ERRATUM TO "THIN ELASTIC FILMS: THE IMPACT OF HIGHER ORDER PERTURBATIONS"

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Abstract. The asymptotic behavior of an elastic thin film penalized by a van der Wals type interfacial energy is investigated when both its thickness and the magnitude of the additional energy vanish in the limit. Keeping track of both mid-plane and out of plane deformations (through the introduction of the Cosserat vector), the resulting behavior strongly depends upon the ratio between thickness and interfacial energy. Non-locality is evidenced for a critical value of that ratio.

Keywords : dimension reduction, Γ-convergence, thin films, relaxation, interfacial energy.

1. Introduction

In the paper [2] the main objective was to identify

$$E_{-}^{\gamma}(u,b;A)=\inf\left\{\liminf_{\varepsilon\to0^{+}}E_{\varepsilon}^{\gamma}(u_{\varepsilon};A):\ u_{\varepsilon}\in W^{2,2}\left(\Omega;\mathbb{R}^{3}\right),\right.$$
$$\left. u_{\varepsilon}\rightharpoonup u \text{ in } W^{1,q}\left(\Omega;\mathbb{R}^{3}\right),\quad \frac{1}{\varepsilon}D_{3}u_{\varepsilon}\rightharpoonup b \text{ in } L^{q}\left(\Omega;\mathbb{R}^{3}\right)\right\},$$

where

$$E_{\varepsilon}^{\gamma}(u;A):=\int_{A\times I}W\left(D_{p}u\left|\frac{1}{\varepsilon}D_{3}u\right.\right)dx$$
$$+\varepsilon^{\gamma}\int_{A\times I}\left(|D_{p}^{2}u|^{2}+\frac{1}{\varepsilon^{2}}|D_{p3}u|^{2}+\frac{1}{\varepsilon^{4}}|D_{33}u|^{2}\right)dx$$

if $u\in W^{2,2}(\Omega;\mathbb{R}^{3})$, and $E_{\varepsilon}^{\gamma}(u;A):=\infty$ otherwise. Here $q>1$, $\Omega=\omega\times I$, $\mathcal{A}(\omega)$ is the family of open subsets of ω, and W satisfies the condition

$(H_1)'$ $W:\mathbb{R}^{3\times3}\to[0,\infty)$ is continuous and there exists $C>0$ such that

$$\frac{1}{C}\left|F\right|^{q}-C\leq W\left(F\right)\leq C\left(1+\left|F\right|^{q}\right)$$

for all $F\in\mathbb{R}^{3\times3}$.

We recall that

$$\mathcal{V}^{\gamma}:=\left\{(u,b)\in W^{1,q}\left(\Omega;\mathbb{R}^{3}\right)\times L^{q}\left(\Omega;\mathbb{R}^{3}\right):\ D_{3}u=0\ \mathcal{L}^{3}\text{ a.e. in }\Omega,\right.$$
$$\left. D_{3}b=0\ \mathcal{L}^{3}\text{ a.e. in }\Omega\text{ if }\gamma<2,\ D_{3}b\in L^{2}\left(\Omega;\mathbb{R}^{3}\right)\text{ if }\gamma=2\right\} \tag{1}$$

and for $(u,b)\in\mathcal{V}^{\gamma}$ and $A\in\mathcal{A}(\omega)$,

$$H^{\gamma}\left(u,b,A\right):=\inf\left\{\liminf_{\varepsilon\to0^{+}}\int_{A\times I}W\left(D_{p}u_{\varepsilon}\left(x\right)|b_{\varepsilon}\left(x\right)\right)dx:\ \{u_{\varepsilon}\}\subset W^{1,q}(\Omega;\mathbb{R}^{3}),\right.$$
$$\left.\{b_{\varepsilon}\}\subset L^{q}\left(\Omega;\mathbb{R}^{3}\right),\ u_{\varepsilon}\rightharpoonup u\text{ in }W^{1,q}(\Omega;\mathbb{R}^{3}),\ b_{\varepsilon}\rightharpoonup b\text{ in }L^{q}(\Omega;\mathbb{R}^{3})\right\}$$

if $\gamma \neq 2$, and

$$H^2(u,b,A) := \inf\Big\{ \liminf_{\varepsilon\to 0^+} \int_{A\times I} \Big[W\left(D_p u_\varepsilon(x)\,|b_\varepsilon(x)\right) + |D_3 b_\varepsilon(x)|^2 \Big]\, dx : \{u_\varepsilon\} \subset W^{1,q}(\Omega;\mathbb{R}^3), \{b_\varepsilon\} \subset L^q\left(\Omega;\mathbb{R}^3\right) \text{ with } D_3 b_\varepsilon \in L^2\left(\Omega;\mathbb{R}^3\right), u_\varepsilon \rightharpoonup u \text{ in } W^{1,q}(\Omega;\mathbb{R}^3),\ b_\varepsilon \rightharpoonup b \text{ in } L^q(\Omega;\mathbb{R}^3) \Big\}.$$

In this erratum we slightly change notation and denote H^γ and H^2 by $H^\gamma_{3,3}$ and $H^2_{3,3}$, respectively. We also introduce $H^\gamma_{2,3}$ and $H^2_{2,3}$ that we define analogously except for the fact that the approximating sequences $\{u_\varepsilon\}$ belong to $W^{1,q}(\omega;\mathbb{R}^3)$.

The proof of Theorem 3.1 has an error in the upper bound. To date this upper bound can only be established if $\gamma < 2$ or if $D_3 b = 0$ $\mathcal{L}^3$ a.e. in Ω. Therefore, Theorem 3.1 should read as

Theorem A (New Theorem 3.1)**.** *Assume that condition $(H_1)'$ is satisfied. Then,*

$$H^\gamma_{3,3}(u,b,A) \le E^\gamma_-(u,b;A) \le H^\gamma_{2,3}(u,b,A) \tag{2}$$

for all $(u,b) \in \mathcal{V}^\gamma$ and $A \in \mathcal{A}(\omega)$. If $\gamma < 2$ or $\gamma \ge 2$ and $D_3 b = 0$ $\mathcal{L}^3$ a.e. in Ω, then

$$E^\gamma_-(u,b;A) = H^\gamma_{3,3}(u,b,A) = H^\gamma_{2,3}(u,b,A) = H^\gamma_{2,2}(u,b,A) = \int_A (Q_2 \times C_2)[W]\left(D_p u(x_\alpha)\,|b(x_\alpha)\right) dx_\alpha.$$

The lower bound in the original Theorem 3.1 is correct. The mistake in the proof of Theorem 3.1 was in trying to show that the upper bound is still $H^\gamma_{3,3}$. The correct upper bound is $H^\gamma_{2,3}$, as justified next, and in general there may be a gap between $H^\gamma_{3,3}$ and $H^\gamma_{2,3}$. However, when $D_3 b = 0$ $\mathcal{L}^3$ a.e. in Ω (a condition that is automatically satisfied when $\gamma < 2$ in the finite energy regime), then indeed

$$H^\gamma_{2,3}(u,b,A) = H^\gamma_{3,3}(u,b,A).$$

To see this, we refer the reader to the proof of Theorem 4.1 that still works unchanged for any $\gamma > 0$ when $D_3 b = 0$ $\mathcal{L}^3$ a.e. in Ω.

To prove the second inequality in (2) we remark that the estimates involving $\{u * \varphi_j, b * \varphi_j\}$ led to (3.2) and (3.3) for (u,b,A), and this analysis relied heavily on the fact that $D_3 u = 0$ $\mathcal{L}^3$ a.e. in Ω. Therefore, given $(u,b) \in \mathcal{V}^\gamma$, $A \in \mathcal{A}(\omega)$, if $\{u_j\} \subset W^{1,q}(\omega;\mathbb{R}^3)$ converges weakly to u in $W^{1,q}(\omega;\mathbb{R}^3)$ and $\{b_j\} \subset L^q(\Omega;\mathbb{R}^3)$ converges weakly to b in $L^q(\Omega;\mathbb{R}^3)$, then we may apply (3.2) and (3.3) to (u_j, b, A), and we deduce that

$$E^\gamma_-(u,b;A) \le \liminf_{j\to\infty} E^\gamma_-(u_j,b_j;A) \le \liminf_{j\to\infty} \int_{A\times I} W\left(D_p u_j(x_\alpha)\,|b_j(x)\right) dx$$

if $\gamma \neq 2$, and if $\gamma = 2$ that

$$E^2_-(u,b;A) \le \liminf_{j\to\infty} E^2_-(u_j,b_j;A) \le \liminf_{j\to\infty} \int_{A\times I} \Big[W\left(D_p u_j(x_\alpha)\,|b_j(x)\right) + |D_3 b_j(x)|^2 \Big]\, dx.$$

Taking the infimum over all such sequences $\{u_j\}$ and $\{b_j\}$ yields

$$E^\gamma_-(u,b;A) \le H^\gamma_{2,3}(u,b,A).$$

Theorem A covers completely the case $\gamma < 2$ but leaves (partially) open the case $\gamma \geq 2$ when $D_3 b \not\equiv 0$ $\mathcal{L}^3$ a.e. in Ω We close the gap in this erratum. To be precise, if $\gamma = 2$ and if W satisfies the additional q-Lipschitz condition

$$|W(F) - W(G)| \leq C\left(1 + |F|^{q-1} + |G|^{q-1}\right)|F - G| \tag{3}$$

for all $F, G \in \mathbb{R}^{3\times 3}$, then we characterize the Γ-limit in Theorem B, while for $\gamma > 2$ we refer to Theorem G.

