

A NOTE ON THE DERIVATION OF RIGID-PLASTIC MODELS

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ABSTRACT. This note is devoted to a rigorous derivation of rigid-plasticity as the limit of elasto-plasticity when the elasticity tends to infinity.

1. INTRODUCTION

Small strain elasto-plasticity is formally modeled as follows. Consider a homogeneous elasto-plastic material occupying a volume $\Omega \subset \mathbb{R}^n$ with Hooke's law (elasticity tensor) \mathbb{C} . Assume that the body is subjected to a time-dependent loading process during a time interval $[0, T]$ with, say, $f(t)$ as body loads, $g(t)$ as surface loads on a part Γ_N of $\partial\Omega$, and $w(t)$ as displacement loads (hard device) on the complementary part Γ_D of $\partial\Omega$. Denoting by $Eu(t)$ the infinitesimal strain at t , that is, the symmetric part of the spatial gradient of the displacement field $u(t)$ at t , small strain elasto-plasticity requires that $Eu(t)$ decompose additively as

$$Eu(t) = e(t) + p(t) \text{ in } \Omega, \text{ with } u(t) = w(t) \text{ on } \Gamma_D$$

where $e(t)$ is the elastic strain and $p(t)$ the plastic strain. The elastic strain is related to the stress tensor $\sigma(t)$ through the constitutive law of linearized elasticity $\sigma(t) = \mathbb{C}e(t)$. In a quasi-static setting, the equilibrium equations read as

$$\operatorname{div} \sigma(t) + f(t) \text{ in } \Omega, \quad \sigma(t)\nu = g(t) \text{ on } \Gamma_N,$$

where ν denotes the outer unit normal to $\partial\Omega$. In plasticity, the stresses are constrained to remain below a yield stress at which permanent strains appear. Specifically, the deviatoric stress $\sigma_D(t)$ must belong to a fixed compact and convex subset K of the deviatoric (trace free) matrices

$$\sigma_D(t) \in K.$$

If $\sigma_D(t)$ lies inside the interior of K , the material behaves elastically ($p(t) = 0$). On the other hand, if $\sigma_D(t)$ reaches the boundary of K (called the yield surface), a plastic flow may develop, so that, after unloading, there will remain a non-trivial permanent plastic strain $p(t)$. Its evolution is described by the so-called flow rule

$$\dot{p}(t) \in N_K(\sigma_D(t))$$

where $N_K(\sigma_D(t))$ is the normal cone to K at $\sigma_D(t)$. By arguments of convex analysis, the flow rule can be equivalently written as Hill's principle of maximum plastic work

$$\sigma_D(t) : \dot{p}(t) = \max_{\tau_D \in K} \tau_D : \dot{p}(t) =: H(\dot{p}(t)),$$

where H is the support function of K , and $H(\dot{p}(t))$ identifies with the plastic dissipation.

In this self-contained note, we propose to show that rigid plasticity – that is the model where one formally sets $\mathbb{C} = \infty$ (and correspondingly $\dot{p}(t) = E\dot{u}(t)$, $\operatorname{div} \dot{u}(t) = 0$) in the system above – can be derived as an asymptotic limit of small strain elasto-plasticity as \mathbb{C} actually gets larger and larger. Rigid-plastic models are particularly useful in order to compute analytical solutions in a plane-strain setting. Indeed, inside the plastic zone, the stress equations can be formally written

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as a non-linear hyperbolic system which is solved by the method of characteristics. The family of characteristics are the so-called *slip lines* along which some combinations of the stress remain constants, while the tangential velocities can jump. It thus seems appropriate to rigorously derive rigid-plasticity in order to investigate the hyperbolic structure of the equations. However, this later task falls outside the scope of the present work.

Notationwise, we denote by $\mathbb{M}_{sym}^{n \times n}$ the set of symmetric $n \times n$ matrices. If A and $B \in \mathbb{M}_{sym}^{n \times n}$, we use the Euclidean scalar product $A : B := \text{tr}(AB)$ and the associated Euclidean norm $|A| := \sqrt{A : A}$. The subset $\mathbb{M}_D^{n \times n}$ of $\mathbb{M}_{sym}^{n \times n}$ stands for trace free symmetric matrices. If $A \in \mathbb{M}_{sym}^{n \times n}$, it can be orthogonally decomposed as

$$A = A_D + \frac{\text{tr } A}{n} I,$$

where $A_D \in \mathbb{M}_D^{n \times n}$, and I is the identity matrix in \mathbb{R}^n . The notation \odot stands for the symmetrized tensor product between vectors in \mathbb{R}^n , *i.e.*, if a and $b \in \mathbb{R}^n$, $(a \odot b)_{ij} = (a_i b_j + a_j b_i)/2$ for all $1 \leq i, j \leq n$. Note in particular that $\frac{1}{\sqrt{2}}|a||b| \leq |a \odot b| \leq |a||b|$.

The Lebesgue measure in \mathbb{R}^n and the $(n-1)$ -dimensional Hausdorff measure are denoted by \mathcal{L}^n and \mathcal{H}^{n-1} , respectively. Given a locally compact set $E \subset \mathbb{R}^n$ and a Euclidean space X , we denote by $\mathcal{M}(E; X)$ (or simply $\mathcal{M}(E)$ if $X = \mathbb{R}$) the space of bounded Radon measures on E with values in X , endowed with the norm $\|\mu\|_{\mathcal{M}(E; X)} := |\mu|(E)$, where $|\mu| \in \mathcal{M}(E)$ is the variation of the measure μ . Moreover, if ν is a non-negative Radon measure over E , we denote by $d\mu/d\nu$ the Radon-Nikodym derivative of μ with respect to ν .

We use standard notation for Lebesgue and Sobolev spaces. In particular, for $1 \leq p \leq \infty$, the L^p -norms of the various quantities are denoted by $\|\cdot\|_p$. If $U \subset \mathbb{R}^n$ is an open set, the space $BD(U)$ of functions of bounded deformation in U is made of all functions $u \in L^1(U; \mathbb{R}^n)$ such that $Eu \in \mathcal{M}(U; \mathbb{M}_{sym}^{n \times n})$, where $Eu := (Du + Du^T)/2$ and Du is the distributional derivative of u . We refer to [14] for general properties of this space. Finally, $H(\text{div}, U)$ stands for the Hilbert space of all $\tau \in L^2(U; \mathbb{M}_{sym}^{n \times n})$ such that $\text{div } \tau \in L^2(U; \mathbb{R}^n)$.

