EXISTENCE OF MINIMIZERS FOR NON-QUASICONVEX FUNCTIONALS ARISING IN OPTIMAL DESIGN

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Abstract

This paper investigates the existence of minimizers for the so-called Kohn-Strang functional with affine boundary conditions. Such a functional, which arises in optimal shape design problems in electrostatics, is not quasi-convex, and therefore existence of minimizers is, in general, guaranteed only for its quasi-convex envelope. Such a quasi-convexification has been computed in two space dimensions in [11]. Recently, necessary and sufficient conditions on the affine boundary conditions for existence of minimizers for the Kohn-Strang functional have been derived in two space dimensions in [7]. We generalize these previous results for arbitrary space dimensions. Our method relies on the homogenization approach for relaxing optimal design problems. We also generalize our results to some variants of the Kohn-Strang functional.

Key words: homogenization, quasiconvexity, rank-one convexity, calculus of variations, relaxation, optimal design.

1 Introduction

Let Ω be a bounded domain of $I\!R^n$. Let $u(x)$ be a vector-valued function from Ω into $I\!R^N$ with derivatives denoted by $Du = (\partial u_i/\partial x_j) \in I\!R^{nN}$. Let ξ be a constant matrix in $I\!R^{nN}$, *i.e.*, ξ has N lines and n columns. Let D_ξ denote the space

$$D_\xi = \left\{ \xi \cdot x + H_0^1(\Omega; I\!R^N) \right\}.$$

This paper is devoted to the question of existence of minimizers in D_ξ for the following functional

$$F(u) = \int_\Omega f\,(Du)\,dx, \tag{1}$$

where the integrand f is a function from $\mathbb{R}^{nN}$ into $\mathbb{R}^{+}$, defined by

$$f(\eta)=\begin{cases}\lambda+\alpha|\eta|^2, & \eta\neq 0,\\ 0, & \eta=0,\end{cases} \tag{2}$$

whith $0<\alpha,\ \lambda<+\infty$. In the case $n=2$, the function f under consideration was introduced by Kohn and Strang in [11] as a model problem in the field of optimal design. Specifically, the associated minimization problem is equivalent to a shape optimization problem in electrostatics.

It is by now well-known that the functional F is not (sequentially) weakly lower semi-continuous on D_ξ. Therefore, the direct method of the calculus of variations does not yield the existence of minimizers for (1) in D_ξ. Rather, one needs to introduce the relaxed functional (see [6])

$$\bar{F}(u)=\int_\Omega Qf\,(Du)\,dx, \tag{3}$$

where Qf is the quasiconvex envelope of f defined by

$$Qf(\eta)=\inf_{\varphi\in H_0^1(Y;\mathbb{R}^N)}\int_Y f(\eta+D\varphi)dy,$$

where $Y=(0,1)^n$ is the unit cube of $\mathbb{R}^n$. Then, $u_0(x)=\xi\cdot x$ is a minimizer of the relaxed functional $\bar{F}$ on D_ξ, and

$$Qf(\xi)=\inf_{u\in D_\xi}\frac{1}{|\Omega|}\int_\Omega f\,(Du)\,dx.$$

When $n=2$, the quasiconvexification Qf has been computed in [11]. The result is

$$Qf(\eta)=\begin{cases}\lambda+\alpha|\eta|^2 & \text{if } |\eta|^2+2|adj_2\eta|\geq\frac{\lambda}{\alpha},\\ 2\sqrt{\alpha\lambda}\left(|\eta|^2+2|adj_2\eta|\right)^{1/2}-2\alpha|adj_2\eta| & \text{otherwise,}\end{cases}$$

where $adj_2\eta$ is the $\frac{N(N-1)}{2}$ vector of the 2×2 minors of $\eta\in\mathbb{R}^{2N}$.

In a recent paper [7] Dacorogna and Marcellini addressed the question of finding conditions for existence or non-existence of minimizers in D_ξ of functionals of the type (1) for a general non-quasiconvex integrand f. As an example, they considered the Kohn-Strang energy, defined in (2), when $n=2$, and derived the following

Theorem 1.1 *Let ξ belong to $\mathbb{R}^{2N}$. A necessary and sufficient condition for (1) to have a minimizer over D_ξ is that, either $f(\xi)=Qf(\xi)$, or rank $\xi=2$.*

The main results of the present paper are generalizations of the computation of the quasi-convexification Qf and of the above theorem to the case $n>2$. For arbitrary n, denoting by $\eta_1, ..., \eta_n$ the square roots of the eigenvalues of $\eta^t \eta$, we prove that (see Theorem 2.2)

$$Qf(\eta) = \begin{cases} \alpha|\eta|^2 + \lambda & \text{if } \sum_{i=1}^n \eta_i \geq \sqrt{\frac{\lambda}{\alpha}}, \\ \alpha|\eta|^2 - \alpha\left(\sum_{i=1}^n \eta_i\right)^2 + 2\sqrt{\lambda\alpha}\sum_{i=1}^n \eta_i & \text{if } \sum_{i=1}^n \eta_i < \sqrt{\frac{\lambda}{\alpha}}. \end{cases}$$

Furthermore, for arbitrary n, we also prove (see Theorem 2.3)

Theorem 1.2 *Let ξ belong to $\mathbb{R}^{nN}$. A sufficient condition for (1) to have a minimizer over D_ξ is that, either $f(\xi) = Qf(\xi)$, or rank $\xi = n$. While a sufficient condition for (1) to have no minimizer over D_ξ is that $f(\xi) > Qf(\xi)$ and rank $\xi = 1$.*

Remark that our theorem does not furnish a necessary and sufficient condition for existence of minimizers, since it does not cover the case $f(\xi) > Qf(\xi)$ and $2 \leq \text{rank } \xi \leq n-1$. However, in such a case we conjecture there are no minimizers for (1) over D_ξ. To support our claim, we prove that in such a case there are no smooth-type minimizers of (1) in D_ξ (see Proposition 2.5).

The existence of possible minimizers for (1) is not merely a question of purely theoretical interest. It also has important consequences in the context of optimal shape design. Let us briefly explore the connection between the Kohn-Strang energy, defined in (2), and optimal shape design (see Section 4 in [11] for more details). For each measurable subset ω of Ω, define

$$E(\omega, \xi) = \inf_{v \in D_{\xi,\omega}} \left\{ \int_\omega \left(\alpha|Dv(x)|^2 + \lambda\right) dx \right\},$$

where $D_{\xi,\omega}$ is the space defined by

$$D_{\xi,\omega} = \{v \in D_\xi \text{ s.t. } Dv(x) = 0 \text{ a.e. in } \Omega \setminus \omega\}.$$

Of course, any function $v \in D_{\xi,\omega}$ satisfies

$$\int_\Omega f(Dv)\, dx \leq \int_\omega \left(\alpha|Dv(x)|^2 + \lambda\right) dx,$$

but it is also true that

$$E(\Omega_v, \xi) \leq \int_\Omega f(Dv)\, dx,$$

where the measurable set $\Omega_v \subset \Omega$ is given by

$$\Omega_v = \{x \in \Omega \text{ s.t. } Dv(x) \neq 0\}.$$

Therefore, we deduce

$$\inf_{\omega \subset \Omega} E(\omega, \xi) = \inf_{u \in D_\xi} \int_\Omega f(Du)\, dx. \tag{4}$$

Whenever the right hand side of (4) admits a minimizer u, the corresponding set Ω_u minimizes the left hand side of (4). The minimization in the left hand side of (4) is a shape optimization problem in electrostatics : find the best arrangement of conductor α and holes so as to minimize the stored electrical energy. Since the seminal counter-examples of Murat [14] and Tartar [16], the generic non-existence of such optimal shapes is well-known. Rather, the problem is relaxed through the introduction, as admissible designs, of composite materials that mimic the behavior of minimizing sequences of shapes. Nevertheless, there could exist boundary conditions, corresponding to a special choice of ξ, for which an optimal shape is feasible. Our results provide conditions on ξ that permit to assert or to rule out the existence of such an optimal shape.

As a final remark, we emphasize that our method is very specific to the type of functionals considered here, *i.e.*, to the so-called Kohn-Strang energy and its various generalizations. Indeed, our key argument is the link between these functionals and the homogenization theory for two-phase composite materials. Of course, there are many other non-quasiconvex functionals for which existence of minimizers has been investigated : we refer to the recent article of Dacorogna and Marcellini [7] and references therein.

The outline of this paper is as follows. Section 2 is dedicated to the computation of $Qf(\eta)$ and to the proof of Theorem 1.2. Section 3 investigates the "dual" problem to (1)-(2), *i.e.*, a functional acting on divergence-free fields. Our motivation in the analysis of this dual problem is twofold : firstly, the conditions for existence of minimizers are quite different, and secondly, it is this type of dual problem, and not (1)-(2) which arises in the context of optimal shape design (see [11] or [1] for details). Finally, section 4 deals with a partial generalization of Theorem 1.2 to the case of non-quadratic Kohn-Strang type functionals.