When $\gamma = 2$, we introduce the functional

$$\overline{\mathcal{W}}_2 : \mathbb{R}^{2\times 3} \times W^{1,2}(I;\mathbb{R}^3) \to [0,\infty)$$

defined for $\overline{F} \in \mathbb{R}^{2\times 3}$ and $b \in W^{1,2}(I;\mathbb{R}^3)$ by

$$\begin{aligned}\overline{\mathcal{W}}_2(\overline{F}|b) := \inf_{\varphi,L}\Bigg\{ \int_Q \Bigg(& W\left(\overline{F} + D_p\varphi(x)|b(x_3) + LD_3\varphi(x)\right) + \left|\frac{1}{L}D_p^2\varphi(x)\right|^2 \\ & + |D_{p3}\varphi(x)|^2 + |b'(x_3) + LD_{33}\varphi(x)|^2\Bigg)\,dx : \\ & L > 0,\ \varphi \in W^{2,2}(Q;\mathbb{R}^3),\ \varphi(\cdot,x_3)\ Q'\text{-periodic for } \mathcal{L}^1 \text{ a.e. } x_3, \\ & \int_{Q'} D_3\varphi(x_\alpha,x_3)\,dx_\alpha = 0 \text{ for } \mathcal{L}^1 \text{ a.e. } x_3 \Bigg\}.\end{aligned} \tag{4}$$

We have the following representation

Theorem B. *Assume that $\gamma = 2$ and that conditions $(H_1)'$ and (3) are satisfied. Then for all $(u,b) \in \mathcal{V}^2$ and $A \in \mathcal{A}(\omega)$,*

$$E_-^2(u,b;A) = \int_A \overline{\mathcal{W}}_2(D_p u(x_\alpha)|b(x_\alpha,\cdot))\,dx_\alpha, \tag{5}$$

where $\overline{\mathcal{W}}_2$ is defined in (4).

To prove the theorem above, we start by showing that under the q-Lipschitz condition (3), minimizing sequences for $\overline{\mathcal{W}}_2$ prefer scales L diverging to infinity.

Proposition C. *Assume that $\gamma = 2$ and that conditions $(H_1)'$ and (3) are satisfied. Then for all $\overline{F} \in \mathbb{R}^{2\times 3}$ and $b \in W^{1,2}(I;\mathbb{R}^3)$ the $\inf_{\varphi,L}$ in the definition of $\overline{\mathcal{W}}_2(\overline{F}|b)$ may be replaced by $\lim_{L\to\infty}\inf_\varphi$.*

Proof. For $\overline{F} \in \mathbb{R}^{2\times 3}$, $b \in W^{1,2}(I;\mathbb{R}^3)$, and $L > 0$ let

$$\begin{aligned}\mathcal{H}_L(\overline{F}|b) := \inf_{\varphi}\Bigg\{ \int_Q \Bigg(& W\left(\overline{F} + D_p\varphi(x)|b(x_3) + LD_3\varphi(x)\right) + \left|\frac{1}{L}D_p^2\varphi(x)\right|^2 \\ & + |D_{p3}\varphi(x)|^2 + |b'(x_3) + LD_{33}\varphi(x)|^2\Bigg)\,dx : \\ & \varphi \in W^{2,2}(Q;\mathbb{R}^3),\ \varphi(\cdot,x_3)\ Q'\text{-periodic for } \mathcal{L}^1 \text{ a.e. } x_3, \\ & \int_{Q'} D_3\varphi(x_\alpha,x_3)\,dx_\alpha = 0 \text{ for } \mathcal{L}^1 \text{ a.e. } x_3 \Bigg\}.\end{aligned}$$

Clearly

$$\overline{\mathcal{W}}_2(\overline{F}|b) \leq \liminf_{L\to\infty} \mathcal{H}_L(\overline{F}|b).$$

We now prove that

$$\limsup_{L\to\infty} \mathcal{H}_L(\overline{F}|b) \leq \overline{\mathcal{W}}_2(\overline{F}|b).$$

Consider a sequence $\{L_n\}$ converging to infinity such that

$$\limsup_{L\to\infty} \mathcal{H}_L(\overline{F}|b) = \lim_{n\to\infty} \mathcal{H}_{L_n}(\overline{F}|b).$$

Let $\varphi \in W^{2,\infty}(Q;\mathbb{R}^3)$ and $L > 0$ be admissible for $\overline{\mathcal{W}}_2(\overline{F}|b)$, and define

$$\varphi_n(x) := \frac{1}{m_n}\varphi(m_n x_\alpha, x_3),$$

where

$$m_n := \left\lceil \frac{L_n}{L} \right\rceil$$

with $\lceil \frac{L_n}{L} \rceil$ the integer part of $\frac{L_n}{L}$. Note that φ_n is admissible for $\mathcal{H}_{Ln}$, and so

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{H}_{L_n}(\overline{F}|b) \le{}& \liminf_{n\to\infty} \left\{ \int_Q \left(W\left(\overline{F} + D_p\varphi_n(x)\,|b(x_3) + L_n D_3\varphi_n(x)\right) + \left|\frac{1}{L_n} D_p^2 \varphi_n(x)\right|^2 \right.\right.\\
&\left.\left. + |D_{p3}\varphi_n(x)|^2 + |b'(x_3) + L_n D_{33}\varphi_n(x)|^2 \right) dx \right\}\\
={}& \liminf_{n\to\infty} \left\{ \int_Q \left(W\left(\overline{F} + D_p\varphi(m_n x_\alpha, x_3) \middle| b(x_3) + \frac{L_n}{m_n} D_3\varphi(m_n x_\alpha, x_3)\right) \right.\right.\\
&+ \left|\frac{m_n}{L_n} D_p^2 \varphi(m_n x_\alpha, x_3)\right|^2\\
&\left.\left. + |D_{p3}\varphi(m_n x_\alpha, x_3)|^2 + \left|b'(x_3) + \frac{L_n}{m_n} D_{33}\varphi(m_n x_\alpha, x_3)\right|^2 \right) dx \right\}\\
={}& \liminf_{n\to\infty} \left\{ \int_Q \left(W\left(\overline{F} + D_p\varphi(m_n x_\alpha, x_3)\,|b(x_3) + L D_3\varphi(m_n x_\alpha, x_3)\right) \right.\right.\\
&+ \left|\frac{1}{L} D_p^2 \varphi(m_n x_\alpha, x_3)\right|^2\\
&\left.\left. + |D_{p3}\varphi(m_n x_\alpha, x_3)|^2 + |b'(x_3) + L D_{33}\varphi(m_n x_\alpha, x_3)|^2 \right) dx \right\} + o(1),
\end{aligned}$$

where in the last equality we used (3) together with the facts that $L_n \to \infty$ and that $\varphi \in W^{2,\infty}(Q;\mathbb{R}^3)$. Since $\varphi(\cdot, x_3)$ is Q'-periodic for $\mathcal{L}^1$ a.e. $x_3 \in I$, it follows from Lebesgue's Dominated Convergence Theorem, Fubini's Theorem, and the Riemman-Lebesgue Lemma that

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{H}_{L_n}(\overline{F}|b) \le{}& \int_Q \left(\left(W\left(\overline{F} + D_p\varphi(x_\alpha, x_3)\,|b(x_3) + L D_3\varphi(x_\alpha, x_3)\right)\right) + \left|\frac{1}{L} D_p^2 \varphi(x_\alpha, x_3)\right|^2 \right.\\
&\left. + |D_{p3}\varphi(x_\alpha, x_3)|^2 + |b'(x_3) + L D_{33}\varphi(x_\alpha, x_3)|^2 \right) dx.
\end{aligned}$$

Using the arbitrariness of φ and L, the density of smooth functions in the set of test functions for (4), and the growth hypothesis $(H_1)'$, we conclude that

$$\lim_{n\to\infty} \mathcal{H}_{L_n}(\overline{F}|b) \le \overline{\mathcal{W}}_2(\overline{F}|b).$$

□

To prove Theorem B, it is enough to show that for any given sequence $\{\varepsilon_n\}$, with $\varepsilon_n \to 0^+$, there exists a subsequence $\{\varepsilon_{n_k}\}$ of $\{\varepsilon_n\}$ such that the Γ-lower limit defined by

$$E^-_{\{\varepsilon_{n_k}\}}(u,b;A) := \inf\Big\{\liminf_{k\to\infty} E^2_{\varepsilon_{n_k}}(u_k;A) : \, u_k \in W^{2,2}\left(\Omega;\mathbb{R}^3\right), \quad u_k \rightharpoonup u \text{ in } W^{1,q}\left(\Omega;\mathbb{R}^3\right), \quad \frac{1}{\varepsilon_{n_k}} D_3 u_{n_k} \rightharpoonup b \text{ in } L^q\left(\Omega;\mathbb{R}^3\right)\Big\} \tag{6}$$

coincides with

$$\int_A \overline{\mathcal{W}}_2(D_p u(x_\alpha)|b(x_\alpha,\cdot))\,dx_\alpha$$

for all $(u,b) \in \mathcal{V}^2$ and $A \in \mathcal{A}(\omega)$.