2. THE ELASTO-PLASTIC MODEL

We now consider a homogeneous elasto-plastic material with Hooke's law given by a fourth order tensor \mathbb{C} satisfying the usual symmetry properties

$$\mathbb{C}_{ijkl} = \mathbb{C}_{jikl} = \mathbb{C}_{klij}, \quad \text{for all } 1 \leq i, j, k, l \leq n, \quad (2.1)$$

and the growth and coercivity assumptions

$$\alpha|\xi|^2 \leq \mathbb{C}\xi : \xi \leq \beta|\xi|^2, \quad \text{for all } \xi \in \mathbb{M}_{sym}^{n \times n}, \quad (2.2)$$

where α and $\beta > 0$.

It occupies the domain Ω , a bounded and connected open subset of \mathbb{R}^n with at least Lipschitz boundary (see Definition 2.1) and outer normal ν . Its boundary $\partial\Omega$ is split into the union of a Dirichlet part Γ_D which is non empty and open in the relative topology of $\partial\Omega$, a Neumann part $\Gamma_N := \partial\Omega \setminus \overline{\Gamma_D}$, and their common relative boundary denoted by $\partial_{|\partial\Omega} \Gamma_D$.

Standard plasticity is characterized by the fact that the deviatoric stress is constrained to stay in a fixed compact and convex subset $K \subset \mathbb{M}_D^{n \times n}$ of deviatoric matrices. We further assume that

$$B(0, c_*) \subset K \subset B(0, c^*), \quad (2.3)$$

where $0 < c_* < c^* < \infty$, and denote by

$$\mathcal{K} := \{\sigma \in L^2(\Omega; \mathbb{M}_{sym}^{n \times n}) : \sigma_D(x) \in K \text{ for a.e. } x \in \Omega\}.$$

The support function of K , defined for any $p \in \mathbb{M}_D^{n \times n}$ by $H(p) := \sup_{\tau \in K} \tau : p$, satisfies, according to (2.3),

$$c_* |p| \leq H(p) \leq c^* |p|, \quad \text{for all } p \in \mathbb{M}_D^{n \times n}.$$

On the Dirichlet part Γ_D of the boundary, the body is subjected to a hard device, *i.e.*, a boundary displacement which is the trace on Γ_D of a function $w \in AC([0, T]; H^1(\Omega; \mathbb{R}^n))$. In addition, the body is subjected to two types of forces: bulk forces $f \in AC([0, T]; L^n(\Omega; \mathbb{R}^n))$, and surface forces $g \in AC([0, T]; L^\infty(\Gamma_N; \mathbb{R}^n))$, the latter acting on the Neumann part Γ_N of the boundary. It is classical to assume a uniform safe load condition (see [12]) which ensures the existence of a plastically, as well as statically admissible state of stress π associated with the pair (f, g) . Specifically, there exists $\pi \in AC([0, T]; L^2(\Omega; \mathbb{M}_{sym}^{n \times n}))$ and some safety parameter $c > 0$ such that

$$\begin{cases} \pi_D(t, x) + B(0, c) \subset K \text{ for a.e. } x \in \Omega \text{ and all } t \in [0, T] \\ \operatorname{div} \pi(t) + f(t) = 0 \text{ in } \Omega, \quad \pi(t)\nu = g(t) \text{ on } \Gamma_N. \end{cases}$$

Given a boundary datum $\hat{w} \in H^1(\Omega; \mathbb{R}^n)$, we define the space of all kinematically admissible triples as

$$\begin{aligned} \mathcal{A}(\hat{w}) := \{ & (u, e, p) \in BD(\Omega) \times L^2(\Omega; \mathbb{M}_{sym}^{n \times n}) \times \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n}) : \\ & Eu = e + p \text{ in } \Omega, \quad p = (\hat{w} - u) \odot \nu \mathcal{H}^{n-1} \llcorner \Gamma_D \text{ on } \Gamma_D \}, \end{aligned}$$

where we still denote by u the trace of u on $\partial\Omega$ (see [2]). We also define the space of all statically admissible stresses as

$$\Sigma := \{ \sigma \in L^2(\Omega; \mathbb{M}_{sym}^{n \times n}) : \operatorname{div} \sigma \in L^n(\Omega; \mathbb{R}^n), \quad \sigma_D \in L^\infty(\Omega; \mathbb{M}_D^{n \times n}), \quad \sigma\nu \in L^\infty(\Gamma_N; \mathbb{R}^n) \},$$

where $\sigma\nu$ is the normal trace of $\sigma \in H(\operatorname{div}, \Omega)$ which is well defined as an element of $H^{-1/2}(\Gamma_N; \mathbb{R}^n)$, the dual space of $H_{00}^{1/2}(\Gamma_N; \mathbb{R}^n)$.

Following [7, Section 6], we introduce the following class of domains for which a meaningful duality pairing between stresses and strains can be defined. Note that the class contains in particular \mathcal{C}^2 -domains where the relative a boundary $\partial_{|\partial\Omega} \Gamma_D$ is a $(n-2)$ -dimensional manifold, and $\partial\Omega$ is of class \mathcal{C}^2 in a neighborhood of $\partial_{|\partial\Omega} \Gamma_D$ [10], as well as hypercubes where Γ_D is one of its faces [7, Section 6].

Definition 2.1. We say that Ω is admissible if for any $\sigma \in \Sigma$, and any $p \in \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})$, with $(u, e, p) \in \mathcal{A}(\hat{w})$ for some $\hat{w} \in H^1(\Omega; \mathbb{R}^n)$, $u \in BD(\Omega)$ and $e \in L^2(\Omega; \mathbb{M}_{sym}^{n \times n})$, the distribution defined for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ by

$$\begin{aligned} \langle [\sigma_D : p], \varphi \rangle := & \int_{\Omega} \varphi \sigma : (E\hat{w} - e) dx - \int_{\Omega} \varphi \operatorname{div} \sigma \cdot (u - \hat{w}) dx \\ & - \int_{\Omega} \sigma : [(u - \hat{w}) \odot \nabla \varphi] dx + \int_{\Gamma_N} \varphi \sigma \nu \cdot (u - \hat{w}) d\mathcal{H}^{n-1} \end{aligned}$$

extends to a bounded Radon measure in \mathbb{R}^n with $\|[\sigma_D : p]\| \leq \|\sigma_D\|_\infty |p|$. In this case, its mass is given by

$$\langle \sigma_D, p \rangle := \langle [\sigma_D : p], 1 \rangle = \int_{\Omega} \sigma : (E\hat{w} - e) dx - \int_{\Omega} \operatorname{div} \sigma \cdot (u - \hat{w}) dx + \int_{\Gamma_N} \sigma \nu \cdot (u - \hat{w}) d\mathcal{H}^{n-1}. \quad (2.4)$$