2 Existence of minimizers for the Kohn-Strang functional

This section is devoted to an analysis of possible minimizers for the functional

$$\int_\Omega f\,(Du)\,dx \tag{5}$$

where Ω is a bounded domain of $\mathbb{R}^n$, and u is affine on the boundary of Ω, *i.e.*,

$$u \in D_\xi = \left\{\xi \cdot x + H_0^1(\Omega;\mathbb{R}^N)\right\}, \xi \in \mathbb{R}^{nN}. \tag{6}$$

The specific function f under consideration was introduced by Kohn and Strang in [11] as a model problem in the field of optimal design ; specifically, for $\eta \in \mathbb{R}^{nN}$,

$$f(\eta) = \begin{cases} \lambda + \alpha|\eta|^2, & \eta \neq 0, \\ 0, & \eta = 0, \end{cases} \tag{7}$$

where $0 < \alpha,\ \lambda < +\infty$.

Only the case $n = 2$ is investigated in [11]. In the two-dimensional setting it is shown in Section 4 of [11] that the minimization problem stems from a shape optimization problem in electrostatics.

In any case the functional defined in (5) is not (sequentially) weakly lower semi-continuous over $H^1(\Omega; \mathbb{R}^N)$ so that minimizers for (5) over D_ξ defined in (6) need not exist. It was shown in [11], Theorem 1.1, that, when $n = 2$, the quasiconvexification of f is

$$Qf(\eta) = \begin{cases} \lambda + \alpha|\eta|^2 & \text{if } |\eta|^2 + 2|adj_2\eta| \geq \frac{\lambda}{\alpha}, \\ 2\sqrt{\alpha\lambda}\left(|\eta|^2 + 2|adj_2\eta|\right)^{1/2} - 2\alpha|adj_2\eta| & \text{otherwise,} \end{cases} \tag{8}$$

where $adj_2\eta$ is the $\frac{N(N-1)}{2}$ vector of the 2×2 minors of $\eta \in \mathbb{R}^{2N}$.

Accordingly the functional

$$\int_\Omega Qf(Du)dx \tag{9}$$

admits (a) minimizer(s) over D_ξ and the minimum value of (9) coincides with the infimum of (5) over D_ξ (see [11], Theorem 1.1).

As mentioned in the introduction, the following result about the existence of a minimizer for (5) (and not only for (9)) over D_ξ is derived in [7], Theorem 6.1 :

Theorem 2.1 ($n = 2$)*: A necessary and sufficient condition for (5) to have a minimizer over D_ξ is that at least one of the following hold*

(i) $\xi = 0$,

(ii) $|\xi|^2 + 2|adj_2\xi| \geq \frac{\lambda}{\alpha}$,

(iii) *rank* $\xi = 2$.

The proof of Theorem 2.1 hinges on the knowledge of the quasiconvexification Qf of f.

We prove below a generalization of Theorem 2.1 to arbitrary n. Our method is closely related to the homogenization approach for the relaxation of functional (5) because it uses decisively the characterization of $Qf(\eta)$ in terms of a finite dimensional minimization problem over the set of effective tensors associated to arbitrary mixtures of a material —with isotropic conductivity α— with voids of arbitrary shapes and sizes.

Let us begin with an explicit formula for the quasi-convexification $Qf(\eta)$ of the original function $f(\eta)$ for arbitrary n.

Theorem 2.2 *Let $0 \le \eta_1 \le ... \le \eta_n$ be the singular values of η (*i.e.*, the square roots of the eigenvalues of $\eta^t\eta$). Then*

$$Qf(\eta) = \begin{cases} \alpha|\eta|^2 + \lambda & \text{if } \sum_{i=1}^n \eta_i \ge \sqrt{\frac{\lambda}{\alpha}}, \\ \alpha|\eta|^2 - \alpha\left(\sum_{i=1}^n \eta_i\right)^2 + 2\sqrt{\lambda\alpha}\sum_{i=1}^n \eta_i & \text{if } \sum_{i=1}^n \eta_i < \sqrt{\frac{\lambda}{\alpha}}. \end{cases} \quad (10)$$

Furthermore, the quasi-convexification $Qf(\eta)$ coincides with the rank-one convex envelope of the original function $f(\eta)$.

Of course, in space dimension $n = 2$ the definitions (8) and (10) of $Qf(\eta)$ are equivalent.

Theorem 2.3 *Let $0 \le \xi_1 \le ... \le \xi_n$ be the singular values of ξ. A sufficient condition for (5), (7) to have a minimizer over D_ξ is that at least one of the following hold*

(i) $\xi = 0$,

(ii) $\sum_{i=1}^n \xi_i \ge \sqrt{\frac{\lambda}{\alpha}}$,

(iii) *rank* $\xi = n$,

while (5) has no minimizers over D_ξ when

(iv) *rank* $\xi = 1$ *and* $|\xi|^2 < \frac{\lambda}{\alpha}$.

Remark 2.4 *Note that sufficient conditions (i), (iv), and (ii) when rank $\xi = 1$, were previously derived in Corollary 5.3 of [7]. Thus the new results are sufficient conditions (ii), when $2 \le$ rank $\xi \le n$, and (iii) for the existence of a minimizer. We however present a complete proof of Theorem 2.3 because our proof of (iv) as a sufficient condition for non existence does not use in an essential manner the rotational invariance of the original functional, in contrast with that given in [7].*

Theorem 2.3 says nothing about the matrices ξ, with intermediate ranks between 2 and $n-1$, when $\sum_{i=1}^n \xi_i < \sqrt{\frac{\lambda}{\alpha}}$ (in contrast to the setting of Section 3 below). In such a case we conjecture that, for a bounded domain Ω, there are no minimizers of (5), (7). To support our claim, we now state a partial result which rules out the existence of "smooth-type" minimizers in such a case.

Proposition 2.5 *Let Ω be a bounded domain. Let $0 \le \xi_1 \le ... \le \xi_n$ be the singular values of ξ. Assume that*

$$2 \le \textit{rank } \xi \le n - 1, \;\; \textit{and } \sum_{i=1}^n \xi_i < \sqrt{\frac{\lambda}{\alpha}}.$$

For each function $u \in D_\xi$, extended by $\xi \cdot x$ outside Ω, define the set

$$Z_u = \{x \in I\!R^n \text{ such that } Du(x) = 0\}\,.$$

Then, (5) admits no minimizer $u \in D_\xi$ such that Z_u is a closed set in $I\!R^n$.

Remark 2.6 *We must however confess our dissatisfaction with the condition on the close character of Z_u. Indeed, it is doubtful whether this condition will be satisfied in general by a minimizer in D_ξ (see, for example, the "confocal ellipsoïds" construction, when rank $\xi = n$, in the proof of Theorem 2.3). Nevertheless, we believe the idea of the proof of Proposition 2.5 interesting enough to be included here. We also refer to Remark 2.12 below for a discussion of this conjecture from a different perspective.*

Proof of Theorem 2.2. The proof is divided into three steps. The first step provides a convenient characterization of the quasiconvexification of f using homogenization theory. In the second step an explicit expression for Qf is obtained. Finally the third step addresses the computation of the rank-one convex envelope of f with the help of the Kohn-Strang algorithm (see section 5C in [11]).

Step 1. The starting idea in our proof is familiar in the context of homogenization whenever the microstructure exhibits voids. A poor conductor is allowed to fill those, which cures the degeneracy of the conductivity tensor and permits direct application of the theory of homogenization. Of course it still remains to show that the algebraic limit, as the conductivity of the poor conductor tends to 0, of the obtained result is indeed the sought result (cf. for example [11], Section 6, or [1], Section 3, for similar considerations).

We introduce, in lieu of (7),

$$f_\beta(\eta) = \inf\left(\lambda + \alpha|\eta|^2, \beta|\eta|^2\right) \tag{11}$$

where $1 \leq \beta < +\infty$. The sequential lower semicontinuous envelope of

$$\int_\Omega f_\beta(Du)dx$$

over D_ξ is obtained by consideration of the new functional

$$\int_\Omega Qf_\beta(Du)dx$$

where Qf_β is the quasiconvexification of f_β, *i.e.*,

$$Qf_\beta(\eta) = \inf_{\varphi \in H^1_\#(Y;I\!R^N)} \int_Y f_\beta(\eta + D\varphi)dy. \tag{12}$$

In (12) Y is a unit cube in $\mathbb{R}^n$ ($Y = (0,1)^n$) and $H^1_\#(Y;\mathbb{R}^N)$ denotes the subspace of $H^1(Y;\mathbb{R}^N)$ of periodic functions. Note that the usual definition of the quasiconvexification of a functional over $\mathbb{R}^{nN}$ involves Dirichlet rather than periodic boundary data for the trial fields (see e.g. [6], Theorem 1.1, p. 201) but that both definitions are equivalent, at least when the functional is non negative, continuous and grows at most quadratically, which is precisely the case here ([5], Conjecture 3.7 and Theorem 3.1).