To choose the subsequence $\{\varepsilon_{n_k}\}$, let $\mathcal{R}(\omega)$ be the countable subfamily of $\mathcal{A}(\omega)$ obtained by taking all finite unions of open squares in ω with faces parallel to the axes, centered at $x_\alpha \in \omega \cap \mathbb{Q}^2$ and with rational edge length. Since $L^1(\Omega;\mathbb{R}^3)$ is a separable metric space, using Kuratowski's Compactness Theorem and a diagonal argument, we may find a subsequence $\{\varepsilon_{n_k}\}$ of $\{\varepsilon_n\}$ such that

$$\Gamma - \lim_{k\to\infty} E^2_{\varepsilon_{n_k}}(u,b;A) \text{ exists for all } (u,b) \in \mathcal{V}^2 \text{ and for all } A' \in \mathcal{R}(\omega). \tag{7}$$

Theorem D. *Assume that condition* $(H_1)'$ *is satisfied and that* $\gamma = 2$. *Then for every* $(u,b) \in \mathcal{V}^2$ *the set function* $E^-_{\{\varepsilon_{n_k}\}}(u,b;\cdot)$ *is the trace of a Radon measure absolutely continuous with respect to* $\mathcal{L}^2\lfloor\omega$.

Proof. The proof is very similar to that of Theorem 4.2 and thus we only indicate the main changes.

Step 1: Fix $(u,b) \in \mathcal{V}^2$. We claim that

$$E^-_{\{\varepsilon_{n_k}\}}(u,b;A_1) \le E^-_{\{\varepsilon_{n_k}\}}(u,b;A_2) + E^-_{\{\varepsilon_{n_k}\}}(u,b;A_1 \setminus \overline{A_3}) \tag{8}$$

for all A_1, A_2, $A_3 \in \mathcal{A}(\omega)$, with $A_3 \subset\subset A_2 \subset A_1$.

Without loss of generality we may assume that the right-hand side of the previous inequality is finite.

Fix $\eta > 0$ and find $\{u_k\} \subset W^{2,2}\left(\Omega;\mathbb{R}^3\right)$ converging weakly to u in $W^{1,q}(\Omega;\mathbb{R}^3)$ and such that $\frac{1}{\varepsilon_{n_k}} D_3 u_k \rightharpoonup b$ in $L^q\left(\Omega;\mathbb{R}^3\right)$, and

$$\liminf_{k\to\infty} E^2_{\varepsilon_{n_k}}(u_k;\left(A_1 \setminus \overline{A_3}\right)) \le E^-_{\{\varepsilon_{n_k}\}}(u,b;A_1 \setminus \overline{A_3}) + \eta.$$

Extract a subsequence $\{n_{k_j}\}$ for which

$$\lim_{j\to\infty} E^2_{\varepsilon_{n_{k_j}}}(u_{k_j};\left(A_1 \setminus \overline{A_3}\right)) \le E^-_{\{\varepsilon_{n_k}\}}(u,b;A_1 \setminus \overline{A_3}) + \eta. \tag{9}$$

Let $A' \in \mathcal{R}(\omega)$ be such that $A_3 \subset\subset A' \subset\subset A_2$. By (7) there exists a sequence $\{v_k\} \subset W^{2,2}\left(\Omega;\mathbb{R}^3\right)$ converging weakly to u in $W^{1,q}(\Omega;\mathbb{R}^3)$ and such that $\frac{1}{\varepsilon_{n_k}} D_3 v_k \rightharpoonup b$ in $L^q\left(\Omega;\mathbb{R}^3\right)$, and

$$E^-_{\{\varepsilon_{n_k}\}}(u,b;A') = \lim_{k\to\infty} E^2_{\varepsilon_{n_k}}(v_k;A').$$

In particular

$$E^-_{\{\varepsilon_{n_k}\}}(u,b;A') = \lim_{j\to\infty} E^2_{\varepsilon_{n_{k_j}}}\left(v_{k_j};A'\right). \tag{10}$$

For every $v \in W^{2,2}(\Omega;\mathbb{R}^3)$, for every Borel set $E \subset \omega$, and for every $j \in \mathbb{N}$ define

$$\mathcal{G}_j(v;E) := \int_{E\times I}\left(1 + |D_p v|^q + \frac{1}{\varepsilon_{n_{k_j}}^q}|D_3 v|^q\right) dx + \int_{E\times I}\left(\varepsilon_{n_{k_j}}^2 |D_p^2 v|^2 + |D_{p3}v|^2 + \frac{1}{\varepsilon_{n_{k_j}}^2}|D_{33}v|^2\right) dx.$$

Due to the coercivity hypothesis $(H_1)'$ we may extract a bounded subsequence from the sequence of measures $\nu_j := \mathcal{G}_j(u_{k_j};\cdot) + \mathcal{G}_j(v_{k_j};\cdot)$ restricted to $A' \setminus \overline{A_3}$ converging $\star$-weakly to some Radon measure ν defined on $A' \setminus \overline{A_3}$.

Find $t > 0$ such that $\nu(S_t) = 0$, where

$$S_t := \{x_\alpha \in A' :\ \operatorname{dist}(x_\alpha, \partial A_3) = t\}.$$

For $\delta > 0$ define

$$L_\delta := \{x_\alpha \in A' :\ \operatorname{dist}(x_\alpha, S_t) < \delta\}.$$

Choose δ so small that $L_\delta \subset A' \setminus \overline{A_3}$. Consider a smooth cut-off function $\varphi_\delta \in C_0^\infty(A_2;[0,1])$ such that $\varphi_\delta = 1$ in

$$\{x_\alpha \in A' :\ \operatorname{dist}(x_\alpha, \partial A_3) < t - \delta\}$$

and $\varphi_\delta = 0$ in

$$\{x_\alpha \in A' :\ \operatorname{dist}(x_\alpha, \partial A_3) > t + \delta\},$$

with

$$\|D_p\varphi_\delta\|_{L^\infty(\omega)} \le C/\delta, \quad \|D_p^2\varphi_\delta\|_{L^\infty(\omega)} \le C/\delta^2.$$

Define

$$\tilde{u}_k(x) := \begin{cases} (1-\varphi_\delta(x_\alpha))u_{k_j}(x) + \varphi_\delta(x_\alpha)v_{k_j}(x) & \text{if } k = k_j \text{ for some } j \in \mathbb{N},\\ u(x_\alpha) + \varepsilon_{n_k}\int_0^{x_3}(b * \psi_k)(x_\alpha, s)\, ds & \text{otherwise,}\end{cases}$$

where $\psi_k = \psi_k(x_\alpha)$ is a standard mollifier. Then $\tilde{u}_k \rightharpoonup u$ in $W^{1,q}(\Omega;\mathbb{R}^3)$ and since φ_δ does not depend on x_3, we also have that $\frac{1}{\varepsilon_{n_k}} D_3\tilde{u}_k \rightharpoonup b$ in $L^q(\Omega;\mathbb{R}^3)$ as $k \to \infty$. Hence

$$E^-_{\{\varepsilon_{n_k}\}}(u,b;A_1) \le \liminf_{k\to\infty} E^2_{\varepsilon_{n_k}}(u_k;A_1) \le \liminf_{j\to\infty} E^2_{\varepsilon_{n_{k_j}}}(\tilde{u}_{k_j};A_1). \tag{11}$$

Thus it remains to estimate the right-hand side of the previous inequality. By the growth condition $(H_1)'$, we have the estimate

$$\begin{aligned} E^2_{\varepsilon_{n_{k_j}}}(\tilde{u}_{k_j};A_1) \le{}& E^2_{\varepsilon_{n_{k_j}}}(u_{k_j};A_1\setminus\overline{A_3}) + E^2_{\varepsilon_{n_{k_j}}}(v_{k_j};A') \\ &+ C\left(\mathcal{G}_j(u_{k_j};L_\delta) + \mathcal{G}_j(v_{k_j};L_\delta)\right) \\ &+ \frac{C}{\delta^q}\int_{L_\delta\times I}|u_{k_j} - v_{k_j}|^q\, dx + \frac{C\varepsilon_{n_{k_j}}^2}{\delta^4}\int_{L_\delta\times I}|u_{k_j} - v_{k_j}|^2\, dx \\ &+ \frac{C\varepsilon_{n_{k_j}}^2}{\delta^2}\int_{L_\delta\times I}|D_p u_{k_j} - D_p v_{k_j}|^2\, dx + \frac{C}{\delta^2}\int_{L_\delta\times I}|D_3 u_{k_j} - D_3 v_{k_j}|^2\, dx. \end{aligned} \tag{12}$$

Since

$$\sup_{j\in\mathbb{N}}\int_{L_\delta\times I}|\varepsilon_{n_{k_j}} D^2 u_{k_j}|^2\, dx < \infty,$$

by Poincaré's inequality we have that

$$\int_{L_\delta \times I} |\varepsilon_{n_{k_j}} Du_{k_j} - c_j|^2 \, dx \leq C_\delta \int_{L_\delta \times I} |\varepsilon_{n_{k_j}} D^2 u_{k_j}|^2 \, dx$$

where

$$c_j := \frac{1}{\mathcal{L}^2 (L_\delta)} \int_{L_\delta \times I} \varepsilon_{n_{k_j}} Du_{k_j} \, dx \to 0.$$

Hence

$$\sup_{j \in \mathbb{N}} \int_{L_\delta \times I} |\varepsilon_{n_{k_j}} Du_{k_j}|^2 \, dx < \infty.$$