For any $e \in L^2(\Omega; \mathbb{M}_{sym}^{n \times n})$, the elastic energy is

$$\mathcal{Q}(e) = \frac{1}{2} \int_{\Omega} \mathbb{C}e : e dx,$$

while, for any $p \in \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})$, the dissipation energy is the convex functional of measure (see [9, 6])

$$\mathcal{H}(p) := \int_{\Omega \cup \Gamma_D} H\left(\frac{dp}{d|p|}\right) d|p|.$$

If $p : [0, T] \rightarrow \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})$, we define the total dissipation between times a and b by

$$\mathcal{V}_{\mathcal{H}}(p; [a, b]) := \sup \left\{ \sum_{i=1}^N \mathcal{H}(p(t_i) - p(t_{i-1})) : N \in \mathbb{N}, a = t_0 < t_1 < \dots < t_N = b \right\}.$$

If additionally $p \in AC([0, T]; \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n}))$, then [4, Theorem 7.1] shows that

$$\mathcal{V}_{\mathcal{H}}(p; [a, b]) = \int_a^b \mathcal{H}(\dot{p}(s)) ds.$$

We finally impose the following initial condition on the evolution: $(u_0, e_0, p_0) \in \mathcal{A}(w(0))$ with $\sigma_0 := \mathbb{C}e_0$ such that

$$\operatorname{div} \sigma_0 + f(0) = 0 \text{ in } \Omega, \quad \sigma_0 \nu = g(0) \text{ on } \Gamma_N, \quad (\sigma_0)_D \in \mathcal{K}.$$

The following existence result has been established in [4, 7].

Theorem 2.2. *Under the previous assumptions, there exist a quasi-static evolution, i.e. a mapping $t \mapsto (u(t), e(t), p(t))$ with the following properties*

$$u \in AC([0, T]; BD(\Omega)), \quad \sigma, e \in AC([0, T]; L^2(\Omega; \mathbb{M}_{sym}^{n \times n})), \quad p \in AC([0, T]; \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})),$$

$$(u(0), e(0), p(0)) = (u_0, e_0, p_0),$$

and for all $t \in [0, T]$,

$$\begin{cases} Eu(t) = e(t) + p(t) \text{ in } \Omega, \\ p(t) = (w(t) - u(t)) \odot \nu \mathcal{H}^{n-1} \llcorner \Gamma_D \text{ on } \Gamma_D, \\ \sigma(t) = \mathbb{C}e(t) \text{ in } \Omega, \end{cases}$$

$$\begin{cases} \operatorname{div} \sigma(t) + f(t) = 0 \text{ in } \Omega, \\ \sigma(t) \nu = g(t) \text{ on } \Gamma_N, \\ \sigma_D(t) \in \mathcal{K}, \end{cases}$$

and for a.e. $t \in [0, T]$,

$$H(\dot{p}(t)) = [\sigma_D(t) : \dot{p}(t)] \text{ in } \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n}). \quad (2.5)$$

Remark 2.3. Equation (2.5) is a measure-theoretic formulation of the usual flow rule of perfect plasticity. Using the definition (2.4) of duality, it can be equivalently written as an energy balance

$$\begin{aligned} \mathcal{Q}(e(t)) + \int_0^t \mathcal{H}(\dot{p}(s)) ds &= \mathcal{Q}(e_0) + \int_0^t \int_{\Omega} \sigma(s) : E\dot{w}(s) dx ds \\ &+ \int_0^t \int_{\Omega} f(s) \cdot (\dot{u}(s) - \dot{w}(s)) dx ds + \int_0^t \int_{\Gamma_N} g(s) \cdot (\dot{u}(s) - \dot{w}(s)) d\mathcal{H}^{n-1} ds, \end{aligned}$$

or equivalently, according to the safe-load condition,

$$\begin{aligned} \mathcal{Q}(e(t)) + \int_0^t \mathcal{H}(\dot{p}(s)) ds - \int_0^t \langle \pi_D(s), \dot{p}(s) \rangle ds + \int_{\Omega} \pi(t) : (Ew(t) - e(t)) dx \\ = \mathcal{Q}(e_0) + \int_{\Omega} \pi(0) : (Ew(0) - e_0) dx + \int_0^t \int_{\Omega} \sigma(s) : E\dot{w}(s) dx ds \\ + \int_0^t \int_{\Omega} \dot{\pi}(s) : (Ew(s) - e(s)) dx ds. \end{aligned} \quad (2.6)$$

3. THE RIGID-PLASTIC MODEL

In order to derive the rigid-plastic model from elasto-plasticity, we assume that

$$\mathbb{C}^\varepsilon = \varepsilon^{-1} \mathbb{C}, \quad \text{where } \mathbb{C} \text{ satisfies (2.1) and (2.2),} \quad (3.1)$$

and $\varepsilon \rightarrow 0^+$. In addition, we suppose that the boundary data are compatible with rigid plasticity, that is

$$\operatorname{div} w(t) = 0 \text{ in } \Omega, \quad (3.2)$$

and, for simplicity, that the initial data satisfy

$$e_0 = \sigma_0 = 0. \quad (3.3)$$

Theorem 3.1. *Let u^ε , e^ε , p^ε and σ^ε be the solutions given by Theorem 2.2. There exist a subsequence (not relabeled), and functions $u \in AC([0, T]; BD(\Omega))$ and $\sigma \in L^2(0, T; L^2(\Omega; \mathbb{M}_{sym}^{n \times n}))$ such that*

$$\begin{aligned} u^\varepsilon(t) &\rightharpoonup u(t) \text{ weakly* in } BD(\Omega), \text{ for all } t \in [0, T], \\ \sigma^\varepsilon &\rightharpoonup \sigma \text{ weakly in } L^2(0, T; L^2(\Omega; \mathbb{M}_{sym}^{n \times n})). \end{aligned}$$

Denoting by $v := \dot{u} \in L_{w*}^\infty(0, T; BD(\Omega))$, then for a.e. $t \in [0, T]$, we have

$$\begin{cases} -\operatorname{div} \sigma(t) = f(t) \text{ in } \Omega, \\ \sigma(t)\nu = g(t) \text{ on } \Gamma_N, \\ \sigma(t) \in \mathcal{K}, \end{cases} \quad \begin{cases} \operatorname{div} v(t) = 0 \text{ in } \Omega, \\ (\dot{w}(t) - v(t)) \cdot \nu = 0 \text{ on } \Gamma_D, \\ H(Ev(t)) = [\sigma_D(t) : Ev(t)] \text{ in } \Omega \cup \Gamma_D. \end{cases} \quad (3.4)$$

The remaining of this paper is devoted to the proof of Theorem 3.1.

