Theorem 3.4. Take *e.g.* $f \in L^2(\Omega; \mathbb{R}^2)$ and consider the minimizer $u^\varepsilon \in H_0^1(\Omega; \mathbb{R}^2)$ for the functional

$$(4.7) \qquad\qquad \mathscr{I}^\varepsilon(v) - 2 \int_\Omega f \cdot v \, dx.$$

That minimizer exists and is unique thanks to the coercivity property in Lemma 4.1, together with the fact that substitution of $\mathbb{L}(x/\varepsilon)$ by $\mathbb{K}(x/\varepsilon)$ in the expression for $\mathscr{I}^\varepsilon$ imparts convexity on the integrand and, even better, strict convexity in view of the presence of the zeroth order term in the expression for $\mathscr{I}^\varepsilon$.

Remark that u^ε is then the unique solution of the Euler-Lagrange equation associated with the minimization of $\mathscr{I}^\varepsilon$ over $H_0^1(\Omega; \mathbb{R}^2)$, that is

$$(4.8) \qquad \begin{aligned} -\text{div}\left(\mathbb{L}(x/\varepsilon)\nabla u^\varepsilon\right) + u^\varepsilon = f &\quad \text{in } \Omega \\ u^\varepsilon = 0 &\quad \text{on } \partial\Omega. \end{aligned}$$

The ε-indexed sequence u^ε is clearly bounded in $L^2(\Omega; \mathbb{R}^2)$ and, thanks to Theorem 3.4, we conclude in particular to the L^2-weak convergence of this sequence of minimizers to the (unique) minimizer u^0 in $H_0^1(\Omega; \mathbb{R}^2)$, or $\mathbf{X}$, depending on the setting, of the Γ-limit

$$(4.9) \qquad\qquad \mathscr{I}^0(v) - 2 \int_\Omega f \cdot v \, dx.$$

In other words u^ε is a recovery sequence for u^0, which, by Γ-convergence, is a minimizer of $\mathscr{I}^0$ (resp. $\mathscr{I}_{1/2}^0$). Further, u^0 is the unique minimizer because those functionals are strictly convex on $H_0^1(\Omega; \mathbb{R}^2)$ (resp. $\mathbf{X}$). Indeed, in both cases the

integral term with $\mathbb{L}^0 \nabla \cdot \nabla$ is quadratic and nonnegative (see (4.18) below in the latter case) while the lower order term is strictly convex.

In cases (i) or (ii) with $\theta \neq 1/2$, it is then immediate, through classical variations, that u^0 is the unique $H_0^1(\Omega; \mathbb{R}^2)$-solution of the Euler-Lagrange equation associated with that functional, that is

$$
\begin{aligned}
-\operatorname{div}\left(\mathbb{L}^0 \nabla u^0\right) + u^0 &= f \quad \text{in } \Omega \\
u^0 &= 0 \qquad \text{on } \partial\Omega.
\end{aligned}
$$
(4.10)

In case (ii) with $\theta = 1/2$, we need to appeal to the precise values of $\mathbb{L}^0$ in (3.34) and to perform the appropriate variations keeping in mind Remark 3.1. We easily conclude that u^0 satisfies

$$
\begin{aligned}
-\operatorname{div}\left(\mathbb{L}^0 \nabla u^0\right) + u^0 &= f \quad \text{in } \Omega \\
u_1^0 = 0, \ u_2^0 \, \nu_1 &= 0 \qquad \text{on } \partial\Omega.
\end{aligned}
$$
(4.11)

where $\mathbb{L}^0 \nabla u^0$ is *a priori* a distribution since $u^0 \in \mathbf{X}$.