It follows by Rellich Kondrachov Theorem and the fact that $u_{k_j} \rightharpoonup u$ in $W^{1,q}(\Omega; \mathbb{R}^3)$ that $\varepsilon_{n_{k_j}} Du_{k_j} \to 0$ in $L^2 \left(L_\delta \times I; \mathbb{R}^{3\times 3}\right)$. Again by Poincaré's inequality we have that

$$\int_{L_\delta \times I} |\varepsilon_{n_{k_j}} u_{k_j} - d_j|^2 \, dx \leq C_\delta \int_{L_\delta \times I} |\varepsilon_{n_{k_j}} Du_{k_j}|^2 \, dx,$$

where

$$d_j := \frac{1}{\mathcal{L}^2 (L_\delta)} \int_{L_\delta \times I} \varepsilon_{n_{k_j}} u_{k_j} \, dx \to 0,$$

and so $\varepsilon_{n_{k_j}} u_{k_j} \to 0$ in $L^2 \left(L_\delta \times I; \mathbb{R}^{3\times 3}\right)$. Similar conclusions hold for v_{k_j}. Hence, letting $j \to \infty$ in (12) and using (9) and (10), we have

$$\begin{aligned} E^-_{\{\varepsilon_{n_k}\}}(u, b; A_1) &\leq E^-_{\{\varepsilon_{n_k}\}}(u, b; A_1 \setminus \overline{A_3}) + E^-_{\{\varepsilon_{n_k}\}}(u, b; A') + \eta + C\nu(\overline{L_\delta}) \\ &\leq E^-_{\{\varepsilon_{n_k}\}}(u, b; A_1 \setminus \overline{A_3}) + E^-_{\{\varepsilon_{n_k}\}}(u, b; A_2) + \eta + C\nu(\overline{L_\delta}) \end{aligned}$$

and letting δ go to zero we obtain

$$\begin{aligned} E^-_{\{\varepsilon_{n_k}\}}(u, b; A_1) &\leq E^-_{\{\varepsilon_{n_k}\}}(u, b; A_2) + E^-_{\{\varepsilon_{n_k}\}}(u, b; A_1 \setminus \overline{A_3}) + 2\eta + C\nu(S_t) \\ &= E^-_{\{\varepsilon_{n_k}\}}(u, b; A_2) + E^-_{\{\varepsilon_{n_k}\}}(u, b; A_1 \setminus \overline{A_3}) + 2\eta. \end{aligned}$$

It suffices to let $\eta \to 0^+$. □

As an immediate consequence of the previous theorem we have

$$E^-_{\{\varepsilon_{n_k}\}}(u, b; A) = \int_A \frac{dE^-_{\{\varepsilon_{n_k}\}}(u, b; \cdot)}{d\mathcal{L}^2} (x_\alpha) \, dx_\alpha,$$

where $\frac{dE^-_{\{\varepsilon_{n_k}\}}(u,b;\cdot)}{d\mathcal{L}^2}$ is the Radon-Nikodym derivative of $E^-_{\{\varepsilon_{n_k}\}}(u, b; \cdot)$ with respect to the Lebesgue measure in $\mathbb{R}^2$.

Remark E. *A proof almost identical to that of Remark 4.3 in* [2] *shows that* $\overline{\mathcal{W}}_2(\cdot, \cdot)$ *is upper semi-continuous on* $\mathbb{R}^{2\times 3} \times W^{1,2}(I; \mathbb{R}^3)$ *equipped with its strong topology.*

We now turn to the proof of Theorem B. The argument is very similar to that of Theorem 4.4 in [2] with the exception that in the proof of the lower bound the additional hypothesis (3) allows us to avoid the use of equi-integrable sequences.

Proof. Fix $(u, b) \in \mathcal{V}^2$ and $A \in \mathcal{A}(\omega)$. As usual, we identify u with a function in $W^{1,q} \left(\omega; \mathbb{R}^3\right)$. Also, for simplicity of notation from now on we write ε_k in place of ε_{n_k}.

Lower bound. We claim that

$$E^-_{\{\varepsilon_k\}} (u, b; A) \geq \int_A \overline{\mathcal{W}}_2(D_p u(x_\alpha)|b(x_\alpha, \cdot)) \, dx_\alpha. \tag{13}$$

Consider any sequence $\{u_k\} \subset W^{2,2}\left(\Omega;\mathbb{R}^3\right)$ such that $u_k \rightharpoonup u$ in $W^{1,q}(\Omega;\mathbb{R}^3)$, $\frac{1}{\varepsilon_k} D_3 u_k \rightharpoonup b$ in $L^q(\Omega;\mathbb{R}^3)$. Extracting a subsequence if necessary, we may assume, without loss of generality, that

$$\liminf_{k\to\infty} E^2_{\varepsilon_k}(u_k;A) = \lim_{k\to\infty} E^2_{\varepsilon_k}(u_k;A) \tag{14}$$

and that the bounded sequence

$$\mu_k := \left(W\left(D_p u_k \left| \frac{1}{\varepsilon_k} D_3 u_k \right.\right) + \varepsilon_k^2 \left|D_p^2 u_k\right|^2 + \left|D_{p3} u_k\right|^2 + \frac{1}{\varepsilon_k^2}\left|D_{33} u_k\right|^2 \right) \mathcal{L}^3 \lfloor (A \times I)$$

satisfies

$$\mu_k \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}\,(A \times I)$$

for some of nonnegative finite Radon measure μ on $A \times I$. Denote by $\hat{\mu}$ the finite Radon measure on A defined by

$$\hat{\mu}(B) := \mu\,(B \times I)\,,$$

for all Borel sets $B \subset A$. We will show below that the Radon-Nikodym derivative of $\hat{\mu}$ with respect to the Lebesgue measure on $\mathbb{R}^2$ satisfies

$$\frac{d\hat{\mu}}{d\mathcal{L}^2}(x_\alpha) \geq \overline{\mathcal{W}}_2(D_p u(x_\alpha)|b(x_\alpha,\cdot)) \tag{15}$$

for $\mathcal{L}^2$ a.e. every point $x_\alpha \in A$.

Note that if (15) holds, then from (14)

$$\begin{aligned}\lim_{k\to\infty} E^2_{\varepsilon_k}(u_k;A) &= \lim_{k\to\infty} \mu_k\,(A\times I) \geq \hat{\mu}(A)\\ &\geq \int_A \frac{d\hat{\mu}}{d\mathcal{L}^2}(x_\alpha)\,dx_\alpha \geq \int_A \overline{\mathcal{W}}_2(D_p u(x_\alpha)|b(x_\alpha,\cdot))\,dx_\alpha.\end{aligned}$$

Taking the infimum over all admissible sequences $\{u_k\}$ we obtain (13).

Step 2: As in Lemma 5.1 in [2], it can be shown that, up to the extraction of a subsequence,

$$\frac{1}{\varepsilon_k} D_3 u_k(\cdot, x_3) \rightharpoonup b(\cdot,x_3) \text{ in } L^q(A;\mathbb{R}^3) \text{ for all } x_3 \in I \tag{16}$$

and that for any Borel subset $B \subset A$ and for all $x_3 \in I$,

$$\sup_{k\in\mathbb{N}} \left| \int_B \frac{1}{\varepsilon_k} D_3 u_k\,(x_\alpha, x_3)\; dx_\alpha \right| < \infty. \tag{17}$$

We now address the proof of (15).

Since $u \in W^{1,q}(\omega;\mathbb{R}^3)$, for $\mathcal{L}^2$ a.e. $x_\alpha^0 \in A$ we have

$$\lim_{\delta\to 0^+} \frac{1}{\delta^{2+q}} \int_{Q'(x_\alpha,\delta)} |u(x_\alpha) - u(x_\alpha^0) - D_p u(x_\alpha^0)(x_\alpha - x_\alpha^0)|^q\,dx_\alpha = 0. \tag{18}$$

Moreover, viewing b as a Bochner integrable function, that is an element of

$$L^q\left(A; L^q\left(I;\mathbb{R}^3\right)\right),$$

for $\mathcal{L}^2$ a.e. $x_\alpha^0 \in A$ we have

$$\lim_{\delta\to 0^+} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{1}{\delta^2} \int_{Q'(x_\alpha^0,\delta)} b(x_\alpha,x_3)\,dx_\alpha - b(x_\alpha^0,x_3) \right|^q dx_3 = 0. \tag{19}$$

Fix a point $x_\alpha^0 \in A$ that satisfies (18), (19), and such that

$$b(x_\alpha^0, \cdot) \in W^{1,2}\left(I; \mathbb{R}^3\right) \tag{20}$$

and

$$\frac{d\hat{\mu}}{d\mathcal{L}^2}(x_\alpha^0) \text{ exists and is finite.}$$

We claim that (15) holds at x_α^0.