Remark 3.2. Although $Eu(t)$ is a measure *a priori* defined in Ω , we tacitly extend it by $(w(t) - u(t)) \odot \nu \mathcal{H}^{n-1} \llcorner \Gamma_D$ on Γ_D so that $Eu(t) \in \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})$.

Remark 3.3. In contrast with the framework of classical elasto-plasticity, that of rigid plasticity only involves the velocity field, and not the displacement field itself. As expressed above, time is merely a parameter, although the associated measurability properties of the various fields are obtained through the limit process $\varepsilon \searrow 0$ and would be difficult to obtain directly from the limit formulation.

3.1. A priori estimates. In this section all constants are independent of ε . We start with an estimate of the stress. Since $\sigma_D^\varepsilon(t) \in K$ in Ω , and K is bounded by (2.3), we first deduce that

$$\sup_{t \in [0, T]} \|\sigma_D^\varepsilon(t)\|_\infty \leq C. \quad (3.5)$$

The following result allows us to bound the hydrostatic stress.

Lemma 3.4. *There exists a bounded sequence $(c^\varepsilon)_{\varepsilon > 0}$ in $L^2(0, T)$ such that for each $\varepsilon > 0$,*

$$\int_0^T \left\| \frac{\operatorname{tr} \sigma^\varepsilon(t)}{n} + c^\varepsilon(t) \right\|_2^2 dt \leq C.$$

Proof. Since the mapping $t \mapsto \sigma^\varepsilon(t)$ belongs to $L^2(0, T; H(\operatorname{div}, \Omega))$, there is a sequence $(\sigma_k^\varepsilon)_{k \in \mathbb{N}}$ of $H(\operatorname{div}, \Omega)$ -valued simple functions such that $\sigma_k^\varepsilon \rightarrow \sigma^\varepsilon$ strongly in $L^2(0, T; H(\operatorname{div}, \Omega))$ as $k \rightarrow +\infty$. For all $k \in \mathbb{N}$ and all $t \in [0, T]$, we have

$$\nabla \left(\frac{\operatorname{tr} \sigma_k^\varepsilon(t)}{n} \right) = \operatorname{div} \sigma_k^\varepsilon(t) - \operatorname{div}(\sigma_k^\varepsilon)_D(t) \text{ in } \Omega$$

which leads to

$$\int_0^T \left\| \nabla \left(\frac{\operatorname{tr} \sigma_k^\varepsilon(t)}{n} \right) \right\|_{H^{-1}(\Omega; \mathbb{R}^n)}^2 dt \leq \int_0^T \|\operatorname{div} \sigma_k^\varepsilon(t)\|_{H^{-1}(\Omega; \mathbb{R}^n)}^2 dt + \int_0^T \|(\sigma_k^\varepsilon)_D(t)\|_2^2 dt.$$

Since $\operatorname{div} \sigma_k^\varepsilon \rightarrow \operatorname{div} \sigma^\varepsilon$ in $L^2(0, T; L^2(\Omega; \mathbb{R}^n))$ and $-\operatorname{div} \sigma^\varepsilon = f \in L^2(0, T; L^2(\Omega; \mathbb{R}^n))$, we deduce that the first integral in the right-hand-side of the previous inequality is uniformly bounded with respect to ε and k . The second integral is bounded as well since $(\sigma_k^\varepsilon)_D \rightarrow \sigma_D^\varepsilon$ in $L^2(0, T; L^2(\Omega; \mathbb{M}_D^{n \times n}))$, and $(\sigma_D^\varepsilon)_{\varepsilon > 0}$ is uniformly bounded in that space in view of (3.5). Consequently, there exists a constant $C > 0$ (independent of k and ε) such that

$$\int_0^T \left\| \nabla \left(\frac{\operatorname{tr} \sigma_k^\varepsilon(t)}{n} \right) \right\|_{H^{-1}(\Omega; \mathbb{R}^n)}^2 dt \leq C.$$

Next, according to [8, Corollary 2.1] (see also [13, Lemma 9] in the case of smooth boundaries), for each $\varepsilon > 0$, $k \in \mathbb{N}$ and $t \in [0, T]$, there exists some $c_k^\varepsilon(t) \in \mathbb{R}$ such that

$$\left\| \frac{\operatorname{tr} \sigma_k^\varepsilon(t)}{n} + c_k^\varepsilon(t) \right\|_2 \leq C_\Omega \left\| \nabla \left(\frac{\operatorname{tr} \sigma_k^\varepsilon(t)}{n} \right) \right\|_{H^{-1}(\Omega; \mathbb{R}^n)},$$

for some constant $C_\Omega > 0$ only depending on Ω . Note that, since the mapping $t \mapsto \operatorname{tr} \sigma_k^\varepsilon(t)$ is a simple $L^2(\Omega)$ -valued function, $t \mapsto c_k^\varepsilon(t)$ is a simple real-valued measurable function as well. Additionally,

$$\int_0^T \left\| \frac{\operatorname{tr} \sigma_k^\varepsilon(t)}{n} + c_k^\varepsilon(t) \right\|_2^2 dt \leq C, \quad (3.6)$$

where $C > 0$ is again independent of k and ε . Setting $\hat{\sigma}_k^\varepsilon := \sigma_k^\varepsilon + c_k^\varepsilon I$ yields

$$\int_0^T \|\hat{\sigma}_k^\varepsilon(t)\|_{H(\operatorname{div}, \Omega)}^2 dt \leq C,$$

and thus,

$$\int_0^T \|\hat{\sigma}_k^\varepsilon(t)\nu\|_{H^{-1/2}(\Gamma_N; \mathbb{R}^n)}^2 dt \leq C.$$

Using that $\sigma_k^\varepsilon \nu \rightarrow \sigma^\varepsilon \nu = g$ in $L^2(0, T; H^{-1/2}(\Gamma_N; \mathbb{R}^n))$ and that $g \in L^2(0, T; L^2(\Gamma_N; \mathbb{R}^n))$, we obtain

$$\begin{aligned} & \int_0^T |c_k^\varepsilon(t)|^2 dt \|\nu\|_{H^{-1/2}(\Gamma_N; \mathbb{R}^n)}^2 \\ & \leq \int_0^T \|\hat{\sigma}_k^\varepsilon(t)\nu\|_{H^{-1/2}(\Gamma_N; \mathbb{R}^n)}^2 dt + \int_0^T \|\sigma_k^\varepsilon(t)\nu\|_{H^{-1/2}(\Gamma_N; \mathbb{R}^n)}^2 dt \leq C, \end{aligned} \quad (3.7)$$

for some constant $C > 0$, independent of k and ε . Therefore, the sequence $(c_k^\varepsilon)_{k \in \mathbb{N}}$ is bounded in $L^2(0, T)$ and a subsequence converges weakly in that space to some $c^\varepsilon \in L^2(0, T)$. Passing to the lower limit in (3.6) implies that

$$\int_0^T \left\| \frac{\operatorname{tr} \sigma^\varepsilon(t)}{n} + c^\varepsilon(t) \right\|_2^2 dt \leq C,$$

while (3.7) shows that $(c^\varepsilon)_{\varepsilon > 0}$ is bounded in $L^2(0, T)$. \square