A simple switch in the minimizations leads to

$$Qf_\beta(\eta) = \inf_{\chi \in L^\infty(Y;\{0,1\})} \left\{ A_\chi \eta^t \cdot \eta + \lambda \int_Y \chi \, dy \right\}, \tag{13}$$

where, denoting by $(e_i)_{1\le i\le n}$ the canonical basis of $\mathbb{R}^n$, A_χ is a $n\times n$ symmetric matrix defined by its entries

$$A_\chi e_i \cdot e_j = \inf_{\varphi \in H^1_\#(Y;\mathbb{R})} \int_Y (\chi\alpha + (1-\chi)\beta)(e_i + D\varphi)\cdot(e_j + D\varphi)dy.$$

The matrix A_χ is the limit in the sense of homogenization —the H-limit— of the sequence

$$A^n_\chi = (\chi(nx)\alpha + (1-\chi(nx))\beta)I_2, \tag{14}$$

where I_2 is the identity matrix on $\mathbb{R}^n$. See [15], Section 5.

For a given $\theta \in L^\infty(\Omega;[0,1])$, the set $\mathcal{G}^\beta_\theta$ of all possible H-limits of sequences of the form

$$(\chi^n(x)\alpha + (1-\chi^n(x))\beta)I_2$$

with

$$\chi^n \rightharpoonup \theta \quad \text{weak-}\star \text{ in } L^\infty(\Omega;[0,1])$$

is known ([17], Theorem 1). It is of the form

$$\mathcal{G}^\beta_\theta = \left\{ A \in L^\infty(\Omega;\mathbb{R}^{n^2}_s) \,|\, A(x) \in G^\beta_{\theta(x)} \text{ a.e.} \right\}$$

where G^β_θ is, for any $0\le\theta\le 1$, a fixed set of matrices which is nothing else than the closure of the set of all H-limits of periodic sequences of the form (14) with $\int_Y \chi\,dy = \theta$ (see [17], Proposition 3 for a constructive proof or [8] for a more general argument). Further, as proved in Theorem 1 of [17], G^β_θ is the set of all symmetric $n\times n$ matrices with eigenvalues $\lambda_1,...,\lambda_n$ satisfying

$$\begin{cases} \underline{a}^\beta(\theta) := \left(\frac{\theta}{\alpha} + \frac{(1-\theta)}{\beta}\right)^{-1} \le \lambda_j \le \overline{a}^\beta(\theta) := \theta\alpha + (1-\theta)\beta, \quad 1\le j\le n, \\ \sum_{j=1}^n (\lambda_j - \alpha)^{-1} \le (\underline{a}^\beta(\theta)-\alpha)^{-1} + (n-1)(\overline{a}^\beta(\theta)-\alpha)^{-1}, \\ \sum_{j=1}^n (\beta-\lambda_j)^{-1} \le (\beta - \underline{a}^\beta(\theta))^{-1} + (n-1)(\beta-\overline{a}^\beta(\theta))^{-1}. \end{cases} \tag{15}$$

Then (13) reads as

$$Qf_\beta(\eta) = \inf_{0\le\theta\le1} \left\{ f^\star_\beta(\theta,\eta) + \lambda\theta \right\}, \tag{16}$$

with

$$f^\star_\beta(\theta,\eta) = \inf_{A\in G^\beta_\theta} \left\{ A\eta^t \cdot \eta \right\}. \tag{17}$$

Elementary order preserving properties of H convergence immediately imply that $f^\star_\beta(\theta,\eta)$ is monotonically increasing with β. Furthermore, the function $f^\star_\beta$ can be checked to be continuous in both its arguments (cf. e.g. [9], Lemma 3.9 for a proof in a more general setting). Thus Qf_β is a continuous function. Let us pause a moment in the proof of Theorem 2.2 in order to link Qf_β to the quasiconvexification of f; this is the object of the next lemma.

Lemma 2.7 *The sequence Qf_β monotonically increases to Qf as β goes to $+\infty$, where Qf is the quasiconvexification of f defined in (7).*

Proof. Since f_β monotonicity increases to f as $\beta \nearrow^{+\infty}$,

$$Qf_\beta \le Qf.$$

The functional Qf is quasiconvex and has at most quadratic growth. It is thus rank-1 convex (cf. [6], p. 105), hence continuous (cf. [6], Theorem 2.3, p. 29). Set

$$g_\beta(\eta) = \min\left\{ f_\beta(\eta), Qf(\eta) \right\}, \quad \eta \in \mathbb{R}^{nN}.$$

The sequence g_β is monotone in β and continuous in η; it converges to Qf as β tends to $+\infty$. Dini's theorem implies that

$$g_\beta \nearrow_{\beta\nearrow+\infty} Qf \text{ uniformly on compact subsets of } \mathbb{R}^{nN}. \tag{18}$$

Note that

$$Qg_\beta \le Qf_\beta \le Qf. \tag{19}$$

We prove that

$$Qf \le \lim_{\beta\to+\infty} Qg_\beta, \tag{20}$$

which establishes the desired result. Indeed take φ in $H^1_0(Y;\mathbb{R}^N)$ such that

$$\begin{aligned} Qg_\beta(\eta) \quad &\ge \textstyle\int_Y g_\beta(\eta + D\varphi)dy - \frac{1}{\beta} \\ &= \textstyle\int_Y Qf(\eta + D\varphi)dy - \frac{1}{\beta} - \int_Y |g_\beta - Qf|(\eta + D\varphi)dy. \end{aligned}$$

Since $g_\beta(\eta) = Qf(\eta)$ as soon as $|\eta|$ is large enough (say $|\eta| \ge M$, M independent of β), (18) implies that, denoting by Y_M the set $\{y \in Y \,|\, |D\varphi(y)| \le M + |\eta|\}$, for any $\varepsilon > 0$,

$$\int_Y |g_\beta - Qf|(\eta + D\varphi)dy = \int_{Y_M} |g_\beta - Qf|(\eta + D\varphi)dy \le \varepsilon,$$

as soon as β is large enough.

Thus, for β large enough

$$Qg_\beta(\eta) \geq \int_Y Qf(\eta + D\varphi)dy - \frac{1}{\beta} - \varepsilon \geq Qf(\eta) - \frac{1}{\beta} - \varepsilon$$

because Qf is quasiconvex. Letting ε tend to 0 and β to $+\infty$ proves (20), which concludes the proof of Lemma 2.7.

Let us resume the proof of Theorem 2.2. Upon setting

$$f^\star(\theta, \eta) = \lim_{\beta \to +\infty} f^\star_\beta(\theta, \eta),$$

we conclude, by virtue of Lemma 2.7, that

$$Qf(\eta) = \inf_{0 \leq \theta \leq 1} \{f^\star(\theta, \eta) + \lambda\theta\}. \tag{21}$$

Since $f^\star_\beta$ is continuous in θ, there exists, for a fixed β, a value θ_β of θ such that

$$Qf_\beta(\eta) = f^\star(\theta_\beta, \eta) + \lambda\theta_\beta. \tag{22}$$

Extract a converging subsequence of θ_β ($0 \leq \theta_\beta \leq 1$), still indexed by β. If $\beta < \beta'$, the monotone character of $f^\star_\beta$ implies

$$Qf(\eta) \geq Qf_{\beta'}(\eta) = f^\star_{\beta'}(\theta_{\beta'}, \eta) + \lambda\theta_{\beta'} \geq f^\star_\beta(\theta_{\beta'}, \eta) + \lambda\theta_{\beta'}.$$

We let β' tend to $+\infty$ and obtain, by virtue of the continuous character of $f^\star_\beta(\cdot, \eta)$,

$$Qf(\eta) \geq f^\star_\beta(\theta, \eta) + \lambda\theta.$$

Thus, letting β tend to $+\infty$,

$$Qf(\eta) \geq f^\star(\theta, \eta) + \lambda\theta,$$

i.e.,

$$Qf(\eta) = f^\star(\theta, \eta) + \lambda\theta$$

and we conclude from (21), (22) that

$$Qf(\eta) = \min_{0 \leq \theta \leq 1} \{f^\star(\theta, \eta) + \lambda\theta\} = \lim_{\beta \nearrow +\infty} \left(\min_{0 \leq \theta \leq 1} \left\{ f^\star_\beta(\theta, \eta) + \lambda\theta \right\} \right). \tag{23}$$

A much more explicit expression for $f^\star(\theta, \eta)$ may be derived with the help of (17) which defines $f^\star_\beta(\theta, \eta)$ as the infimum of a linear functional over the set G^β_θ. In view of equation (15), let us define G^∞_θ as the algebraic limit, as $\beta \nearrow^{+\infty}$, of

G_θ^β, *i.e.*, the set of symmetric $n \times n$ matrices with (possibly infinite) eigenvalues $\lambda_1, ..., \lambda_n$ satisfying

$$\left\{ \begin{array}{l} \lambda_i \geq \frac{\alpha}{\theta}, \quad 1 \leq i \leq n, \\ \sum_{i=1}^n (\lambda_i - \alpha)^{-1} \leq \frac{\theta}{(1-\theta)\alpha} \end{array} \right. \tag{24}$$

This yields

$$f^\star(\theta, \eta) = \inf_{A \in G_\theta^\infty} \{A\eta^t \cdot \eta\},$$

which completes the first step in the proof.