Consider a sequence $\{\delta_i\}$, with $\delta_i \to 0^+$ such that

$$\lim_{i\to\infty} \frac{1}{(\delta_i)^2} \int_{Q'(x_\alpha^0,\delta_i)} b(x_\alpha, x_3)\, dx_\alpha = b(x_\alpha^0, x_3) \tag{21}$$

for $\mathcal{L}^1$ a.e. $x_3 \in I$, and

$$\mu\left(\partial\left(Q'(x_\alpha^0, \delta_i) \times I\right)\right) = 0.$$

From the definition of $\hat{\mu}$ together with that of the Radon-Nikodym derivative, we obtain

$$\begin{aligned}
\frac{d\hat{\mu}}{d\mathcal{L}^2}(x_\alpha^0) &= \lim_{i\to\infty} \frac{\hat{\mu}(Q'(x_\alpha^0, \delta_i))}{(\delta_i)^2} = \lim_{i\to\infty} \frac{\mu\left(Q'(x_\alpha^0, \delta_i) \times I\right)}{(\delta_i)^2} \\
&= \lim_{i\to\infty} \lim_{k\to\infty} \frac{1}{(\delta_i)^2} \left(\int_{Q'(x_\alpha^0,\delta)\times I} W\left(D_p u_k(x) \middle| \frac{1}{\varepsilon_k} D_3 u_k(x)\right) dx \right. \\
&\left. + \int_{Q'(x_\alpha^0,\delta)\times I} \left(\varepsilon_k^2 \left|D_p^2 u_k(x)\right|^2 + \left|D_{p3} u_k(x)\right|^2 + \frac{1}{\varepsilon_k^2} \left|D_{33} u_k(x)\right|^2\right) dx \right) \\
&= \lim_{i\to\infty} \lim_{k\to\infty} \left(\int_Q W\left(D_p v_{k,i}(y) \middle| \frac{\delta_i}{\varepsilon_k} D_3 v_{k,i}(y)\right) dy \right. \\
&\left. + \int_Q \left(\left(\frac{\varepsilon_k}{\delta_i}\right)^2 \left|D_p^2 v_{k,i}(y)\right|^2 + \left|D_{p3} v_{k,i}(y)\right|^2 + \left(\frac{\delta_i}{\varepsilon_k}\right)^2 \left|D_{33} v_{k,i}(y)\right|^2 \right) dy \right),
\end{aligned}$$

where for $y \in Q$,

$$\begin{aligned}
v_{k,i}(y) &:= \frac{u_k(x_\alpha^0 + \delta_i y_\alpha, y_3) - u(x_\alpha^0)}{\delta_i}, \\
u_0(y_\alpha) &:= D_p u(x_\alpha^0) \cdot y_\alpha, \quad b_0(y_3) := b(x_\alpha^0, y_3).
\end{aligned}$$

Note that, since $u_k \to u$ in $L^q(\Omega; \mathbb{R}^3)$ and by (18), we have

$$\begin{aligned}
&\lim_{i\to\infty} \lim_{k\to\infty} \int_Q \left|v_{k,i}(y) - u_0(y_\alpha)\right|^q dy \\
&= \lim_{i\to\infty} \lim_{k\to\infty} \frac{1}{(\delta_i)^{2+q}} \int_{Q'(x_\alpha^0,\delta_i)\times I} \left|u_k(x) - u(x_\alpha^0) - D_p u(x_\alpha^0) \cdot (x_\alpha - x_\alpha^0)\right|^q dx \\
&= \lim_{i\to\infty} \frac{1}{(\delta_i)^{2+q}} \int_{Q'(x_\alpha^0,\delta_i)} \left|u(x_\alpha) - u(x_\alpha^0) - D_p u(x_\alpha^0) \cdot (x_\alpha - x_\alpha^0)\right|^q dx_\alpha = 0.
\end{aligned}$$

On the other hand, in view of (16), for all $y_3 \in I$,

$$\begin{aligned}
\lim_{k\to\infty} \int_{Q'} \frac{\delta_i}{\varepsilon_k} D_3 v_{k,i}(y_\alpha, y_3)\, dy_\alpha &= \lim_{k\to\infty} \frac{1}{(\delta_i)^2} \int_{Q'(x_\alpha^0,\delta_i)} \frac{1}{\varepsilon_k} D_3 u_k(x_\alpha, y_3)\, dx_\alpha \\
&= \frac{1}{(\delta_i)^2} \int_{Q'(x_\alpha^0,\delta_i)} b(x_\alpha, y_3)\, dx_\alpha,
\end{aligned}$$

and so by (17) it follows from Lebesgue's Dominated Convergence Theorem,

$$\lim_{k\to\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\int_{Q'}\frac{\delta_i}{\varepsilon_k}D_3v_{k,i}(y_\alpha,y_3)\,dy_\alpha-b_0(y_3)\right|^q dy_3$$
$$=\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{1}{(\delta_i)^2}\int_{Q'(x_\alpha^0,\delta_i)}b(x_\alpha,x_3)\,dx_\alpha-b(x_\alpha^0,x_3)\right|^q dx_3.$$

By (19) we have

$$\lim_{i\to\infty}\lim_{k\to\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\int_{Q'}\frac{\delta_i}{\varepsilon_k}D_3v_{k,i}(y_\alpha,y_3)\,dy_\alpha-b_0(y_3)\right|^q dy_3=0.$$

By a standard diagonalization argument, we may extract subsequences $v_i := v_{k_i,i}$ and $\varepsilon_i := \frac{\varepsilon_{k_i}}{\delta_i} \to 0^+$ such that

$$\infty>\frac{d\hat{\mu}}{d\mathcal{L}^2}(x_\alpha^0)=\lim_{i\to\infty}\left(\int_Q W\left(D_pv_i\left|\frac{1}{\varepsilon_i}D_3v_i\right.\right)\right.\tag{22}$$
$$\left.+\int_Q\left(\varepsilon_i^2\left|D_p^2v_i\right|^2+\left|D_{p3}v_i\right|^2+\frac{1}{\varepsilon_i^2}\left|D_{33}v_i\right|^2\right)dy\right),$$
$$\lim_{i\to\infty}\int_Q|v_i-u_0|^q\,dy=0,\tag{23}$$
$$\lim_{i\to\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\int_{Q'}\frac{1}{\varepsilon_i}D_3v_i(y_\alpha,y_3)\,dy_\alpha-b_0(y_3)\right|^q dy_3=0.\tag{24}$$

Reasoning as in Theorem D, we can assume, without loss of generality, that $v_i = u_0$ in a neighborhood of $\partial Q' \times I$.

For $y \in Q$ define

$$\varphi_i(y):=v_i(y)-u_0(y_\alpha)-\int_{Q'}(v_i-u_0)(w_\alpha,y_3)\,dw_\alpha.$$

Then

$$D_p\varphi_i(y)=D_pv_i(y)-D_pu(x_\alpha^0),\tag{25}$$
$$\frac{1}{\varepsilon_i}D_3\varphi_i(y)=\frac{1}{\varepsilon_i}D_3v_i(y)-\int_{Q'}\frac{1}{\varepsilon_i}D_3v_i(w_\alpha,y_3)\,dw_\alpha,\tag{26}$$
$$\frac{1}{\varepsilon_i}D_{33}\varphi_i(y)=\frac{1}{\varepsilon_i}D_{33}v_i(y)-\int_{Q'}\frac{1}{\varepsilon_i}D_{33}v_i(w_\alpha,y_3)\,dw_\alpha,\tag{27}$$

and note that

$$\int_{Q'}D_3\varphi_i(y_\alpha,y_3)\,dy_\alpha=0\text{ for all }y_3\in I.\tag{28}$$

Since $\varphi_i(\cdot, y_3)$ is Q'-periodic for $\mathcal{L}^1$ a.e. y_3, it follows that φ_i is admissible in the definition of $\overline{\mathcal{W}}_2(D_p u(x_\alpha^0)|b(x_\alpha^0, \cdot))$. Moreover,

$$\begin{aligned}
&\int_Q W\left(D_p v_i(y)\left|\frac{1}{\varepsilon_i}D_3 v_i(y)\right.\right) dy \\
&+\int_Q \left(\varepsilon_i^2 \left|D_p^2 v_i(y)\right|^2 + |D_{p3} v_i(y)|^2 + \frac{1}{\varepsilon_i^2}|D_{33} v_i(y)|^2\right) dy \\
&= \int_Q W\left(D_p u(x_\alpha^0) + D_p\varphi_i(y)\left| b_0(y_3) + \frac{1}{\varepsilon_i}D_3\varphi_i(y) + z_i(y_3)\right.\right) \\
&+\int_Q \left(\varepsilon_i^2 \left|D_p^2 \varphi_i(y)\right|^2 + |D_{p3}\varphi_i(y)|^2 + \left|b_0'(y_3) + \frac{1}{\varepsilon_i}D_{33}\varphi_i(y) + z_i'(y_3)\right|^2\right) dy,
\end{aligned} \tag{29}$$

where

$$z_i(y_3) := \int_{Q'} \frac{1}{\varepsilon_i} D_3 v_i(w_\alpha, y_3)\, dw_\alpha - b_0(y_3). \tag{30}$$

Since by (28),

$$\int_{Q'} D_{33}\varphi_i(y_\alpha, y_3)\, dy_\alpha = D_3\left(\int_{Q'} D_3\varphi_i(y_\alpha, y_3)\, dy_\alpha\right) = 0 \text{ for all } y_3 \in I,$$

it follows that

$$\begin{aligned}
\int_Q \left|b_0' + \frac{1}{\varepsilon_i}D_{33}\varphi_i + z_i'\right|^2 dy &\geq \int_Q \left|b_0' + \frac{1}{\varepsilon_i}D_{33}\varphi_i\right|^2 dy + 2\int_Q \left(b_0' + \frac{1}{\varepsilon_i}D_{33}\varphi_i\right)\cdot z_i'\, dy \\
&= \int_Q \left|b_0' + \frac{1}{\varepsilon_i}D_{33}\varphi_i\right|^2 dy + 2\int_Q b_0' \cdot z_i'\, dy.
\end{aligned} \tag{31}$$

We claim that

$$z_i' \rightharpoonup 0 \text{ in } W^{1,2}\left(I;\mathbb{R}^3\right). \tag{32}$$

If the claim holds, then letting $i \to \infty$ in the previous inequality yields

$$\limsup_{i\to\infty} \int_Q \left|b_0' + \frac{1}{\varepsilon_i}D_{33}\varphi_i + z_i'\right|^2 dy \geq \limsup_{i\to\infty}\int_Q \left|b_0' + \frac{1}{\varepsilon_i}D_{33}\varphi_i\right|^2 dy. \tag{33}$$