We have thus proved the following

Proposition 4.2. *In the setting of Theorem 3.4, the unique minimizer for* (4.7) *converges weakly in $L^2(\Omega; \mathbb{R}^2)$ to the unique minimizer for* (4.9) *which further satisfies the Euler-Lagrange equation* (4.10) *(or* (4.11) *in the Gutiérrez case).*

Remark 4.3. Note, for implicit use in the next and final subsection, that all results (suitably modified) in this subsection remain true in the context of Corollary 3.10, that is if the term $\int_\Omega |v|^2 \, dx$ is replaced by $\int_\Omega a(x/\varepsilon)|v|^2 \, dx$. and the linear term $\int_\Omega f \cdot v \, dx$ is replaced by $\int_\Omega b(x/\varepsilon) f \cdot v \, dx$. ¶

4.2. **Wave propagation in the setting of Theorem 3.4.** We now consider a typical problem of elasto-dynamics at fixed ε. Consider $(f, g) \in H_0^1(\Omega; \mathbb{R}^2) \times L^2(\Omega; \mathbb{R}^2)$ and the following system for $u^\varepsilon(t)$, $t \in [0, \infty)$,

$$
\begin{aligned}
\rho(x/\varepsilon)\frac{\partial^2 u^\varepsilon}{\partial t^2} - \operatorname{div}\left(\mathbb{L}(x/\varepsilon)\nabla u^\varepsilon\right) &= 0 \quad \text{in } \Omega \times [0, \infty) \\
u^\varepsilon &= 0 \quad \text{on } \partial\Omega \times [0, \infty) \\
u^\varepsilon(0) = f, \quad \frac{\partial u^\varepsilon}{\partial t}(0) &= g \qquad \text{in } \Omega.
\end{aligned}
$$
(4.12)

In (4.12), $\rho(y)$ is the mass density, that is

$$
\rho(y) = \rho_i \ \text{in } \mathscr{Z}_i, \ i = 1, 2, \quad 0 < \rho_1, \rho_2.
$$

Then, in view of Lemma 4.1 and Remark 4.3, it is classical that this problem has a unique solution $\left(u^\varepsilon, \dfrac{\partial u^\varepsilon}{\partial t}\right) \in C^0([0, \infty); H_0^1(\Omega; \mathbb{R}^2) \times L^2(\Omega; \mathbb{R}^2))$. Since

$$
v \mapsto \int_\Omega \mathbb{L}(x/\varepsilon)\nabla v \cdot \nabla v \, dx \geq 0, \quad v \in H_0^1(\Omega; \mathbb{R}^2),
$$

we immediately deduce from energy conservation that

$$
\frac{\partial u^\varepsilon}{\partial t} \text{ is bounded in } L^\infty((0, \infty); L^2(\Omega; \mathbb{R}^2)).
$$

For a subsequence (that we will not relabel), there exists u^0 such that

$$(4.13) \qquad \begin{cases} u^\varepsilon \overset{\star}{\rightharpoonup} u^0, & \text{weakly-* in } W_{\text{loc}}^{1,\infty}([0,\infty); L^2(\Omega; \mathbb{R}^2)) \\ \dfrac{\partial u^\varepsilon}{\partial t} \overset{\star}{\rightharpoonup} \dfrac{\partial u^0}{\partial t}, & \text{weakly-* in } L^\infty([0,\infty); L^2(\Omega; \mathbb{R}^2)). \end{cases}$$

Furthermore, the Laplace transform

$$\hat{u}^\varepsilon(p) := \int_0^\infty u^\varepsilon(t) e^{-pt} \, dt, \ p > 0,$$

of u^ε, which is well defined since $\|u^\varepsilon(t)\|_{L^2(\Omega; \mathbb{R}^2)}$ has at most linear growth in t, satisfies

$$p^2 \rho(x/\varepsilon) \hat{u}^\varepsilon(p) - \text{div} \left(\mathbb{L}(x/\varepsilon) \nabla \hat{u}^\varepsilon(p) \right) = \rho(x/\varepsilon)(pf + g) \quad \text{in } \Omega$$

$$\hat{u}^\varepsilon(p) = 0 \qquad\qquad\qquad \text{on } \partial\Omega.$$

Recalling (4.8), we infer that $\hat{u}^\varepsilon(p)$ is the unique $H_0^1(\Omega; \mathbb{R}^2)$-mimimizer of

$$\int_\Omega \left\{ \mathbb{L}(x/\varepsilon) \nabla v \cdot \nabla v + p^2 \rho(x/\varepsilon) |v|^2 - 2\rho(x/\varepsilon)(pf + g) \cdot v \right\} dx.$$