Step 2. To compute $f^\star(\theta, \eta)$, we first remark that

$$f^\star(\theta, \eta) = \inf_{A \in G_\theta^\infty} \operatorname{tr}(AH),$$

where $H = \eta^t \eta$ is a $n \times n$ symmetric matrix. Let us denote by $0 \leq \eta_1 \leq ... \leq \eta_n$ the singular values of η, *i.e.*, the square roots of the eigenvalues of H. A well-known result of von Neumann (see e.g. [13]) yields

$$f^\star(\theta, \eta) = \inf_{\lambda_1 \leq ... \leq \lambda_n} \left\{ \sum_{i=1}^n \lambda_i \eta_{n+1-i}^2 \right\}, \tag{25}$$

where $(\lambda_1, ..., \lambda_n)$ are the ordered eigenvalues of A, and the infimum is taken on the convex set defined in (24). Therefore, the infimum in (25) is a minimum if infinite values of λ_i are allowed. If the minimizer in (25) is such that, for some index i_0,

$$\lambda_{i_0} = \frac{\alpha}{\theta},$$

then, the constraints in (24) implies that all the others eigenvalues λ_i are infinite for $i \neq i_0$. This can happen only if $\eta_i = 0$ for $i \neq i_0$, *i.e.*, if η has rank one. Let us assume for the moment that the rank of the tensor η is strictly greater than one. Then, when writing the optimality conditions for minimizers in (25), the constraints $\lambda_i \geq \frac{\alpha}{\theta}$, which are not saturated, do not play any part. Therefore, the optimality conditions read

$$\exists C > 0 \text{ such that } \eta_{n+1-i}^2 = \frac{C^2}{(\lambda_i - \alpha)^2} \; 1 \leq i \leq n. \tag{26}$$

Taking into account that, at the minimum, $\sum_{i=1}^n (\lambda_i - \alpha)^{-1} = \frac{\theta}{(1-\theta)\alpha}$, a straightforward calculation yields

$$C = \frac{(1-\theta)\alpha}{\theta} \sum_{i=1}^n \eta_i,$$

and

$$f^\star(\theta, \eta) = \alpha \sum_{i=1}^n \eta_i^2 + \frac{(1-\theta)\alpha}{\theta} \left(\sum_{i=1}^n \eta_i \right)^2. \tag{27}$$

Formula (27) is immediately seen to hold true also if η has rank one. Then, a simple minimization over θ in (23) leads to the desired formula (10).

Step 3. According to Kohn and Strang [11], section 5C, the rank-one convex envelope Rf of f, *i.e.*, the largest rank-one convex function below f, is the limit as p goes to infinity of the sequence of functions f^p defined by

$$f^0(\eta) = f(\eta), \quad f^{p+1}(\eta) = \inf_{\substack{\eta=\theta\eta_1+(1-\theta)\eta_2,\ 0\leq\theta\leq 1\\ \mathrm{rank}(\eta_1-\eta_2)\leq 1}} \left(\theta f^p(\eta_1) + (1-\theta)f^p(\eta_2)\right).$$

The precise computation of $\lim_{p\to+\infty} f^p$ is cumbersome. Rather, we construct a sequence $f \geq g^p \geq f^p$ such that $g^n = Qf$. Since Qf is a rank-one convex function and f^p monotonically decreases to Rf, this proves that $g^n = f^n = Qf = Rf$. Let us define the sequence $(g^p)_{0\leq p\leq n}$ by

$$g^p(\eta) = \begin{cases} \alpha|\eta|^2 - \alpha\left(\sum_{i=1}^n \eta_i\right)^2 + 2\sqrt{\lambda\alpha}\sum_{i=1}^n \eta_i & \text{if } \sum_{i=1}^n \eta_i < \sqrt{\frac{\lambda}{\alpha}} \text{ and } \mathrm{rank}(\eta) \leq p,\\ \alpha|\eta|^2 + \lambda & \text{otherwise.}\end{cases}$$

Obviously g^p is a decreasing sequence such that $g^0 = f$ and $g^n = Qf$. It remains to prove that $g^p \geq f^p$ for $0 \geq p \geq n$. We proceed by induction on p. It is true for $p = 0$; let us assume it is also true up to order p. Then $g^{p+1}(\eta) = g^p(\eta)$ whenever the rank of η is different from $p + 1$. Thus $g^{p+1}(\eta) \geq f^{p+1}(\eta)$ if $\mathrm{rank}(\eta) \neq p+1$. When $\mathrm{rank}(\eta) = p+1$, the polar decomposition of η allows us to write

$$\eta = \sum_{i=1}^{p+1} \eta_i e_i \otimes h_i,$$

where (η_i) are the singular values of η and (e_i), (h_i) are orthonormal families of vectors in $\mathbb{R}^N$ and $\mathbb{R}^n$ respectively. Then,

$$f^{p+1}(\eta) \leq \inf_{0\leq\theta\leq 1}\left(\theta f^p(\eta_1) + (1-\theta)f^p(\eta_2)\right) \leq \inf_{0\leq\theta\leq 1}\left(\theta g^p(\eta_1) + (1-\theta)g^p(\eta_2)\right),$$

where

$$\eta_1 = \eta + \frac{1-\theta}{\theta}\eta_{p+1}e_{p+1}\otimes h_{p+1}, \text{ and } \eta_2 = \eta - \eta_{p+1}e_{p+1}\otimes h_{p+1}.$$

Since the rank of η_1 is $p + 1$, $g^p(\eta_1) = f(\eta_1)$, and a tedious minimization over θ yields

$$g^{p+1}(\eta) = \inf_{0\leq\theta\leq 1}\left(\theta g^p(\eta_1) + (1-\theta)g^p(\eta_2)\right),$$

which proves that $f^{p+1} \leq g^{p+1}$ for any p. This completes the proof of Theorem 2.2.

Remark 2.8 *In [11], section 5C, Kohn and Strang already proved, in the case $n = 2$, that Qf coincides with the rank-one convex envelope of f. Our proof that it is also true in higher dimensions $n \geq 2$ is a generalization of their two-dimensional proof, once the quasiconvexification Qf of f has been computed.*

Remark 2.9 *Our computation of the quasiconvex envelope Qf does not use in an essential manner the knowledge of the entire G-closure,* i.e., *the set G_θ^∞ (see (15)). It is enough to be able to minimize $\{A\eta^t \cdot \eta\}$ over all A's in G_θ^∞ (see the second step in the proof of Theorem 2.2). At the price of tedious computations, this latter task can be performed without the explicit knowledge of G_θ^∞ by using the so-called Hashin-Shtrikman variational principle (see [3] or [2]).*

Proof of Theorem 2.3. If $\xi = 0$ or $\sum_{i=1}^n \xi_i \geq \sqrt{\frac{\lambda}{\alpha}}$, then $f(\xi) = Qf(\xi)$ which proves that $\xi \cdot x$ is a minimizer for (5) over D_ξ. The rest of the proof is divided into two steps. The first step addresses the case where rank $\xi = 1$ while the second one examines the case where rank $\xi = n$.

Step 1. If rank $\xi = 1$, we deduce from Theorem 2.2 (or more precisely from (23)) that

$$Qf(\xi) = \min_{0 \leq \theta \leq 1} \left\{ \frac{\alpha}{\theta} |\xi|^2 + \lambda\theta \right\}. \tag{28}$$

Either $\alpha|\xi|^2 \geq \lambda$, in which case the minimum is obtained for $\theta = 1$ and

$$Qf(\xi) = \alpha|\xi|^2 + \lambda = f(\xi),$$

which proves sufficient condition ii) for the existence of a minimizer. Or $\alpha|\xi|^2 < \lambda$ in which case the minimum is obtained for

$$\theta_{\min} = \left(\frac{\alpha}{\lambda}\right)^{1/2} |\xi|,$$

and

$$Qf(\xi) = 2(\alpha\lambda)^{1/2}|\xi|.$$

In such a case assume that u is a minimizer for (5) over D_ξ. Since, by the very definition of the quasiconvexification Qf of f, $\xi \cdot x$ is a minimizer for

$$\int_\Omega Qf(Du)dx$$

over D_ξ,

$$Qf(\xi) = \frac{1}{|\Omega|} \int_\Omega f(Du)dx.$$

Define the set

$$\Omega_u = \{x \in \Omega \,|\, Du(x) \neq 0\}.$$

Then

$$Qf(\xi) = \frac{\alpha}{|\Omega|} \int_{\Omega_u} |Du(x)|^2 dx + \lambda \frac{|\Omega_u|}{|\Omega|}. \tag{29}$$

Note that

$$\frac{1}{|\Omega|} \int_\Omega Du\, dx = \xi = \frac{|\Omega_u|}{|\Omega|} \frac{1}{|\Omega_u|} \int_{\Omega_u} Du\, dx. \tag{30}$$