To prove (32) note that, up to a subsequence, from (22) and (24) we may assume that $z_i(y_3) \to 0$ for $\mathcal{L}^1$ a.e. $y_3 \in I$ and that

$$\sup_i \int_Q \left|\frac{1}{\varepsilon_i}D_{33}v_i\right|^2 dy < \infty.$$

Hence by Hölder Inequality

$$\begin{aligned}
\int_{-\frac{1}{2}}^{\frac{1}{2}} |z_i'(y_3)|^2\, dy_3 &\leq 2\int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\left|\int_{Q'} \frac{1}{\varepsilon_i}D_{33}v_i(w_\alpha, y_3)\, dw_\alpha\right|^2 + |b_0'(y_3)|^2\right] dy_3 \\
&\leq 2\int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\int_{Q'} \left|\frac{1}{\varepsilon_i}D_{33}v_i(w_\alpha, y_3)\right|^2 dw_\alpha + |b_0'(y_3)|^2\right] dy_3,
\end{aligned}$$

and so also by (20)

$$\sup_i \int_{-\frac{1}{2}}^{\frac{1}{2}} |z_i'(y_3)|^2 \, dy_3 < \infty.$$

By extracting a further subsequence, if necessary, we have shown (32). In particular, $z_i \to 0$ uniformly. Moreover, by the coercivity hypothesis $(H_1)'$ and (22)

$$\sup_i \int_Q \left(|D_p \varphi_i(y)|^q + \left| \frac{1}{\varepsilon_i} D_3 \varphi_i(y) \right|^q \right) dy < \infty.$$

Hence, using the q-Lipschitz condition (3) we obtain

$$\int_Q W \left(D_p u(x_\alpha^0) + D_p \varphi_i \middle| b_0 + \frac{1}{\varepsilon_i} D_3 \varphi_i + z_i \right) dy$$
$$\geq \int_Q W \left(D_p u(x_\alpha^0) + D_p \varphi_i \middle| b_0 + \frac{1}{\varepsilon_i} D_3 \varphi_i \right) dy + o(1).$$

In turn, using also (22), (33) we have that

$$\frac{d\hat{\mu}}{d\mathcal{L}^2}(x_\alpha^0) \geq \limsup_{i \to \infty} \left(\int_Q \left[W \left(D_p u(x_\alpha^0) + D_p \varphi_i \middle| b_0 + \frac{1}{\varepsilon_i} D_3 \varphi_i \right) \right] dy \right.$$
$$\left. + \int_Q \left(\varepsilon_i^2 \left| D_p^2 \varphi_i \right|^2 + |D_{p3} \varphi_i|^2 + \left| b_0' + \frac{1}{\varepsilon_i} D_{33} \varphi_i \right|^2 \right) dy \right).$$

Since by construction φ_i are admissible functions in the definition of $\overline{\mathcal{W}}(D_p u(x_\alpha^0)|b(x_\alpha^0,\cdot))$ (see (28)), it follows that

$$\frac{d\hat{\mu}}{d\mathcal{L}^2}(x_\alpha^0) \geq \overline{\mathcal{W}}_2(D_p u(x_\alpha^0)|b(x_\alpha^0,\cdot)),$$

and the proof of (15) is complete.

Upper bound. We first prove the upper bound

$$E^-_{\{\varepsilon_k\}}(u, b; A) \leq \int_A \overline{\mathcal{W}}_2(D_p u(x_\alpha)|b(x_\alpha,\cdot)) \, dx_\alpha \tag{34}$$

when $u(x_\alpha) = \overline{F} x_\alpha + c$ for all $x_\alpha \in Q'$ and for some $\overline{F} \in \mathbb{R}^{3\times 2}$, $c \in \mathbb{R}^3$, and $b \in W^{1,2}(I;\mathbb{R}^3)$. For $\eta > 0$ fixed, choose $\varphi \in W^{2,\infty}(Q;\mathbb{R}^3)$, and $L > 0$, with $\varphi(\cdot, x_3)$ Q'-periodic and $\int_{Q'} D_3\varphi(x_\alpha, x_3) \, dx_\alpha = 0$ for $\mathcal{L}^1$ a.e. $x_3 \in I$, such that

$$\int_Q \left[W\left(\overline{F} + D_p\varphi(x)|b(x_3) + L D_3\varphi(x)\right) + |b'(x_3) + L D_{33}\varphi(x)|^2 \right] dx \tag{35}$$
$$\leq \overline{\mathcal{W}}_2(\overline{F}|b) + \eta.$$

Extend $\varphi(\cdot, x_3)$ periodically with period Q' and for $x \in \Omega$ define

$$u_k(x_\alpha, x_3) := \overline{F} x_\alpha + c + \varepsilon_k \int_0^{x_3} b(s) \, ds + L\varepsilon_k \varphi\left(\frac{x_\alpha}{L\varepsilon_k}, x_3 \right). \tag{36}$$

Since b and φ are bounded, it follows that $\{u_k\}$ converges uniformly to u. In addition for $x \in \Omega$ we have that

$$D_p u_k(x) = \overline{F} + D_p \varphi\left(\frac{x_\alpha}{L\varepsilon_k}, x_3 \right),$$
$$\frac{1}{\varepsilon_k} D_3 u_k(x) = b(x_3) + L D_3 \varphi\left(\frac{x_\alpha}{L\varepsilon_k}, x_3 \right).$$

Hence $\{D_p u_k\}$ is bounded in L^∞, and so $u_k \rightharpoonup u$ in $W^{1,q}\left(\Omega;\mathbb{R}^3\right)$, while by the Riemman-Lebesgue Lemma, Fubini's Theorem, and the fact that

$$\int_{Q'} D_3\varphi(x_\alpha, x_3)\, dx_\alpha = 0$$

for $\mathcal{L}^1$ a.e. $x_3 \in I$, we have that $\frac{1}{\varepsilon_k} D_3 u_k \rightharpoonup b$ in $L^q\left(\Omega;\mathbb{R}^3\right)$. This proves that the sequence $\{u_k\}$ is admissible for $E^-_{\{\varepsilon_k\}}(u, b; A)$, and we have

$$\begin{aligned} E^-_{\{\varepsilon_k\}}(u, b; A) &\leq \liminf_{k\to\infty} E^2_{\varepsilon_k}(u_k; A) \\ &= \liminf_{k\to\infty} \left(\int_{A\times I} W\left(D_p u_k \middle| \frac{1}{\varepsilon_k} D_3 u_k\right) dx \right. \\ &\quad \left. + \int_{A\times I} \varepsilon_k^2 \left(\left|D_p^2 u_k\right|^2 + \frac{1}{\varepsilon_k^2}\left|D_{p3}u_k\right|^2 + \frac{1}{\varepsilon_k^4}\left|D_{33}u_k\right|^2 \right) dx \right). \end{aligned} \tag{37}$$

We have that

$$\begin{aligned} &\int_{A\times I} W\left(D_p u_k(x) \middle| \frac{1}{\varepsilon_k} D_3 u_k(x)\right) dx \\ &= \int_{A\times I} W\left(\overline{F} + D_p\varphi\left(\frac{x_\alpha}{L\varepsilon_k}, x_3\right) \middle| b(x_3) + LD_3\varphi\left(\frac{x_\alpha}{L\varepsilon_k}, x_3\right)\right) dx. \end{aligned} \tag{38}$$

On the other hand, for $x \in \Omega$ we have that

$$\begin{aligned} D_p^2 u_k(x) &= \frac{1}{L\varepsilon_k} D_p^2\varphi\left(\frac{x_\alpha}{L\varepsilon_k}, x_3\right), \\ D_{p3}u_k(x) &= D_{p3}\varphi\left(\frac{x_\alpha}{L\varepsilon_k}, x_3\right), \\ D_{33}u_k(x) &= \varepsilon_k b'(x_3) + L\varepsilon_k D_{33}\varphi\left(\frac{x_\alpha}{L\varepsilon_k}, x_3\right), \end{aligned}$$

and so

(39)

$$\begin{aligned} &\int_{A\times I} \varepsilon_k^2 \left(\left|D_p^2 u_k\right|^2 + \frac{1}{\varepsilon_k^2}\left|D_{p3}u_k\right|^2 + \frac{1}{\varepsilon_k^4}\left|D_{33}u_{n,k}\right|^2 \right) dx \\ &= \int_{A\times I} \left(\frac{1}{L^2}\left|D_p^2\varphi\left(\frac{x_\alpha}{L\varepsilon_k}, x_3\right)\right|^2 + \left|D_{p3}\varphi\left(\frac{x_\alpha}{L\varepsilon_k}, x_3\right)\right|^2 + \left|b'(x_3) + LD_{33}\varphi\left(\frac{x_\alpha}{L\varepsilon_k}, x_3\right)\right|^2 \right) dx. \end{aligned}$$