But then, applying Proposition 4.2 (and Remark 4.3), we conclude that, at least in the settings validated in Theorem 3.4 and Corollary 3.10,

$$\hat{u}^\varepsilon(p) \rightharpoonup \check{u}^0(p), \ \text{weakly in } L^2\Omega; \mathbb{R}^2),$$

where $\check{u}^0(p)$ is the unique H_0^1 (or $\mathbf{X}$ in case (ii))-minimizer of

$$\int_\Omega \left\{ \mathbb{L}^0 \nabla v \cdot \nabla v + p^2 \bar{\rho} |v|^2 - 2\bar{\rho}(pf + g) \cdot v \right\} dx$$

with

$$(4.14) \qquad\qquad\qquad \bar{\rho} := \theta \rho_1 + (1 - \theta)\rho_2.$$

Now, in view of (4.13),

$$(4.15) \qquad\qquad\qquad \check{u}^0(p) = \hat{u}^0(p), \ p > 0.$$

In case (i) or case (ii) with $\theta \neq 1/2$, the system of elasto-dynamics

$$(4.16) \qquad \begin{aligned} \bar{\rho} \frac{\partial^2 \bar{u}^0}{\partial t^2} - \text{div} \left(\mathbb{L}^0 \nabla \bar{u}^0 \right) &= 0 && \text{in } \Omega \times [0, \infty) \\ \bar{u}^0 &= 0 && \text{on } \partial\Omega \times [0, \infty) \\ \bar{u}^0(0) = f, \quad \frac{\partial \bar{u}^0}{\partial t}(0) &= g && \text{in } \Omega \end{aligned}$$

has a unique solution $\left(\bar{u}^0, \dfrac{\partial \bar{u}^0}{\partial t} \right)$ in $C^0([0,\infty); H_0^1(\Omega; \mathbb{R}^2) \times L^2(\Omega; \mathbb{R}^2))$, as can be easily checked by $e.g.$ extending all functions to 0 ouside Ω and recalling Remark 3.7 which implies coercivity of

$$\int_{\mathbb{R}^2} \left\{ \mathbb{L}^0 \nabla v \cdot \nabla v + \bar{\rho} |v|^2 \right\} dx$$

over $H_0^1(\mathbb{R}^2; \mathbb{R}^2)$.

The system

$$\bar{\rho}\,\frac{\partial^2 \bar{u}^0}{\partial t^2} - \operatorname{div}\left(\mathbb{L}^0 \nabla \bar{u}^0\right) = 0 \quad \text{in } \Omega \times [0, \infty)$$

(4.17)
$$\bar{u}^0_1 = 0, \ \bar{u}^0_2\,\nu_1 = 0 \qquad \text{on } \partial\Omega \times [0, \infty)$$

$$\bar{u}^0(0) = f, \quad \frac{\partial \bar{u}^0}{\partial t}(0) = g \quad \text{in } \Omega$$

also possesses a unique solution in case (ii) with $\theta = 1/2$, which is admittedly less classical.

The result can be obtained through various methods. For example, one can remark that the operator

$$\mathcal{L}^0 := \begin{pmatrix} 0 & 1 \\ \operatorname{div}\left(\mathbb{L}^0 \nabla \cdot\right) & 0 \end{pmatrix}$$

with domain $D(\mathcal{L}^0) := \left\{ (v, w) \in \mathbf{X} \times L^2(\Omega; \mathbb{R}^2) : w \in \mathbf{X},\, \operatorname{div}\left(\mathbb{L}^0 \nabla v\right) \in L^2(\Omega; \mathbb{R}^2) \right\}$ is skew self-adjoint on the Hilbert space $\mathbf{X} \times L^2(\Omega; \mathbb{R}^2)$.