Jensen's inequality implies, in view of (29), (30), that

$$\begin{aligned} Qf(\xi) &\geq \tfrac{\alpha|\Omega_u|}{|\Omega|}\left(\tfrac{1}{|\Omega_u|}\int_{\Omega_u} Du(x)\,dx\right)^2 + \lambda\tfrac{|\Omega_u|}{|\Omega|} \\ &= \alpha\tfrac{|\Omega|}{|\Omega_u|}|\xi|^2 + \lambda\tfrac{|\Omega_u|}{|\Omega|}, \end{aligned}$$

which, together with (28), implies that

$$\theta_{\min} = \frac{|\Omega_u|}{|\Omega|},$$

and

$$\frac{1}{|\Omega|}\int_\Omega |Du(x)|^2 dx = \frac{|\Omega|}{|\Omega_u|}|\xi|^2,$$

or, equivalently,

$$\int_{\Omega_u} |Du(x)|^2 dx = \frac{|\Omega|^2}{|\Omega_u|}|\xi|^2. \tag{31}$$

If ν is a fixed element of $\mathbb{R}^{nN}$, (31) leads to

$$\int_{\Omega_u} |Du(x) - \nu|^2 dx = \frac{|\Omega|^2}{|\Omega_u|}\left|\xi - \nu\frac{|\Omega_u|}{|\Omega|}\right|^2,$$

and a choice of $\nu = \frac{|\Omega|}{|\Omega_u|}\xi$ yields

$$\int_{\Omega_u} \left|Du(x) - \frac{|\Omega|}{|\Omega_u|}\xi\right|^2 dx = 0.$$

Consequently,

$$Du(x) = \begin{cases} \frac{|\Omega|}{|\Omega_u|}\xi & , \text{ a.e. on } \Omega_u, \\ 0 & , \text{ a.e. on } \Omega \setminus \Omega_u. \end{cases}$$

But a vector valued function cannot have a gradient that only takes two values on Ω unless Ω_u and $\Omega\setminus\Omega_u$ are made of parallel strips normal to their difference (cf. Proposition 1 of [4]). These layers necessarily meet the boundary of Ω and the boundary value of u cannot be affine all along that boundary. There are thus no minimizers u for (5) over D_ξ. The proof of sufficient condition iv) for the non existence of minimizers is complete.

Step 2. If rank $\xi = n$, then $H = \xi^t\xi$ is a positive definite $n \times n$ matrix. Denote by $0 < \xi_1^2 \leq ... \leq \xi_n^2 < +\infty$ the eigenvalues of H.

By virtue of (23)

$$Qf(\xi) = f^\star(\theta, \xi) + \lambda\theta$$

for some $\theta \in [0, 1]$. Further $\theta \neq 0$ otherwise

$$Qf(\xi) = f^\star(0, \xi) = \lim_{\beta\to+\infty} f^\star_\beta(0, \xi) = \lim_{\beta\to+\infty} \beta|\xi|^2 = +\infty$$

because $\xi \neq 0$.

If $\theta = 1$ then

$$f^\star(1,\xi) = \alpha|\xi|^2,$$

and

$$Qf(\xi) = \alpha|\xi|^2 + \lambda,$$

from which it is immediately concluded that $\xi \cdot x$ is a minimizer for (5) over D_ξ.

Assume from now onward that $\theta \notin \{0,1\}$ and recall from (25) that

$$f^\star(\theta,\xi) = \inf_{A \in G_\theta^\infty} tr(AH) = \inf_{\lambda_1 \leq \ldots \leq \lambda_n} \left\{ \sum_{j=0}^{n-1} \xi_{n-j}^2 \lambda_{j+1} \right\}, \tag{32}$$

where the eigenvalues λ_j belong to the convex set defined in (24).

The infimum in (32) is attained and this at a point $(\lambda_1^0, ..., \lambda_n^0)$ such that

$$\left\{ \begin{array}{l} \lambda_j^0 > \frac{\alpha}{\theta}, \quad 1 \leq j \leq n, \\ \sum_{j=1}^n (\lambda_j^0 - \alpha)^{-1} = \frac{\theta}{(1-\theta)\alpha}, \end{array} \right.$$

since rank ξ is strictly greater than one (cf. step 2 in the proof of Theorem 2.2).

We now revisit the explicit construction proposed in Proposition 6 of [17] with the help of the so-called coated ellipsoïd. Consider two ellipsoïds B_{ρ^-} and B_{ρ^+} with equations

$$\sum_{j=1}^n \frac{x_j^2}{\rho^\pm + m_j} = 1,\ \rho^- < \rho^+,\ m = (m_1, ..., m_n) \in I\!R^n,\ \rho^- + \inf_j (m_j) > 0. \tag{33}$$

In (33) x_j denotes the j^{th} component of x in the orthonormal basis $\{e_i\}_{i=1,\ldots,n}$ generated by the eigendirections of H.

For each vector $\zeta \in I\!R^n$, the real-valued solution $u(\zeta, B_{\rho^+})$ of

$$\left\{ \begin{array}{ll} -\operatorname{div}\big(A(x) Du(\zeta, B_{\rho^+})\big) = 0 & \text{on } I\!R^n \setminus B_{\rho^-} \\ u(\zeta, B_{\rho^+}) = 0 & \text{on } \partial B_{\rho^-}, \\ u(\zeta, B_{\rho^+}) = \zeta \cdot x & \text{on } I\!R^n \setminus B_{\rho^+}, \end{array} \right. \tag{34}$$

with

$$A(x) = \left\{ \begin{array}{ll} \alpha I & \text{on } B_{\rho^+} \setminus B_{\rho^-}, \\ \sum_{i=1}^n \lambda_i e_i \otimes e_i & \text{on } I\!R^n \setminus B_{p^+},\ \frac{\alpha}{\theta} < \lambda_1 \leq ... \leq \lambda_n < +\infty, \end{array} \right.$$

is of the form (see Proposition 6 of [17])

$$u(\zeta, B_{\rho^+})(x) = \sum_{i=1}^n \zeta_i x_i f_i(\rho) \tag{35}$$

with

$$f_i(\rho) = \left[\int_{\rho^-}^{\rho^+} \frac{dt}{g_m(t)(t+m_i)}\right]^{-1} \int_{\rho^-}^{\rho} \frac{dt}{g_m(t)(t+m_i)}, \; 1 \le i \le n. \tag{36}$$

In (36)

$$g_m(\rho) = \prod_{k=1}^{n} (\rho + m_k)^{1/2}$$

is the volume of the ellipsoïd B_ρ with equation

$$\sum_{j=1}^{n} \frac{x_j^2}{\rho + m_j} = 1. \tag{37}$$

Further

$$\sum_{i=1}^{n} \frac{1}{\lambda_i - \alpha} = \left(\frac{g_m(\rho_+)}{g_m(\rho^-)} - 1\right) \frac{1}{\alpha}$$

so that, if we choose ρ_-/ρ_+ in a manner such that

$$g_m(\rho_-) = (1-\theta) g_m(\rho_+), \tag{38}$$

we get

$$\sum_{i=1}^{n} \frac{1}{\lambda_i - \alpha} = \frac{\theta}{(1-\theta)\alpha}, \quad \frac{\alpha}{\theta} < \lambda_1 \le ... \le \lambda_n < +\infty. \tag{39}$$

Finally when m spans $I\!R^n_+$, all points $(\lambda_1, ..., \lambda_n)$ satisfying (39) can be obtained as can be shown through a degree argument (see Proposition 6 of [17]). Before completing the proof of Theorem 2.3, we make a few useful comments on the above construction.

Remark 2.10 *The ellipsoïds $B_{\rho^\pm}$ corresponding to a given point $(\lambda_1, ..., \lambda_n)$ satisfying (38) can always be rescaled through multiplication of ρ^+, ρ^- and the m_i's by a small number so that B_{ρ^+} lies inside the unit cube Q. Then*

$$\begin{aligned} \alpha \int_{B_{\rho^+}} |Du(\zeta, B_{\rho^+})|^2 dx &= \alpha \int_{\partial B_{\rho^+}} u \tfrac{\partial u}{\theta n} dH^{n-1} \\ &= -\int_{\partial(I\!R^n - B_{\rho^+})} u\, A\, Du \cdot \overrightarrow{n}\, dH^{n-1} \\ &= \int_{\partial Q} u\, A\, Du \cdot \overrightarrow{n}\, dH^{n-1} - \int_{Q/B_{\rho^+}} A\, Du \cdot Du\, dx \end{aligned}$$

where, in the last two equalities, $\overrightarrow{n}$ represents the outward normal to the hypersurface over which integration is performed. But, according to the third equality in (34), $u = \zeta \cdot x$ on $\overline{Q} \setminus B_{\rho^+}$ so that

$$\begin{cases} \int_{\partial Q} uA\, Du \cdot \overrightarrow{n}\, dH^{n-1} &= \sum_{j=1}^{n} \lambda_j \zeta_j^2, \\ \int_{Q \setminus B_\rho} A\, Du \cdot Du\, dx &= \left(\sum_{j=1}^{n} \lambda_j \zeta_j^2\right)(1 - |B_{\rho^+}|). \end{cases}$$

Thus

$$\alpha \int_{B_{\rho^+}} |Du(\zeta, B_{\rho^+})|^2 dx = \left(\sum_{j=1}^{n} \lambda_j \zeta_j^2 \right) |B_{\rho^+}|.$$