As a consequence of the previous result and of (3.5), we deduce that

$$\int_0^T \|\sigma^\varepsilon(t)\|_2^2 dt \leq C. \quad (3.8)$$

Next, according to the energy balance (2.6), [4, Lemma 3.2], assumptions (3.2)–(3.3), and Cauchy-Schwarz inequality, we infer that

$$\begin{aligned} \frac{1}{2} \int_\Omega \mathbb{C}^\varepsilon e^\varepsilon(t) : e^\varepsilon(t) dx &\leq \int_\Omega \pi(t) : (e^\varepsilon(t) - Ew(t)) dx + \int_\Omega \pi(0) : Ew(0) dx \\ &\quad + \int_0^t \int_\Omega \sigma_D^\varepsilon(s) : E\dot{w}(s) dx ds + \int_0^t \int_\Omega \dot{\pi}(s) : (Ew(s) - e^\varepsilon(s)) dx ds \\ &\leq C \left(\sup_{t \in [0, T]} \|\pi(t)\|_2 + \int_0^T \|\dot{\pi}(s)\|_2 ds \right) \left(\sup_{t \in [0, T]} \|e^\varepsilon(t)\|_2 + \sup_{t \in [0, T]} \|Ew(t)\|_2 \right) \\ &\quad + \sup_{t \in [0, T]} \|\sigma_D^\varepsilon(t)\|_\infty \int_0^T \|E\dot{w}(s)\|_2 ds, \end{aligned}$$

which implies, according to the assumption (3.1) on \mathbb{C}^ε , (3.5) and Young's inequality, that

$$\sup_{t \in [0, T]} \|e^\varepsilon(t)\|_2 \leq C\sqrt{\varepsilon}. \quad (3.9)$$

Using again the energy balance (2.6), Cauchy-Schwarz inequality, (3.5) and (3.9), we find that

$$\begin{aligned} \int_0^t \mathcal{H}(\dot{p}^\varepsilon(s)) ds - \int_0^t \langle \pi_D(s), \dot{p}^\varepsilon(s) \rangle ds &\leq \int_\Omega \pi(t) : (e^\varepsilon(t) - Ew(t)) dx + \int_\Omega \pi(0) : Ew(0) dx \\ &\quad + \int_0^t \int_\Omega \sigma_D^\varepsilon(s) : E\dot{w}(s) dx ds + \int_0^t \int_\Omega \dot{\pi}(s) : (Ew(s) - e^\varepsilon(s)) dx ds \leq C. \end{aligned}$$

Applying [4, Lemma 3.2] again yields

$$\int_0^T \|\dot{p}^\varepsilon(s)\|_{\mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})} ds \leq C, \quad (3.10)$$

and thus

$$\sup_{t \in [0, T]} \|p^\varepsilon(t)\|_{\mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})} \leq C. \quad (3.11)$$

For the displacement, Poincaré-Korn's inequality (see [14, Chap. 2, Rmk. 2.5(ii)]) yields

$$\begin{aligned} \|u^\varepsilon(t)\|_{BD(\Omega)} &\leq c \left(\int_{\Gamma_D} |u^\varepsilon(t)| d\mathcal{H}^{n-1} + \|Eu^\varepsilon(t)\|_{\mathcal{M}(\Omega; \mathbb{M}_{sym}^{n \times n})} \right) \\ &\leq c \left(\int_{\Gamma_D} |w(t)| d\mathcal{H}^{n-1} + \int_{\Gamma_D} |u^\varepsilon(t) - w(t)| d\mathcal{H}^{n-1} + \|Eu^\varepsilon(t)\|_{\mathcal{M}(\Omega; \mathbb{M}_{sym}^{n \times n})} \right) \\ &\leq c \left(\|w(t)\|_{L^1(\Gamma_D; \mathbb{R}^n)} + \|p^\varepsilon(t)\|_{\mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})} + \|e^\varepsilon(t)\|_2 \right) \leq C, \end{aligned} \quad (3.12)$$

where we have used (3.9) and (3.11) in the last inequality.

3.2. Convergences. According to the stress estimate (3.8), there exist a subsequence (not relabeled) and $\sigma \in L^2(0, T; L^2(\Omega; \mathbb{M}_{sym}^{n \times n}))$ such that

$$\sigma^\varepsilon \rightharpoonup \sigma \text{ weakly in } L^2(0, T; L^2(\Omega; \mathbb{M}_{sym}^{n \times n})). \quad (3.13)$$

Consequently, since for all $t \in [0, T]$, we have $-\operatorname{div} \sigma^\varepsilon(t) = f(t)$ in Ω and $\sigma^\varepsilon(t)\nu = g(t)$ on Γ_N , we infer that for a.e. $t \in [0, T]$,

$$-\operatorname{div} \sigma(t) = f(t) \text{ in } \Omega, \quad \sigma(t)\nu = g(t) \text{ on } \Gamma_N.$$

In addition, since $\sigma_D^\varepsilon(t) \in \mathcal{K}$ for all $t \in [0, T]$, then

$$\sigma_D(t) \in \mathcal{K} \text{ for a.e. } t \in [0, T].$$

We then apply Helly's selection principle (see [11, Theorem 3.2]) which ensures, thanks to (3.10), the existence of a further subsequence (independent of time and still not relabeled) such that

$$p^\varepsilon(t) \rightharpoonup p(t) \text{ weakly}^* \text{ in } \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n}), \text{ for all } t \in [0, T], \quad (3.14)$$

for some $p \in BV([0, T]; \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n}))$.