To that end, one should endow $\mathbf{X}$ with the inner product

$$\langle\!\langle u, v \rangle\!\rangle_{\mathbf{X}} := \int_\Omega \mathbb{L}^0 \nabla u \cdot \nabla v \ dx.$$

Because Ω is bounded and $C_c^\infty(\Omega; \mathbb{R}^2)$ is dense in $\mathbf{X}$, it is easily checked that this new inner product generates a norm $|\!|\!|\cdot|\!|\!|_{\mathbf{X}}$ with the property that

(4.18)
$$|\!|\!|\cdot|\!|\!|_{\mathbf{X}} \sim \|\cdot\|_{\mathbf{X}}$$

where $\|\cdot\|_{\mathbf{X}}$ is the norm associated with the inner product $\langle\cdot,\cdot\rangle_{\mathbf{X}}$ defined in Remark 3.2. Indeed, take $\varphi \in C_c^\infty(\Omega; \mathbb{R}^2)$. Then, with the help of (3.34),

$$\int_\Omega \mathbb{L}^0 \nabla\varphi \cdot \nabla\varphi \, dx = \int_{\mathbb{R}^2} \left\{ (\bar{\lambda} + 2\bar{\mu}_1)\,\xi_1^2 |\hat{\varphi}_1|^2 + 2(\bar{\lambda} + \bar{\mu}_2)\,\xi_1\xi_2\,\mathrm{Re}(\hat{\varphi}_1\bar{\hat{\varphi}}_2) \right.$$

$$\left. + \bar{\mu}_2\left(\xi_2^2|\hat{\varphi}_1|^2 + \xi_1^2|\hat{\varphi}_2|^2\right) \right\} d\xi =$$

$$\int_{\mathbb{R}^2} \left\{ (\bar{\lambda} + 2\bar{\mu}_1)\,\xi_1^2 |\hat{\varphi}_1|^2 + \bar{L}\begin{pmatrix} \xi_2\hat{\varphi}_1 \\ \xi_1\hat{\varphi}_2 \end{pmatrix} \cdot \overline{\begin{pmatrix} \xi_2\hat{\varphi}_1 \\ \xi_1\hat{\varphi}_2 \end{pmatrix}} \right\} d\xi$$

where $\bar{L} := \begin{pmatrix} \bar{\mu}_2 & \bar{\lambda}+\bar{\mu}_2 \\ \bar{\lambda}+\bar{\mu}_2 & \bar{\mu}_2 \end{pmatrix}$. But, in view of (3.34), the quantities $\bar{\lambda} + 2\bar{\mu}_1$, $-\bar{\lambda}$, $\bar{\mu}_2$, $\bar{\lambda} + 2\bar{\mu}_2$ are positive, so that the integrand in the right hand-side of the second equality above is bounded below by

$$c\int_{\mathbb{R}^2}\left(\xi_1^2|\hat{\varphi}_1|^2 + \xi_2^2|\hat{\varphi}_1|^2 + \xi_1^2|\hat{\varphi}_2|^2\right)d\xi = c\left(\|\nabla\varphi_1\|_{L^2(\Omega;\mathbb{R}^2)}^2 + \left\|\frac{\partial\varphi_2}{\partial x_1}\right\|_{L^2(\Omega)}^2\right) \geq c'_\Omega\|\varphi\|_{\mathbf{X}}^2$$

for some $c > 0$ and some Ω-dependent $c'_\Omega > 0$. Note that the last inequality above holds true precisely by virtue of the Poincaré inequality in the direction x_1 applied to the extension $\check{\varphi}_2$ of φ_2 defined in Remark 3.2.

It then suffices to apply Stone's theorem for unitary groups of operators (see *e.g.* [14, Chapter IX-9]).