Remark 2.11 *Everywhere inside $B_{\rho^+} \setminus B_{\rho^-}$ one has*

$$Du(\zeta, B_{\rho^+})(x) \neq 0.$$

Indeed, by virtue of the form (35) of $u(\zeta, B_{\rho^+})$,

$$\frac{\partial u}{\partial x_j} = \zeta_j f_j(\rho) + \sum_{i=1}^{n} \zeta_i x_i f_i'(\rho) \frac{\partial \rho}{\partial x_j}.$$

But, from (37),

$$\sum_{i=1}^{n} \frac{x_i^2}{(\rho + m_i)^2} \frac{\partial \rho}{\partial x_j} = 2 \frac{x_j}{\rho + m_j},$$

thus

$$\frac{\partial u}{\partial x_j} = \zeta_j f_j(\rho) + \frac{2}{\left(\sum_{i=1}^{n} \frac{x_i^2}{(\rho+m_i)^2} \right)} \sum_{i=1}^{n} \zeta_i x_i f_i'(\rho) \frac{x_j}{(\rho + m_j)}.$$

If

$$\frac{\partial u}{\partial x_j} = 0, \quad j = 1, ..., n,$$

it implies that

$$\sum_{j=1}^{n} \zeta_j (\rho + m_j) f_j'(\rho) \frac{\partial u}{\partial x_j} = \sum_{j=1}^{n} \zeta_j^2 (\rho + m_j) f_j'(\rho) f_j(\rho) + \frac{2 \left(\sum_{i=1}^{n} \zeta_i x_i f_i'(\rho) \right)^2}{\left(\sum_{i=1}^{n} \frac{x_i^2}{(\rho+m_i)^2} \right)} = 0,$$

which is impossible since $f_j(\rho)$ and $f_j'(\rho)$ are positive for $\rho^- < \rho < \rho^+$.

Coming back to the proof of Theorem 2.3, we consider the point $(\lambda_1^0, ..., \lambda_n^0)$ satisfying (38) such that the infimum in (32) is attained. According to Remark 2.10, there exists, say for a given ρ^- (hence a given ρ^+ determined by (38)), rescaled versions of B_{ρ^-} and B_{ρ^+} lying inside the unit cube and corresponding to $(\lambda_1^0, ..., \lambda_n^0)$. Vitali's covering theorem implies the existence of a countable family $\mathcal{G}$ of disjoint homothetics, of ratio less than or equal to 1, of B_{ρ^+} such that

$$\left| \Omega - \bigcup_{B^+ \in \mathcal{G}} B^+ \right| = 0. \tag{40}$$

Define, for $x \in \Omega$,

$$u(\zeta)(x) = \begin{cases} u(\zeta, B^+)(x) & \text{if } x \in B^+, \ B^+ \in \mathcal{G}, \\ \zeta \cdot x & \text{otherwise.} \end{cases} \tag{41}$$

Then, according to Remark 2.10, for B^+ in $\mathcal{G}$,

$$\alpha \int_{B^+} |Du(\zeta)|^2 dx = \left(\sum_{j=1}^{n} \lambda_j \zeta_j^2 \right) |B^+|.$$

Thus, by virtue of (40),

$$\alpha \int_{\Omega} |Du(\zeta)|^2 dx = \left(\sum_{j=1}^{n} \lambda_j \zeta_j^2 \right) |\Omega|. \tag{42}$$

Choose ζ to be successively $\xi_1, ..., \xi_N$ (the lines of the matrix ξ), and define

$$u_\xi = (u(\xi_1), ..., u(\xi_N)).$$

Then, in view of Remark 2.11, and upon denoting by B^- the homothetics of B_{ρ^-}, u_ξ, an element of D_ξ, satisfies

$$\begin{array}{lll} u_\xi(x) & = 0 \text{ on } B^- & (B^- \subset B^+,\ B^+ \in \mathcal{G}), \\ Du_\xi(x) & \neq 0 \text{ on } B^+ & B^+ \in \mathcal{G}, \end{array}$$

hence

$$\begin{aligned} \textstyle\int_\Omega f(Du_\xi(x))dx &= \left(\textstyle\sum_{i=1}^{N} \sum_{j=1}^{n} \lambda_j \xi_{ji}^2 \right) |\Omega| + \lambda \textstyle\sum_{B^+ \in \mathcal{G}} |B^+ \setminus B^-| \\ &= (tr(AH) + \lambda\theta)|\Omega| \\ &= \left(\textstyle\sum_{j=0}^{n-1} \xi_{n-j}^2 \lambda_{j+1}^0 + \lambda\theta \right) |\Omega| = Qf(\xi)|\Omega|. \end{aligned} \tag{43}$$

But

$$\inf \left\{ \int_\Omega f(Du(x))dx \,|\, u \in D_\xi \right\} = Qf(\xi)|\Omega|, \tag{44}$$

which in view of (43) permits us to conclude that u_ξ is a minimizer for (5) over D_ξ and proves (iii) in Theorem 2.3. The proof of that theorem is complete.

Remark 2.12 *Let us examine briefly the confocal ellipsoïds construction adapted to a matrix ξ of rank p with $2 \leq p \leq n-1$. Let us denote by $0 = \xi_1^2 = ... = \xi_{n-p}^2 < \xi_{n-p+1}^2 \leq ... \leq \xi_n^2 < +\infty$ the eigenvalues of the matrix $H = \xi^t \xi$, and by $e_1, ..., e_n$ the corresponding eigendirections. In the computation of $f^\star(\theta, \xi)$, the optimality condition (26) in the minimization over the eigenvalues $(\lambda_i)_{1 \leq i \leq n}$ of the homogenized tensor implies that the $(n-p)$ largest eigenvalues λ_i are equal to $+\infty$. According to Proposition 6 in [17] the corresponding values of the ellipsoïds parameters m_i are also equal to $+\infty$, implying that the domain B_{ρ^-} and B_{ρ^+} defined by equation (33) are cylinders obtained by translation in the directions $e_1, ..., e_{n-p}$ of p-dimensional ellipsoïds with axes given by $e_{n-p+1}, ..., e_n$. It can easily be checked that a solution $u(\zeta, B_{\rho^+})$ of (34) can still be defined, which* *does not depend on the variables $x_1, ..., x_{n-p}$. Therefore, Step 2 of the proof of Theorem 2.3 can be generalized if we assume that Ω is a cylindric (unbounded) domain defined by translation in the $e_1, ..., e_{n-p}$ directions of a p-dimensional domain in the subspace generated by $e_{n-p+1}, ..., e_n$. In other words, this proves the existence of a minimizer for (5) when $2 \leq rank\ \xi \leq n-1$ and $\sum_{i=1}^{n} \xi_i < \frac{\lambda}{\alpha}$ if the domain Ω is an unbounded cylindrical domain, aligned with some of the eigendirections of $H = \xi^t \xi$.*

However, when Ω has no such special properties (in particular if it is a bounded domain), we conjecture that there exists no minimizer for (5) over D_ξ when $2 \leq rank\ \xi \leq n-1$ and $\sum_{i=1}^{n} \xi_i < \frac{\lambda}{\alpha}$. We make such a claim because we believe that the optimality condition (26), which forces the $(n - rank\ \xi)$ largest eigenvalues λ_i of the homogenized tensor to be equal to $+\infty$, implies that possible minimizers do not depend on the corresponding $(n - rank\ \xi)$ variables x_i, a fact that would violate the boundary condition.

Proof of Proposition 2.5. Let $u \in D_\xi$ be a minimizer for (5) such that Z_u is closed. Let $d = d(Z_u, \partial\Omega)$ denote the Euclidean distance between Z_u and $\partial\Omega$. Since $Du(x) = \xi \neq 0$ for all points x of the closed set $\mathbb{R}^n \setminus \Omega$, the distance d is strictly positive. This implies the existence, for any boundary condition on $\partial\Omega$, of a test function, satisfying the boundary condition, and such that its gradient vanishes on Z_u. In other words, for any $u_0 \in H^1(\Omega)$,

$$\inf_{\substack{\phi \in \{u_0 + H_0^1(\Omega)\} \\ D\phi = 0 \text{ in } Z_u}} \int_\Omega \alpha |D\phi(x)|^2 dx \ < \ +\infty. \tag{45}$$

Let $Y = (0,1)^n$ be the unit cube in $\mathbb{R}^n$. By Vitali's covering theorem, there exists a countable family $(\Omega_i)_{i \geq 1}$ of disjoint homothetics, of ratio less than or equal to 1, of Ω such that

$$\left| Y - \bigcup_{i \geq 1} \Omega_i \right| = 0.$$

Let $(Z_u)_i$ be the associated family of homothetics of Z_u. Denote by Z the set

$$Z = \bigcup_{i \geq 1} (Z_u)_i.$$

Let $\chi(x)$ be the characteristic function of the set $Y \setminus Z$. Define the homogenized tensor A_χ associated to the characteristic function $\chi(x)$ by

$$A_\chi \zeta^t \cdot \zeta = \inf_{\substack{\varphi \in H^1_\#(Y) \\ \zeta + D\varphi = 0 \text{ in } Z}} \int_Y \alpha |\zeta + D\varphi|^2 dy, \tag{46}$$

where ζ is any vector in $\mathbb{R}^n$. Let us prove that A_χ is a bounded matrix in $\mathbb{R}^{n^2}$. Let ϕ be an admissible test function for (45) with boundary condition