Next according to (3.9), we have that

$$e^\varepsilon \rightarrow 0 \text{ strongly in } L^\infty(0, T; L^2(\Omega; \mathbb{M}_{sym}^{n \times n})). \quad (3.15)$$

Finally, as a consequence of the displacement estimate (3.12), for each $t \in [0, T]$, there exists a further subsequence $(u^{\varepsilon_j}(t))_{j \in \mathbb{N}}$ (now possibly depending on t) such that $u^{\varepsilon_j}(t) \rightharpoonup u(t)$ weakly* in $BD(\Omega)$, for some $u(t) \in BD(\Omega)$. Note that by (3.14)–(3.15), for a.e. $t \in [0, T]$, one has $Eu(t) = p(t)$ in Ω and, by [4, Lemma 2.1], $p(t) = (w(t) - u(t)) \odot \nu \mathcal{H}^{n-1} \llcorner \Gamma_D$ on Γ_D which shows that $u(t)$ is uniquely determined, and thus that the full sequence

$$u^\varepsilon(t) \rightharpoonup u(t) \text{ weakly}^* \text{ in } BD(\Omega), \text{ for all } t \in [0, T]. \quad (3.16)$$

In particular, since $Eu(t) = p(t) \in \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})$, we also deduce that

$$\operatorname{div} u(t) = 0 \text{ in } \Omega, \quad (w(t) - u(t)) \cdot \nu = 0 \text{ on } \Gamma_D. \quad (3.17)$$

3.3. Flow rule. According to the energy balance (2.6) and the fact that the plastic strain $p^\varepsilon \in AC([0, T]; \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n}))$, we can integrate by parts in time, so that for all $t \in [0, T]$,

$$\begin{aligned} \mathcal{V}_{\mathcal{H}}(p^\varepsilon; [0, t]) + \int_{\Omega} \pi(t) : (Ew(t) - e^\varepsilon(t)) dx - \langle \pi_D(t), p^\varepsilon(t) \rangle \\ \leq \int_{\Omega} \pi(0) : Ew(0) dx - \langle \pi_D(0), p_0 \rangle + \int_0^t \int_{\Omega} \sigma_D^\varepsilon(s) : E\dot{w}(s) dx ds \\ + \int_0^t \int_{\Omega} \dot{\pi}(s) : (Ew(s) - e^\varepsilon(s)) dx ds - \int_0^t \langle \dot{\pi}_D(s), p^\varepsilon(s) \rangle ds. \end{aligned}$$

Since by (3.14)–(3.16) $p^\varepsilon(t) \rightharpoonup Eu(t)$ weakly* in $\mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})$ for a.e. $t \in [0, T]$, Reshetnyak lower semicontinuity theorem, (3.13), (3.15), (3.16) and the definition (2.4) of duality ensures that

$$\begin{aligned} \mathcal{V}_{\mathcal{H}}(Eu; [0, t]) + \int_{\Omega} \pi(t) : Ew(t) dx - \langle \pi_D(t), Eu(t) \rangle \\ \leq \int_{\Omega} \pi(0) : Ew(0) dx - \langle \pi_D(0), Eu_0 \rangle + \int_0^t \int_{\Omega} \sigma_D(s) : E\dot{w}(s) dx ds \\ + \int_0^t \int_{\Omega} \dot{\pi}(s) : Ew(s) dx ds - \int_0^t \langle \dot{\pi}_D(s), Eu(s) \rangle ds. \quad (3.18) \end{aligned}$$

We now show the converse inequality. Since $\sigma_D \in L^1(0, T; L^2(\Omega; \mathbb{M}_D^{n \times n}))$, while $u - w \in L^1(0, T; L^{\frac{n}{n-1}}(\Omega; \mathbb{R}^n))$, and $u - w \in L^1(0, T; L^1(\Gamma_N; \mathbb{R}^n))$, [5, Lemma 7.5] implies the existence of a subdivision $0 = t_0 < t_1 < \dots < t_k = t$ of the time interval $[0, t]$ such that

$$\sum_{i=1}^k \chi_{[t_{i-1}, t_i]}(\sigma_D(t_i), u(t_i) - w(t_i), u(t_i) - w(t_i)) \rightarrow (\sigma_D, u - w, u - w)$$

and

$$\sum_{i=1}^k \chi_{[t_{i-1}, t_i]}(\sigma_D(t_{i-1}), u(t_{i-1}) - w(t_{i-1}), u(t_{i-1}) - w(t_{i-1})) \rightarrow (\sigma_D, u - w, u - w)$$

strongly in $L^1(0, T; L^2(\Omega; \mathbb{M}_D^{n \times n})) \times L^1(0, T; L^{\frac{n}{n-1}}(\Omega; \mathbb{R}^n)) \times L^1(0, T; L^1(\Gamma_N; \mathbb{R}^n))$, as $\max_{1 \leq i \leq k} (t_i - t_{i-1}) \rightarrow 0$. According to Proposition 3.9 in [7] and to the fact that Ω is admissible, we infer that for each $1 \leq i \leq k$,

$$\begin{aligned} \mathcal{H}(Eu(t_i) - Eu(t_{i-1})) &\geq \langle \sigma_D(t_i), Eu(t_i) - Eu(t_{i-1}) \rangle \\ &= \int_{\Omega} \sigma_D(t_i) : (Ew(t_i) - Ew(t_{i-1})) dx + \int_{\Omega} f(t_i) \cdot (u(t_i) - u(t_{i-1}) - w(t_i) + w(t_{i-1})) dx \\ &\quad + \int_{\Gamma_N} g(t_i) \cdot (u(t_i) - u(t_{i-1}) - w(t_i) + w(t_{i-1})) d\mathcal{H}^{n-1}. \end{aligned}$$

Summing up for $i = 1, \dots, k$, and performing discrete integration by parts yields

$$\begin{aligned} \mathcal{V}_{\mathcal{H}}(Eu, [0, t]) &\geq \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \int_{\Omega} \sigma_D(t_i) : E\dot{w}(s) dx ds \\ &\quad - \sum_{i=1}^{k-1} \int_{t_i}^{t_{i+1}} \int_{\Omega} \dot{f}(s) \cdot (u(t_i) - w(t_i)) dx ds - \sum_{i=1}^{k-1} \int_{t_i}^{t_{i+1}} \int_{\Gamma_N} \dot{g}(s) \cdot (u(t_i) - w(t_i)) d\mathcal{H}^{n-1} ds \\ &\quad + \int_{\Omega} f(t) \cdot (u(t) - w(t)) dx + \int_{\Gamma_N} g(t) \cdot (u(t) - w(t)) d\mathcal{H}^{n-1} \\ &\quad - \int_{\Omega} f(t_1) \cdot (u_0 - w(0)) dx - \int_{\Gamma_N} g(t_1) \cdot (u_0 - w(0)) d\mathcal{H}^{n-1}. \end{aligned}$$