Since, for $\mathcal{L}^1$ a.e. $x_3 \in I$ the function

$$\begin{aligned} &W\left(\overline{F} + D_p\varphi(\cdot, x_3) \left| b(x_3) + LD_3\varphi(\cdot, x_3)\right.\right) \\ &+ \frac{1}{L^2}\left|D_p^2\varphi(\cdot,, x_3)\right|^2 + \left|D_{p3}\varphi(\cdot,, x_3)\right|^2 + \left|b'(x_3) + LD_{33}\varphi(\cdot,, x_3)\right|^2 \end{aligned}$$

is Q'-periodic, it converges weakly in $L^1(A)$ to its mean, that is, to

$$\begin{aligned} &\int_{Q'} \left(W\left(\overline{F} + D_p\varphi(x_\alpha, x_3) \left| b(x_3) + LD_3\varphi(x_\alpha, x_3)\right.\right) \right. \\ &\quad \left. \frac{1}{L^2}\left|D_p^2\varphi\left(\frac{x_\alpha}{L\varepsilon_k}, x_3\right)\right|^2 + \left|D_{p3}\varphi\left(\frac{x_\alpha}{L\varepsilon_k}, x_3\right)\right|^2 + \left|b'(x_3) + LD_{33}\varphi\left(\frac{x_\alpha}{L\varepsilon_k}, x_3\right)\right|^2 \right) dx_\alpha. \end{aligned}$$

Lebesgue's Dominated Convergence Theorem and Fubini's Theorem imply that

$$\lim_{k\to\infty}\int_{A\times I} W\left(\overline{F}+D_p\varphi\left(\frac{x_\alpha}{L\varepsilon_k},x_3\right)\middle|b\left(x_3\right)+LD_3\varphi\left(\frac{x_\alpha}{L\varepsilon_k},x_3\right)\right)\,dx$$
$$=\mathcal{L}^2(A)\int_Q W\left(\overline{F}+D_p\varphi\left(x_\alpha,x_3\right)|b\left(x_3\right)+LD_3\varphi\left(x_\alpha,x_3\right)\right)\,dx$$

and

(40)

$$\lim_{k\to\infty}\int_{A\times I}\left(L^2\left|D_p^2\varphi\left(\frac{x_\alpha}{L\varepsilon_k},x_3\right)\right|^2+\left|D_{p3}\varphi\left(\frac{x_\alpha}{L\varepsilon_k},x_3\right)\right|^2+\left|b'\left(x_3\right)+LD_{33}\varphi\left(\frac{x_\alpha}{L\varepsilon_k},x_3\right)\right|^2\right)\,dx$$
$$=\mathcal{L}^2(A)\int_Q\left(\frac{1}{L^2}\left|D_p^2\varphi\left(x_\alpha,x_3\right)\right|^2+\left|D_{p3}\varphi\left(x_\alpha,x_3\right)\right|^2+\left|b'\left(x_3\right)+LD_{33}\varphi\left(x_\alpha,x_3\right)\right|^2\right)\,dx,$$

which, in view of (35), (37), (38), and (39), finally yields

$$E^-_{\{\varepsilon_{n_k}\}}(u,b;A)\le\mathcal{L}^2(A)[\overline{\mathcal{W}}_2(\overline{F}|b)+\eta].$$

Letting η tend to 0, we conclude

$$E^-_{\{\varepsilon_{n_k}\}}(u,b;A)\le\mathcal{L}^2(A)\overline{\mathcal{W}}_2(\overline{F}|b). \tag{41}$$

This proves (34) when $u(x_\alpha)=\overline{F}x_\alpha+c$ for all $x_\alpha\in Q'$ and $b\in W^{1,2}\left(I;\mathbb{R}^3\right)$. The general case follows as in Step 2 and 3 of Theorem 4.4. We omit the details. ☐

In order to address the case $\gamma>2$ we first recall the result obtained in [1], where

$$\mathcal{I}_\varepsilon(u;A):=\int_{A\times I}W\left(D_pu|\frac{1}{\varepsilon}D_3u\right)\,dx$$

if $u\in W^{1,q}(\Omega;\mathbb{R}^3)$, and $\mathcal{I}_\varepsilon(u;A):=\infty$ otherwise. Fix a countable dense family $\{\theta_i\}_{i\in\mathbb{N}}$ in $L^{q'}(I;\mathbb{R}^3)$, where q' is the conjugate exponent of q. For $\overline{F}\in\mathbb{R}^{2\times3}$, $b\in L^q(I;\mathbb{R}^3)$ we define

$$\mathcal{Q}_\infty W(\overline{F}|b):=\sup_n\mathcal{Q}_nW(\overline{F}|b)=\lim_{nj\to\infty}\mathcal{Q}_nW(\overline{F}|b),$$

where

$$\begin{aligned}\mathcal{Q}_nW(\overline{F}|b):=\inf_{\varphi,L}\Bigg\{&\int_Q W\left(\overline{F}+D_p\varphi(x)|b\left(x_3\right)+LD_3\varphi\left(x\right)\right)\,dx:\\ &L>0,\ \varphi\in W^{1,q}(Q;\mathbb{R}^3),\ \varphi(\cdot,x_3)\ Q'\text{-periodic for }\mathcal{L}^1\text{ a.e. }x_3,\\ &\left|\int_Q LD_3\varphi(x_\alpha,x_3)\theta_i\left(x_3\right)\,dx\right|\le\frac{1}{n},\ i=1,\cdots,n\Bigg\}.\end{aligned}\tag{42}$$

Remark F. *If W satisfies $(H_1)'$, then a density argument shows that in the definition above it is possible to restrict admissible functions $\varphi\in W^{1,q}(Q;\mathbb{R}^3)$ to functions $\varphi\in C^\infty(\overline{Q};\mathbb{R}^3)$*

The main result in [1] is that under condition $(H_1)'$,

$$\begin{aligned}\mathcal{I}(u,b;A) &:= \inf\Big\{\liminf_{\varepsilon\to 0^+}\mathcal{I}_\varepsilon(u_\varepsilon;A) :\ u_\varepsilon\in W^{1,q}\left(A\times I;\mathbb{R}^3\right),\\ &\qquad u_\varepsilon \rightharpoonup u \text{ in } W^{1,q}\left(A\times I;\mathbb{R}^3\right),\quad \frac{1}{\varepsilon}D_3u_\varepsilon \rightharpoonup b \text{ in } L^q\left(A\times I;\mathbb{R}^3\right)\Big\} \qquad (43)\\ &= \int_A \mathcal{Q}_\infty W(D_pu(x_\alpha)|b(x_\alpha,\cdot))\,dx_\alpha\end{aligned}$$

for all $(u,b)\in W^{1,q}\left(A;\mathbb{R}^3\right)\times L^q\left(A\times I;\mathbb{R}^3\right)$ and $A\in\mathcal{A}(\omega)$.

Next we show that the Γ-liminf $E_-^\gamma(u,b;A)$ coincides with $\mathcal{I}(u,b;A)$. This establishes that in the case $\gamma>2$ the second order perturbation plays no role.

Theorem G. *Assume that $\gamma>2$ and that condition $(H_1)'$ is satisfied. Then for all $(u,b)\in\mathcal{V}^\gamma$ and $A\in\mathcal{A}(\omega)$,*

$$E_-^\gamma(u,b;A) = \int_A \mathcal{Q}_\infty W(D_pu(x_\alpha)|b(x_\alpha,\cdot))\,dx_\alpha.$$

Proof. For any given sequence $\{\varepsilon_n\}$, with $\varepsilon_n\to 0^+$, we extract a subsequence $\{\varepsilon_{n_k}\}$ such that

$$\Gamma-\lim_{k\to\infty} E^\gamma_{\varepsilon_{n_k}}(u,b;A) \text{ exists for all } (u,b)\in\mathcal{V}^\gamma \text{ and for all } A'\in\mathcal{R}(\omega),$$

and we define the Γ-lower limit

$$\begin{aligned}E^-_{\{\varepsilon_{n_k}\}}(u,b;A) &:= \inf\Big\{\liminf_{k\to\infty} E^\gamma_{\varepsilon_{n_k}}(u_k;A) :\ u_k\in W^{2,2}\left(\Omega;\mathbb{R}^3\right),\\ &\qquad u_k \rightharpoonup u \text{ in } W^{1,q}\left(\Omega;\mathbb{R}^3\right),\quad \frac{1}{\varepsilon_{n_k}}D_3u_{n_k} \rightharpoonup b \text{ in } L^q\left(\Omega;\mathbb{R}^3\right)\Big\}.\end{aligned}$$

As in the case $\gamma=2$, it suffices to show that

$$E^-_{\{\varepsilon_{n_k}\}}(u,b;A) = \int_A \mathcal{Q}_\infty W(D_pu(x_\alpha)|b(x_\alpha,\cdot))\,dx_\alpha.$$

In the sequel, to simplify the notation, we write ε_k in place of ε_{n_k}.
Lower bound. In view of (43) we deduce immediately that

$$E^-_{\{\varepsilon_k\}}(u,b;A) \geq \mathcal{I}(u,b;A) = \int_A \mathcal{Q}_\infty W(D_pu(x_\alpha)|b(x_\alpha,\cdot))\,dx_\alpha.$$

Upper bound. We first prove the upper bound

$$E^-_{\{\varepsilon_k\}}(u,b;A) \leq \int_A \mathcal{Q}_\infty W(D_pu(x_\alpha)|b(x_\alpha,\cdot))\,dx_\alpha \qquad (44)$$

when $u(x_\alpha) = \overline{F}x_\alpha + c$ for all $x_\alpha\in Q'$ and for some $\overline{F}\in\mathbb{R}^{3\times 2}$, $c\in\mathbb{R}^3$, and $b\in L^q\left(I;\mathbb{R}^3\right)$. In this case the right-hand side reduces to $\mathcal{Q}_\infty W(\overline{F}|b(\cdot))\mathcal{L}^2(A)$. In view of $(H_1)'$ we have that

$$\mathcal{Q}_\infty W(\overline{F}|b(\cdot)) < \infty. \qquad (45)$$

By definition of $\mathcal{Q}_n W(\overline{F}|b(\cdot))$ (see (42) and Remark F) we may find admissible $\varphi_n\in C^\infty(\overline{Q};\mathbb{R}^3)$ and $L_n>0$ such that

$$\int_Q W\left(\overline{F}+D_p\varphi_n(x)|b(x_3)+L_nD_3\varphi_n(x)\right)dx \leq \mathcal{Q}_nW(\overline{F}|b(\cdot))+\frac{1}{n}. \qquad (46)$$