$u_0(x) = \zeta \cdot x$. In each Ω_i we define a test function φ to be the sum of the homothetics of ϕ and of $-\zeta \cdot x$. Since φ is equal to 0 on the boundary of each Ω_i, pasting these contributions together we obtain a function of $H^1_\#(Y)$. Therefore,

$$A_\chi \zeta^t \cdot \zeta < +\infty \quad \forall \zeta \in I\!R^n. \tag{47}$$

If ζ is a line ξ_i of ξ, for $1 \le i \le N$, we can obtain a better bound for $A_\chi \xi_i^t \cdot \xi_i$. Using $\varphi(x) = -\xi_i \cdot x + u_i(x)$ as a test function in (46), where u_i is the i^{th} line of u, yields

$$A_\chi \xi^t \cdot \xi \le \int_\Omega \alpha |Du(x)|^2 dx.$$

If u is a minimizer for (5), it satisfies

$$\frac{1}{|\Omega|} \int_\Omega \alpha |Du(x)|^2 dx + \lambda \frac{|\Omega \setminus Z_u|}{|\Omega|} = Qf(\xi).$$

Therefore,

$$A_\chi \xi^t \cdot \xi + \lambda \frac{|\Omega \setminus Z_u|}{|\Omega|} \le Qf(\xi) = \min_{0 \le \theta \le 1} \left\{ \min_{A \in G_\theta^\infty} A\xi^t \cdot \xi + \lambda \theta \right\}. \tag{48}$$

Since $\int_Y \chi(x) dx = \frac{|\Omega \setminus Z_u|}{|\Omega|}$, we deduce that $\theta = \frac{|\Omega \setminus Z_u|}{|\Omega|}$ and $A = A_\chi$ realize the minimum in the right hand side of (48). On the other hand, the optimality condition on A in the computation of $Qf(\xi)$ (see (23), (25), and (26)) shows that at least one eigenvalue of A is equal to $+\infty$ because rank $\xi < n$. This is a contradiction with (47). Thus, there is no minimizer $u \in D_\xi$ such that Z_u is closed.

3 Existence of minimizers for a Kohn-Strang type functional defined on divergence free fields

This section, which parallels Section 2, is devoted to an analysis of possible minimizers for the functional

$$\int_\Omega f(\sigma) dx \tag{49}$$

where Ω is a bounded domain of $I\!R^n$, and σ is a divergence free field "which is affine" on the boundary of Ω, *i.e.*,

$$\sigma \in \Sigma_\xi = \left\{ \sigma \in L^2(\Omega; I\!R_s^{nN}) \mid \operatorname{div} \sigma = 0 \text{ on } \Omega \text{ and } \sigma \cdot \vec{n} = \xi \cdot \vec{n} \right\}, \tag{50}$$

where $\vec{n}$ denotes the outward unit normal to Ω at a point of $\partial\Omega$ and div σ is the N-vector whose components are $\left\{ \sum_{j=1}^n \frac{\partial \sigma_{ij}}{\partial x_j} \right\}_{1 \le i \le N}$. The specific function

f under consideration is similar to that introduced in Section 2. It is taken to be, for $\eta \in \mathbb{R}^{nN}$,

$$f(\eta) = \inf\{\lambda + \beta^{-1}|\eta|^2, 0\} \tag{51}$$

with $0 < \beta < +\infty$ and $\lambda < +\infty$.

As in Section 2, the function $f(\eta)$ is in truth the limit of the function

$$f_\alpha(\eta) = \inf\{\lambda + \beta^{-1}|\eta|^2, \alpha^{-1}|\eta|^2\} \tag{52}$$

when $\alpha > 0$ tends to zero. In contrast to the setting of Section 2 we shall also prove a partial result for the non-degenerate function f_α.

Once again the functional defined in (49) is not (sequentially) weakly lower semicontinuous over $L^2(\Omega; \mathbb{R}^{nN})$ so that minimizers for (49) over Σ_ξ defined in (50) need not exist. It is widespread belief that the lower semicontinuous envelope of a functional defined on divergence free fields has for integrand the quasi-convexification of the original integrand. In other words, if

$$Qf(\eta) = \inf_{s \in \Sigma_\#} \left\{ \int_Y f(\eta + s(y))dy \right\}$$

with

$$\Sigma_\# = \left\{ s \in L^2(Y; \mathbb{R}^N) \mid \operatorname{div} s = 0 \text{ in } Y,\ s \cdot \vec{n} \text{ antiperiodic on } \partial Y, \int_Y s\, dy = 0 \right\},$$

then the functional

$$\int_\Omega Qf(\sigma)dx \tag{53}$$

admits (a) minimizer(s) over Σ_ξ and the minimum value of (53) coïncides with the infimum of (49). The equivalence between sequential weak lower semicontinuity and quasiconvexity in the context of divergence free fields has recently been established in [10]. We will not concern ourselves in this section with a complete proof of the proposed form for the lower semicontinuous envelope.

Let us begin with an explicit formula for the quasi-convexification $Qf(\eta)$ of the original function $f(\eta)$.

Theorem 3.1 *Let $0 \le \eta_1 \le ... \le \eta_n$ be the singular values of η (*i.e.*, the square roots of the eigenvalues of $\eta^t\eta$). Define the function $\rho(\eta)$ by*

$$\rho(\eta) = \sqrt{\sum_{i=p+1}^{n} \eta_i^2 + \frac{1}{p-1}\left(\sum_{i=1}^{p} \eta_i\right)^2} \quad \text{if } \eta_{p+1} > \frac{1}{p-1}\sum_{i=1}^{p} \eta_i \ge \eta_p \tag{54}$$

(there exists a unique $p \in \{2, ..., n\}$ such that $\eta_{p+1} > \frac{1}{p-1}\sum_{i=1}^{p} \eta_i \ge \eta_p$ with the notation $\eta_{n+1} = +\infty$). Then

$$Qf(\eta) = \begin{cases} \frac{1}{\beta}|\eta|^2 + \lambda & \text{if } \rho(\eta) \ge \sqrt{\lambda\beta}, \\ \frac{1}{\beta}|\eta|^2 - \frac{1}{\beta}\rho(\eta)^2 + 2\sqrt{\frac{\lambda}{\beta}}\rho(\eta) & \text{if } \rho(\eta) < \sqrt{\lambda\beta}. \end{cases} \tag{55}$$

Remark 3.2 *In space dimension 2, formula (54) simplifies in $\rho(\eta) = \eta_1 + \eta_2$ and the function Qf introduced in Section 2 is recovered upon setting $\alpha = \frac{1}{\beta}$.*

In space dimension 3, there are two regimes in formula (54)

$$\rho(\eta) = \begin{cases} \frac{\eta_1+\eta_2+\eta_3}{\sqrt{2}} & \text{if } \eta_3 \leq \eta_1 + \eta_2 \\ \sqrt{(\eta_1+\eta_2)^2 + \eta_3^2} & \text{if } \eta_3 > \eta_1 + \eta_2. \end{cases}$$

Let us also remark that, in any space dimension, when rank $(\eta) \leq n-1$, i.e., *when $\eta_1 = 0$, one has $\rho(\eta) = |\eta|$. In this case, we deduce that*

$$Qf(\eta) = Cf(\eta) = \begin{cases} \frac{1}{\beta}|\eta|^2 + \lambda & \text{if } |\eta| \geq \sqrt{\lambda\beta} \\ 2\sqrt{\frac{\lambda}{\beta}}|\eta| & \text{if } |\eta| \leq \sqrt{\lambda\beta} \end{cases}$$

if rank $(\eta) \leq n-1$. This last remark is at the root of the next theorem.

Theorem 3.3 *A sufficient condition for (49), (51) to have a minimizer over Σ_ξ is that at least one of the following conditions holds*

(i) $\xi = 0$

(ii) $\rho(\xi) \geq \sqrt{\lambda\beta}$

(iii) *rank* $\xi = n$ *and* $\xi_n < \frac{1}{n-1}\sum_{i=1}^n \xi_i$

while (49) has no minimizers over Σ_ξ when

(iv) $1 \leq$ *rank* $\xi \leq n-1$ *and* $|\xi| < \sqrt{\lambda\beta}$.