Passing to the limit as $\max_{1 \leq i \leq k} (t_i - t_{i-1}) \rightarrow 0$, and invoking the dominated convergence theorem yields

$$\begin{aligned} \mathcal{V}_{\mathcal{H}}(Eu, [0, t]) &\geq \int_0^t \int_{\Omega} \sigma_D(s) : E\dot{w}(s) dx ds \\ &\quad - \int_0^t \int_{\Omega} \dot{f}(s) \cdot (u(s) - w(s)) dx ds - \int_0^t \int_{\Gamma_N} \dot{g}(s) \cdot (u(s) - w(s)) d\mathcal{H}^{n-1} ds \\ &\quad + \int_{\Omega} f(t) \cdot (u(t) - w(t)) dx + \int_{\Gamma_N} g(t) \cdot (u(t) - w(t)) d\mathcal{H}^{n-1} \\ &\quad - \int_{\Omega} f(0) \cdot (u_0 - w(0)) dx - \int_{\Gamma_N} g(0) \cdot (u_0 - w(0)) d\mathcal{H}^{n-1}, \end{aligned}$$

and using the definition (2.4) of duality

$$\begin{aligned} \mathcal{V}_{\mathcal{H}}(Eu; [0, t]) &+ \int_{\Omega} \pi(t) : Ew(t) dx - \langle \pi_D(t), Eu(t) \rangle \\ &\geq \int_{\Omega} \pi(0) : Ew(0) dx - \langle \pi_D(0), Eu_0 \rangle + \int_0^t \int_{\Omega} \sigma_D(s) : E\dot{w}(s) dx ds \\ &\quad + \int_0^t \int_{\Omega} \dot{\pi}(s) : Ew(s) dx ds - \int_0^t \langle \dot{\pi}_D(s), Eu(s) \rangle ds. \end{aligned}$$

Thus, combining with (3.18) leads to the equality in the previous inequality, or still, integrating by parts with respect to time

$$\begin{aligned} \mathcal{V}_{\mathcal{H}}(Eu; [0, t]) &= \langle \pi_D(t), Eu(t) \rangle - \langle \pi_D(0), Eu_0 \rangle \\ &\quad + \int_0^t \int_{\Omega} (\sigma_D(s) - \pi_D(s)) : E\dot{w}(s) dx ds - \int_0^t \langle \dot{\pi}_D(s), Eu(s) \rangle ds. \end{aligned} \quad (3.19)$$

According to [4, Lemma 3.2], for all $0 \leq t_1 \leq t_2 \leq T$,

$$\begin{aligned} c \|Eu(t_2) - Eu(t_1)\|_{\mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})} &\leq \mathcal{H}(Eu(t_2) - Eu(t_1)) - \langle \pi_D(t_2), Eu(t_2) - Eu(t_1) \rangle \\ &\leq \mathcal{V}_{\mathcal{H}}(Eu, [t_1, t_2]) - \langle \pi_D(t_2), Eu(t_2) - Eu(t_1) \rangle. \end{aligned}$$

In view of (3.19), we get that

$$\begin{aligned} c \|Eu(t_2) - Eu(t_1)\|_{\mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})} &\leq \langle \pi_D(t_2) - \pi_D(t_1), Eu(t_1) \rangle \\ &\quad + \int_{t_1}^{t_2} \int_{\Omega} (\sigma_D(s) - \pi_D(s)) : E\dot{w}(s) dx ds - \int_{t_1}^{t_2} \langle \dot{\pi}_D(s), Eu(s) \rangle ds. \end{aligned}$$

Since $Eu = p$ and $p \in BV([0, T]; \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n}))$, we get that $Eu \in L^\infty(0, T; \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n}))$, and thus

$$\begin{aligned} c \|Eu(t_2) - Eu(t_1)\|_{\mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})} &\leq \int_{t_1}^{t_2} \left\{ \|Eu(t_1)\|_{\mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})} \|\dot{\pi}_D(s)\|_\infty \right. \\ &\quad \left. + (\|\pi_D(s)\|_2 + \|\sigma_D(s)\|_2) \|E\dot{w}(s)\|_2 + \|\dot{\pi}_D(s)\|_\infty \|Eu(s)\|_{\mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})} \right\} ds. \end{aligned}$$

The integrand being sommable, it ensures that the strain $Eu \in AC([0, T]; \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n}))$, and by the Poincaré-Korn inequality that $u \in AC([0, T]; BD(\Omega))$. Thus, integrating by part with respect to time and space in the energy equality (3.19),

$$\begin{aligned} \int_0^t \mathcal{H}(E\dot{u}(s)) ds &= \mathcal{V}_{\mathcal{H}}(Eu, [0, t]) = \int_0^t \int_{\Omega} \sigma_D(s) : E\dot{w}(s) dx ds \\ &\quad + \int_0^t \int_{\Omega} f(s) \cdot (\dot{u}(s) - \dot{w}(s)) dx ds + \int_0^t \int_{\Gamma_N} g(s) \cdot (\dot{u}(s) - \dot{w}(s)) d\mathcal{H}^{n-1} ds, \end{aligned}$$

and deriving this equality with respect to time yields, thanks to (2.4), for a.e. $t \in [0, T]$,

$$\mathcal{H}(E\dot{u}(t)) = \langle \sigma_D(t), E\dot{u}(t) \rangle.$$

Since, by [7, Proposition 3.9], $H(E\dot{u}(t)) \geq [\sigma_D(t) : E\dot{u}(t)]$ in $\mathcal{M}(\Omega \cup \Gamma_D)$, we finally deduce that $H(E\dot{u}(t)) = [\sigma_D(t) : E\dot{u}(t)]$ in $\mathcal{M}(\Omega \cup \Gamma_D)$.

Denoting by $v = \dot{u}$ the velocity, we proved that $v \in L_{w*}^\infty(0, T; BD(\Omega))$, and recalling (3.17), we have for a.e. $t \in [0, T]$,

$$\operatorname{div} v(t) = 0 \text{ in } \Omega, \quad (\dot{w}(t) - v(t)) \cdot \nu = 0 \text{ on } \Gamma_D,$$

and

$$H(Ev(t)) = [\sigma_D(t) : Ev(t)] \text{ in } \Omega \cup \Gamma_D.$$

4. UNIQUENESS AND REGULARITY ISSUES FOR THE STRESS WITH A VON MISES YIELD CRITERION

We now specialize to the case where $K := \{\tau_D \in \mathbb{M}_D^{n \times n} : |\tau_D| \leq 1\}$. In such a setting, it is known (see [3]) when elasto-plasticity is considered the stress field is unique and belongs to $H_{\text{loc}}^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$. These properties fail in the case of rigid-plasticity as demonstrated below.