Note that in view of (45) and $(H_1)'$ we have that

$$\sup_n \|(D_p\varphi_n|L_nD_3\varphi_n)\|_{L^q(Q;\mathbb{R}^{3\times 3})} < \infty. \tag{47}$$

Extend $\varphi(\cdot, x_3)$ to $\mathbb{R}^2$ periodically with period Q' and extend b to $\mathbb{R}$ by zero, and for $x \in \Omega$ define

$$u_{k,n}(x_\alpha, x_3) := \overline{F}x_\alpha + c + \varepsilon_k \int_0^{x_3} (b * \rho_k)(s)\, ds + L_n\varepsilon_k\varphi_n\left(\frac{x_\alpha}{L_n\varepsilon_k}, x_3\right),$$

where

$$\rho_k(t) := \frac{1}{\delta_k}\rho\left(\frac{t}{\delta_k}\right), \quad \delta_k := \varepsilon_k^{\frac{\gamma-2}{4}},$$

and $\rho \in C_c^\infty(\mathbb{R})$, $\rho \geq 0$, and $\int_{\mathbb{R}} \rho(t)\, dt = 1$. Note that

$$\lim_{n\to\infty}\lim_{k\to\infty} \|u_{k,n} - u\|_{L^q(\Omega;\mathbb{R}^3)} = 0. \tag{48}$$

Moreover, for $x \in \Omega$ we have that

$$D_pu_{k,n}(x) = \overline{F} + D_p\varphi_n\left(\frac{x_\alpha}{L_n\varepsilon_k}, x_3\right),$$
$$\frac{1}{\varepsilon_k}D_3u_{k,n}(x) = (b*\rho_k)(x_3) + L_nD_3\varphi_n\left(\frac{x_\alpha}{L_n\varepsilon_k}, x_3\right).$$

Since $\varphi_n(\cdot, x_3)$ is Q'-periodic for $\mathcal{L}^1$ a.e. x_3, by (47) it follows that

$$\sup_{k,n\in\mathbb{N}} \left\|\left(D_pu_{k,n}|\frac{1}{\varepsilon_k}D_3u_{k,n}\right)\right\|_{L^q(\Omega;\mathbb{R}^{3\times 3})} < \infty. \tag{49}$$

For every $i \in \mathbb{N}$, by (42)

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{k\to\infty}\left|\int_\Omega \left(\frac{1}{\varepsilon_k}D_3u_{k,n}(x) - b(x_3)\right)\theta_i(x_3)\, dx\right| \\
&= \lim_{n\to\infty}\lim_{k\to\infty}\left|\int_\Omega \left((b*\rho_k)(x_3) - b(x_3) + L_nD_3\varphi_n\left(\frac{x_\alpha}{L_n\varepsilon_k}, x_3\right)\right)\theta_i(x_3)\, dx\right| \\
&= \lim_{n\to\infty}\left|\int_{\omega\times I}\int_{Q'} L_nD_3\varphi_n(y_\alpha, x_3)\, dy_\alpha\theta_i(x_3)\, dx\right| = 0,
\end{aligned} \tag{50}$$

where we have used the Riemman-Lebesgue Lemma and the fact that for $n \geq i$,

$$\left|\int_Q L_nD_3\varphi_n(x_\alpha, x_3)\theta_i(x_3)\, dx\right| \leq \frac{1}{n}.$$

Also by (46), $(H_1)'$, the (generalized) Lebesgue Dominated Convergence Theorem, and the Riemman-Lebesgue Lemma we have that

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{k\to\infty}\int_{A\times I} W\left(D_pu_{k,n}(x)\left|\frac{1}{\varepsilon_k}D_3u_{k,n}(x)\right.\right) dx \\
&= \lim_{n\to\infty}\lim_{k\to\infty}\int_{A\times I} W\left(\overline{F} + D_p\varphi_n\left(\frac{x_\alpha}{L_n\varepsilon_k}, x_3\right)\left|(b*\rho_k)(x_3) + L_nD_3\varphi_n\left(\frac{x_\alpha}{L_n\varepsilon_k}, x_3\right)\right.\right) dx \\
&= \lim_{n\to\infty}\int_{A\times I}\int_{Q'} W\left(\overline{F} + D_p\varphi_n(y_\alpha, x_3)|b(x_3) + L_nD_3\varphi_n(y_\alpha, x_3)\right) dy_\alpha\, dx \\
&\leq \mathcal{L}^2(A)\lim_{n\to\infty}\mathcal{Q}_nW(\overline{F}|b(\cdot)) = \mathcal{L}^2(A)\,\mathcal{Q}_\infty W(\overline{F}|b(\cdot)).
\end{aligned} \tag{51}$$

Finally,

$$\lim_{n\to\infty}\lim_{k\to\infty}\int_{A\times I}\varepsilon_k^{\gamma}\left(\left|D_p^2 u_{k,n}\right|^2+\frac{1}{\varepsilon_k^2}\left|D_{p3}u_{k,n}\right|^2+\frac{1}{\varepsilon_k^4}\left|D_{33}u_{k,n}\right|^2\right)dx$$

(52)

$$=\lim_{n\to\infty}\lim_{k\to\infty}\int_{A\times I}\varepsilon_k^{\gamma}\left(\left|\frac{1}{L_n\varepsilon_k}D_p^2\varphi_n\left(\frac{x_\alpha}{L_n\varepsilon_k},x_3\right)\right|^2+\frac{1}{\varepsilon_k^2}\left|D_{p3}\varphi_n\left(\frac{x_\alpha}{L_n\varepsilon_k},x_3\right)\right|^2\right.$$

$$\left.+\frac{1}{\varepsilon_k^4}\left|\varepsilon_k\left(b*\rho_k'\right)(x_3)+L_n\varepsilon_k D_{33}\varphi_n\left(\frac{x_\alpha}{L_n\varepsilon_k},x_3\right)\right|^2\right)dx=0,$$

where we have used the facts that $\gamma>2$,

$$\lim_{k\to\infty}\int_{A\times I}\left|D^2\varphi_n\left(\frac{x_\alpha}{L_n\varepsilon_k},x_3\right)\right|^2 dx=\mathcal{L}^2(A)\int_{Q'\times I}\left|D^2\varphi_n(y_\alpha,x_3)\right|^2 dy_\alpha\,dx_3,$$

and

$$\lim_{k\to\infty}\int_I\varepsilon_k^{\gamma-2}\left|b*\rho_k'(x_3)\right|^2 dx_3=0,$$

because

$$\begin{aligned}\|b*\rho_k'\|_{L^\infty(I;\mathbb{R}^3)}&\le C\,\|\rho_k'\|_{L^\infty(I)}\,\|b\|_{L^q(I;\mathbb{R}^3)}\\&\le\frac{C}{\delta_k^2}\|b\|_{L^q(I;\mathbb{R}^3)}=\frac{C}{\varepsilon_k^{\frac{\gamma-2}{2}}}\|b\|_{L^q(I;\mathbb{R}^3)}.\end{aligned}$$

Hence, recalling (48), (49), (50), (51), and (52), we may find a diagonal sequence

$$u_n:=u_{k_n,n}$$

such that $u_n\rightharpoonup u$ in $W^{1,q}\left(\Omega;\mathbb{R}^3\right)$ and $\frac{1}{\varepsilon_{k_n}}D_3u_n\rightharpoonup b$ in $L^q\left(\Omega;\mathbb{R}^3\right)$ and

$$\limsup_{n\to\infty}E^{\gamma}_{\varepsilon_{k_n}}(u_n;A)\le\mathcal{L}^2(A)\,\mathcal{Q}_\infty W(\overline{F}|b(\cdot)).$$

Define

$$\tilde{u}_k(x):=\begin{cases}u_{k_n,n}(x) & \text{if } k=k_n \text{ for some } n\in\mathbb{N},\\ \overline{F}x_\alpha+c+\varepsilon_k\int_0^{x_3}(b*\rho_k)(s)\,ds & \text{otherwise.}\end{cases}$$

Since the sequence $\{\tilde{u}_k\}$ is admissible for $E^-_{\{\varepsilon_k\}}(u,b;A)$, we have

$$E^-_{\{\varepsilon_k\}}(u,b;A)\le\liminf_{n\to\infty}E^{\gamma}_{\varepsilon_{k_n}}(u_n;A)\le\mathcal{L}^2(A)\,\mathcal{Q}_\infty W(\overline{F}|b(\cdot)).$$

Just as in the proof of Theorem D, it can be shown that for every $(u,b)\in\mathcal{V}^\gamma$, $E^-_{\{\varepsilon_k\}}(u,b;\cdot)$ is the trace of a Radon measure absolutely continuous with respect to $\mathcal{L}^2\lfloor\omega$. Therefore, to establish the inequality (44) for arbitrary $(u,b)\in\mathcal{V}^\gamma$ we may proceed as in proof of Steps 2 and 3 of Theorem 4.4. We omit the details. ☐

Acknowledgement H. *The authors would like to thank Vincent Millot for bringing to their attention the error in the proof of the original Theorem 3.1.*

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