Thus, in both cases (i) and (ii), the Laplace transform for any $p > 0$ of the unique solution $\bar{u}^0$ to (4.16) is precisely $\check{u}^0(p)$ which in turn, thanks to (4.15), is the Laplace transform $\hat{u}^0(p)$ of u^0 given through convergences (4.13). Thus $\bar{u}^0 = u^0$

and, in view of the uniqueness of the limit function u^0, there is no need to extract subsequences.

We have proved that the following theorem holds true:

Theorem 4.4. *In the setting of Theorem 3.4 and given*

$$(f, g) \in H_0^1(\Omega; \mathbb{R}^2) \times L^2(\Omega; \mathbb{R}^2)$$

as initial conditions, the unique solution

$$\left(u^\varepsilon, \frac{\partial u^\varepsilon}{\partial t} \right) \in C^0([0, \infty); H_0^1(\Omega; \mathbb{R}^2) \times L^2(\Omega; \mathbb{R}^2))$$

to system (4.12) *converges weakly-* in*

$$W_{\text{loc}}^{1,\infty}((0, \infty); L^2(\Omega; \mathbb{R}^2)) \times L^\infty((0, \infty); L^2(\Omega; \mathbb{R}^2))$$

to

$$\left(u^0, \frac{\partial u^0}{\partial t} \right) \in C^0([0, \infty); H_0^1(\Omega; \mathbb{R}^2)(\text{resp. } \mathbf{X}) \times L^2(\Omega; \mathbb{R}^2)),$$

the unique solution to system (4.16) *with $\bar{\rho}$ defined by* (4.14) *in cases* (i) *or* (ii) *with $\theta \neq 1/2$ (resp. system* (4.17) *for case* (ii) *with $\theta = 1/2$).*

We conclude this study in the company of Gutiérrez. In that case, we know that $\mathbb{L}_{2222}^0 = 0$.

As far as the two-phase laminate at fixed ε is concerned, the elasto-dynamic problem (4.12) cannot have a non-zero plane wave solution of the type

$$u^\varepsilon(t, x) = F^\varepsilon(t + \xi^\varepsilon \cdot x) + G^\varepsilon(t - \xi^\varepsilon \cdot x) \quad \text{for some } \xi^\varepsilon \in \mathbb{R}^2 \setminus \{0\},$$

because of the Dirichlet boundary condition satisfied by $u^\varepsilon(t, \cdot)$ on $\partial\Omega$, and this whatever the initial conditions f and g might be. The same applies to the elasto-dynamic problem (4.16) associated with the homogenized material in cases (i) and (ii) with $\theta \neq 1/2$.

On the contrary, in the Gutiérrez setting (ii) with $\theta = 1/2$, starting *e.g.* with the initial conditions

$$(4.19) \qquad f(x) = 2 \sin x_1 \ \vec{e}_2, \ g(x) = 0 \quad \text{for } x \in \Omega = (0, \pi)^2,$$

it is easy to check that the function $u^0 = (0, u_2^0) \in C^0([0, \infty); \mathbf{X})$, with
(4.20)
$$u_2^0(t, x_1) = \sin\left(\sqrt{\mathbb{L}_{1212}^0/\bar{\rho}} \ t + x_1 \right) - \sin\left(\sqrt{\mathbb{L}_{1212}^0/\bar{\rho}} \ t - x_1 \right) \text{ for } t \in [0, \infty), \ x_1 \in (0, \pi)$$

(where $\mathbb{L}_{1212}^0 = 2\mu_1\mu_2/(\mu_1 + \mu_2)$), is a transverse plane wave solution to the homogenized problem (4.17). This is so because the space $\mathbf{X}$ replaces $H_0^1(\Omega; \mathbb{R}^2)$ and thus does not require any boundary condition for u_2 on the horizontal sides of the square. Note that Remark 3.2 equally applies in this case although Ω is not C^1 because the density of $C_c^\infty(\Omega; \mathbb{R}^2)$ into $\mathbf{X}$ can be easily established for such a special geometry.