Example 4.1. Let us consider a two-dimensional body occupying the square $\Omega = (0, 1)^2$ in its reference configuration (the generalization to the n -dimensional case is obvious). We also assume that the boundary conditions are of pure Dirichlet type with a rigid body motion $\dot{w}(x) = Ax + b$ (where $A \in \mathbb{M}^{n \times n}$ is such that $A^T = -A^T$, and $b \in \mathbb{R}^n$) as boundary datum.

Then, defining $v(x) = Ax + b$ for all $x \in \Omega$ ensures that $Ev = 0$ in Ω . In particular, all equations on v are satisfied. Now define the stress as

$$\sigma(x) = \begin{pmatrix} f(x_2) & c \\ c & g(x_1) \end{pmatrix}$$

where $c \in \mathbb{R}$, $f, g \in L^\infty(0, 1)$ so that $\text{div } \sigma = 0$ in Ω . In particular

$$\sigma_D(x) = \begin{pmatrix} \frac{f(x_2) - g(x_1)}{2} & c \\ c & \frac{g(x_1) - f(x_2)}{2} \end{pmatrix}$$

and $|\sigma_D(x)|^2 \leq 2c^2 + |f(x_2)|^2 + |g(x_1)|^2$ for a.e. $x \in \Omega$. Assuming that $\sqrt{2c^2 + \|f\|_\infty^2 + \|g\|_\infty^2} < 1/2$, we deduce that the one parameter family $\sigma^\lambda := \lambda\sigma$ still satisfies $\text{div } \sigma^\lambda = 0$ and $|\sigma_D^\lambda| < 1$ in Ω provided that $|\lambda| \leq 2$.

In general, a certain amount of uniqueness holds true as shown below. It uses a notion of precise representative for the stress field first introduced in [1] (see also [4]).

Proposition 4.2. *Let $(\sigma^1, v^1), (\sigma^2, v^2) \in L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}) \times BD(\Omega)$ be two solutions of the rigid-plastic model (3.4) at a given time $t = t_0$. Then,*

- *There exist two $|Ev^1|$ -measurable functions $\hat{\sigma}_D^1$ and $\hat{\sigma}_D^2 \in L_{|Ev^1|}^\infty(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})$ such that $\hat{\sigma}_D^1 = \sigma^1$ and $\hat{\sigma}_D^2 = \sigma^2$ \mathcal{L}^n -a.e. in $\Omega \cup \Gamma_D$, and*

$$\hat{\sigma}_D^1 = \hat{\sigma}_D^2 \quad |Ev^1| \text{-a.e. in } \Omega \cup \Gamma_D;$$

- *There exist two $|Ev^2|$ -measurable functions $\tilde{\sigma}_D^1$ and $\tilde{\sigma}_D^2 \in L_{|Ev^2|}^\infty(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})$ such that $\tilde{\sigma}_D^1 = \sigma^1$ and $\tilde{\sigma}_D^2 = \sigma^2$ \mathcal{L}^n -a.e. in $\Omega \cup \Gamma_D$, and*

$$\tilde{\sigma}_D^1 = \tilde{\sigma}_D^2 \quad |Ev^2| \text{-a.e. in } \Omega \cup \Gamma_D.$$

Proof. Since $(\sigma^1, v^1), (\sigma^2, v^2)$ are two solutions of the rigid-plastic model (3.4), the following inequalities in $\mathcal{M}(\Omega \cup \Gamma_D)$ hold true

$$[\sigma_D^1 : Ev^1] = |Ev^1| \geq [\sigma_D^2 : Ev^1], \quad [\sigma_D^2 : Ev^2] = |Ev^2| \geq [\sigma_D^1 : Ev^2].$$

As a consequence,

$$[(\sigma_D^1 - \sigma_D^2) : Ev^1] \geq 0, \quad [(\sigma_D^2 - \sigma_D^1) : Ev^2] \geq 0,$$

and thus,

$$[(\sigma_D^1 - \sigma_D^2) : (Ev^1 - Ev^2)] \geq 0.$$

In addition, by definition (2.4) of the duality pairing, the total mass of the measure on the left-hand side of the previous inequality is given by

$$\langle \sigma_D^1 - \sigma_D^2, Ev^1 - Ev^2 \rangle = 0.$$

It thus follows that

$$[(\sigma_D^1 - \sigma_D^2) : Ev^1] = 0, \quad [(\sigma_D^2 - \sigma_D^1) : Ev^2] = 0,$$

or still that

$$[\sigma_D^1 : Ev^1] = |Ev^1| = [\sigma_D^2 : Ev^1], \quad [\sigma_D^2 : Ev^2] = |Ev^2| = [\sigma_D^1 : Ev^2]. \quad (4.1)$$

Arguing as in [4], since \mathcal{L}^n and $E^s v^1$ are mutually singular Borel measures, it is possible to find two disjoint Borel sets A and $B \subset \Omega \cup \Gamma_D$ such that $A \cup B = \Omega \cup \Gamma_D$, and $\mathcal{L}^n(B) = |E^s v^1|(A) = 0$. Then, defining (for $i = 1, 2$)

$$\hat{\sigma}_D^i := \begin{cases} \sigma_D^i & \mathcal{L}^n\text{-a.e. in } A, \\ \frac{dEv^1}{d|Ev^1|} & |E^s v^1|\text{-a.e. in } B, \end{cases}$$

it follows that $\hat{\sigma}_D^1$ and $\hat{\sigma}_D^2 \in L^\infty_{|Ev^1|}(\Omega \cup \Gamma_D; \mathbb{M}_D^{n \times n})$, and

$$\hat{\sigma}_D^1 : \frac{dEv^1}{d|Ev^1|} |Ev^1| = [\sigma_D^1 : Ev^1] = |Ev^1| = [\sigma_D^2 : Ev^1] = \hat{\sigma}_D^2 : \frac{dEv^1}{d|Ev^1|} |Ev^1|.$$

By definition, we have that $\hat{\sigma}_D^1 = \hat{\sigma}_D^2 |E^s v^1|\text{-a.e. in } \Omega \cup \Gamma_D$. In addition, taking the absolutely continuous part in (4.1) yields (see [4, 7]),

$$\sigma_D^1 : E^a v^1 = [\sigma_D^1 : Ev^1]^a = |E^a v^1| = [\sigma_D^2 : Ev^1]^a = \sigma_D^2 : E^a v^1.$$

Thus $\sigma_D^1 = \sigma_D^2 \mathcal{L}^n\text{-a.e. in } \{|E^a v^1| > 0\}$ and finally $\hat{\sigma}_D^1 = \hat{\sigma}_D^2 |Ev^1|\text{-a.e. in } \Omega \cup \Gamma_D$ as requested. \square

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