Forgetting now about boundary conditions, we would like to investigate the kind of plane waves that the Gutiérrez material can withstand on the whole plane. To

this effect, we find it more convenient to write a (2D orthorhombic) stress-strain relation for that material in the form

$$\begin{cases} \sigma_{11} = \bar{\lambda} \operatorname{div} u + 2\bar{\mu}_1 \, \mathbf{E}_{11}(u) \\ \sigma_{12} = 2\bar{\mu}_2 \, \mathbf{E}_{12}(u) \\ \sigma_{22} = \bar{\lambda} \operatorname{div} u + 2\bar{\mu}_3 \, \mathbf{E}_{22}(u), \end{cases}$$

where $\bar{\lambda}$, $\bar{\mu}_1$, $\bar{\mu}_2$ have been defined in (3.34) and $\bar{\mu}_3 := \mu_1$, so that, in particular, $\bar{\lambda} + 2\bar{\mu}_3 = 0$, while $\bar{\lambda} + 2\bar{\mu}_1, \bar{\lambda} + 2\bar{\mu}_2 > 0$ as well as the $\bar{\mu}_i$'s for $i = 1, 2, 3$. We seek a plane wave solution of the form

$$u(t, x) = e^{i(k \cdot x - \omega t)} \, \eta, \quad \eta, k \in \mathbb{R}^2, \ |k| = 1, \ \omega \in \mathbb{R},$$

of the first equation in (4.16). After some algebra, this amounts to finding the eigenvalues, i.e., the $(\bar{\rho}\omega^2)$'s, of the symmetric matrix

$$A(k) := \begin{pmatrix} (\bar{\lambda} + 2\bar{\mu}_1)k_1^2 + \bar{\mu}_2 k_2^2 & (\bar{\lambda} + \bar{\mu}_2)k_1 k_2 \\ (\bar{\lambda} + \bar{\mu}_2)k_1 k_2 & (\bar{\lambda} + 2\bar{\mu}_3)k_2^2 + \bar{\mu}_2 k_1^2 \end{pmatrix},$$

in which case the corresponding eigenvectors are the directions of propagation, i.e., the η's. In our setting, and even in the case where $\bar{\lambda} + 2\bar{\mu}_3 > 0$ (which could be obtained by increasing the value of $\bar{\lambda}$), it is an easy task to check that, provided that

$$(\bar{\lambda} + 2\bar{\mu}_1)(\bar{\lambda} + 2\bar{\mu}_3) - \bar{\lambda}(\bar{\lambda} + 2\bar{\mu}_2) \geq 0,$$

the two eigenvalues of $A(k)$ are always nonnegative. Further, the case

(4.21) $$\bar{\lambda} + 2\bar{\mu}_3 = 0,$$

which is directly in the spirit of the Cauchy material (see (1.5)), is the only case for which one of the eigenvalues can be zero. This happens for, and only for $k_1 = 0$. There, the plane wave is a transversal (shear) wave oscillating in the direction e_2 and propagating in the direction e_1.

So, in essence, the singular behavior of the Gutiérrez material resides both in the possibility of propagating plane waves on a bounded domain with Dirichlet boundary conditions – as demonstrated through (4.19), (4.20) – and in the existence of one, and only one direction in which longitudinal waves cannot propagate, namely the direction of lamination.

It is somewhat tempting to view the Gutiérrez material as a contemporary version of labile aether, that is of an elastic material that does not support longitudinal waves as demonstrated by (4.21). See [13, Chapter 5] for a description of Cauchy's, Green's, and Thomson's attempts in this direction. However, our material merely seems to represent its putative manifestation in a 2D orthorhombic crystal because it only prevents the existence of longitudinal waves in one direction.

But ultimately the main difference is that aether is of course three-dimensional, a setting for which a similar analysis is wanting at present. Gutiérrez has also produced in [9], through multiple layering, a 3D material that loses strict strong ellipticity. It is our unsubstantiated hope that the present analysis can be extended to that case as well.

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