Remark 3.4 *Theorem 3.3 leaves open the case rank $\xi = n$, $\rho(\xi) < \sqrt{\lambda\beta}$, and $\xi_n \geq \frac{1}{n-1}\sum_{i=1}^n \xi_i$. We conjecture that in such a case there are no minimizers of (49). As in Proposition 2.5 we could have stated a result ruling out "smooth-type" minimizers in this case. In the spirit of Remark 2.12 we point out that in such a case some of the eigenvalues of the homogenized tensor A, entering the computation of Qf (see (3.14) below), are zero. We believe it implies that possible minimizers do not depend on the variables of the corresponding eigendirections, a fact that would violate the affine boundary condition.*

Theorem 3.5 *Consider the minimization of the functional*

$$\int_\Omega f_\alpha(\sigma)dx \tag{56}$$

over Σ_ξ, defined in (50), with f_α defined in (52). If rank $\xi < n$, then (56) has a minimizer over Σ_ξ if and only if

$$\sqrt{\frac{(\beta-\alpha)}{\lambda}}\,|\xi| \notin]\alpha, \beta[.$$

Proof of Theorem 3.1. An argument identical to that which led to (16) would demonstrate that, for the function f_α defined in (52),

$$Qf_\alpha(\eta) = \inf_{0\le\theta\le1} \{f_\alpha^\star(\theta,\eta) + \lambda(1-\theta)\} \tag{57}$$

with

$$f_\alpha^\star(\theta,\eta) = \inf_{A\in G_\theta^\alpha} \left\{A^{-1}\eta^t\cdot\eta\right\}, \tag{58}$$

and where G_θ^α is identical to the set G_θ^β defined by (15). Note that the superscript has changed since α is now the varying parameter.

Since f_α monotonically increases to f as α goes to 0, an argument identical to that of Lemma 2.7 yields the monotone convergence of Qf_α to Qf as α goes to 0 and,

$$Qf(\eta) = \inf_{0\le\theta\le1} \{f^\star(\theta,\eta) + \lambda(1-\theta)\}, \tag{59}$$

with

$$f^\star(\theta,\eta) = \inf_{A\in G_\theta^0} \{A^{-1}\eta^t\cdot\eta\}, \tag{60}$$

and where G_θ^0 is the algebraic limit of G_θ^α as α goes to 0, *i.e.*, the set of symmetric $n\times n$ matrices with eigenvalues $\lambda_1,...,\lambda_n$ satisfying

$$\begin{cases} 0\le\lambda_i\le(1-\theta)\beta \\ \sum_{i=1}^n(\beta-\lambda_i)^{-1}\le\frac{\theta-1+n}{\theta\beta} \end{cases} \tag{61}$$

Since $A^{-1}\eta^t\cdot\eta = tr(A^{-1}\eta^t\eta)$ where $\eta^t\eta$ is a $n\times n$ matrix, denoting by $0\le\eta_1\le...\le\eta_n$ the singular values of η and by $0\le\lambda_1\le...\le\lambda_n$ the eigenvalues of A, a well-known result of von Neumann (see e.g. [13]) states that

$$\inf_{A\in G_\theta^0}\{A^{-1}\eta^t\cdot\eta\} = \inf\sum_{i=1}^n\frac{\eta_i^2}{\lambda_i} \tag{62}$$

where the infimum in the right hand side of (62) has to be taken over real values $(\lambda_i)_{1\le i\le n}$ satisfying

$$\begin{cases} \lambda_i\le...\le\lambda_n \\ 0\le\lambda_i\le(1-\theta)\beta \\ \sum_{i=1}^n(\beta-\lambda_i)^{-1}\le\frac{\theta-1+n}{\theta\beta} \end{cases}$$

To compute this infimum, let us assume for the moment that none of the constraints $0\le\lambda_i\le(1-\theta)\beta$ is active and that the only saturated constraint is

$$\sum_{i=1}^n(\beta-\lambda_i)^{-1} = \frac{\theta-1+n}{\theta\beta}.$$

In this case, a possible minimizer must satisfy the following Euler-Lagrange equation :

$$\frac{\eta_i^2}{\lambda_i^2} = \frac{C^2}{(\beta - \lambda_i)^2}, \quad 1 \le i \le n, \tag{63}$$

where $C > 0$ is a Lagrange multiplier.

An easy calculation yields

$$\lambda_i = \frac{\beta \eta_i}{\eta_i + \frac{\theta}{1-\theta} \frac{1}{n-1} \sum_{i=1}^n \eta_i}, \quad 1 \le i \le n. \tag{64}$$

Note that the (λ_i)'s are ordered. One must check that $0 \le \lambda_i \le (1-\theta)\beta$, for all $1 \le i \le n$, which is equivalent to

$$\eta_n \le \frac{1}{n-1} \sum_{i=1}^n \eta_i. \tag{65}$$

Note that $0 < \lambda_i < (1-\theta)\beta$, for all $1 \le i \le n$, is equivalent to the strict inequality in (65). If (65) is satisfied, the value of the minimum in (62) is

$$f^\star(\theta, \eta) = \frac{1}{\beta}|\eta|^2 + \frac{\theta}{\beta(1-\theta)(n-1)} \left(\sum_{i=1}^n \eta_i \right)^2. \tag{66}$$

If (65) is not satisfied, then one of the constraint $0 \le \lambda_i \le (1-\theta)\beta$ is saturated. Since $\lambda_i = 0$ can achieve the minimum in (62) only if $\eta_i = 0$, we consider the case when one of the eigenvalue λ_i is equal to $(1-\theta)\beta$. Let us assume that $\lambda_n = (1-\theta)\beta$ and that all the other eigenvalues satisfy $0 < \lambda_i < (1-\theta)\beta$, $1 \le i \le n-1$. The minimization in the right hand side of (62) becomes a $(n-1)$-dimensional problem with the single active constraint

$$\sum_{i=1}^{n-1} (\beta - \lambda_i)^{-1} = \frac{\theta - 2 + n}{\theta \beta}.$$

A computation similar to the previous one yields

$$\lambda_i = \frac{\beta \eta_i}{\eta_i + \frac{\theta}{1-\theta} \frac{1}{n-2} \sum_{i=1}^{n-1} \eta_i}, \quad 1 \le i \le n-1.$$

One must check again that $0 \le \lambda_i \le (1-\theta)\beta$, for all $1 \le i \le n-1$, which is equivalent to

$$\eta_{n-1} \le \frac{1}{n-2} \sum_{i=1}^{n-1} \eta_i. \tag{67}$$

If (67) is satisfied (but not (65)), the value of the infimum in (62) is

$$f^\star(\theta, \eta) = \frac{1}{\beta}|\eta|^2 + \frac{\theta}{(1-\theta)\beta}\eta_n^2 + \frac{\theta}{(1-\theta)\beta(n-2)} \left(\sum_{i=1}^{n-1} \eta_i \right)^2.$$

An easy induction argument shows that the minimum in (62) is

$$f^\star(\theta,\eta)=\frac{1}{\beta}|\eta|^2+\frac{\theta}{(1-\theta)\beta}\sum_{i=p+1}^{n}\eta_i^2+\frac{\theta}{(1-\theta)\beta}\frac{1}{p-1}\left(\sum_{i=1}^{p}\eta_i\right)^2$$

if η satisfies the following condition, denoted by (H_p),

$$\eta_p\le\frac{1}{p-1}\sum_{i=1}^{p}\eta_i,$$

and does not satisfy all previous conditions (H_q) for $p+1\le q\le n$. It is easily seen that, if (H_p) is not satisfied, all previous conditions (H_q) for $p+1\le q\le n$ are not satisfied either, and that (H_2) is always true. Therefore, with the notation $\eta_{n+1}=+\infty$, then exists a unique $p\in\{2,3,...,n\}$ such that

$$\eta_p\le\frac{1}{p-1}\sum_{i=1}^{p}\eta_i<\eta_{p+1}. \tag{68}$$

Introducing the function

$$\rho(\eta)=\sqrt{\sum_{i=p+1}^{n}\eta_i^2+\frac{1}{p-1}\left(\sum_{i=1}^{p}\eta_i\right)^2}$$

with p defined by (68), we finally obtain

$$f^\star(\theta,\eta)=\frac{1}{\beta}|\eta|^2+\frac{\theta}{(1-\theta)\beta}\rho(\eta)^2.$$

An easy optimization in θ leads to the announced formula for $Qf(\eta)$.

Remark 3.6 *In the spirit of Remark 2.9, we emphasize that our computation of the quasiconvex envelope Qf does not use in an essential manner the knowledge of the entire G-closure,* i.e., *the set G_θ^0. Indeed, the minimum of $\{A^{-1}\eta^t\cdot\eta\}$ over all A's in G_θ^0 can be computed without the explicit knowledge of G_θ^0 by using the so-called Hashin-Shtrikman variational principle (see [3] or [2]).*

Proof of Theorem 3.3. For any matrix ξ, if $\rho(\xi)\ge\sqrt{\lambda\beta}$, then $f(\xi)=Qf(\xi)$, and $\sigma(x)=\xi$ is a minimizer of (49) over Σ_ξ. Now, let ξ be a matrix of rank n such that

$$\xi_n<\frac{1}{n-1}\sum_{i=1}^{n}\eta_i.$$

Then, the homogenized matrix A which achieves the minimum in the left hand side of (62) has eigenvalues $(\lambda_i)_{1\le i\le n}$ satisfying

$$0<\lambda_i<(1-\theta)\beta,\quad 1\le i\le n.$$

Since the (λ_i)'s do not reach the values 0 and $(1-\theta)\beta$, one can repeat the argument of Section 2 concerning the confocal ellipsoïds construction (the parameters $(m_i)_{1\le i\le N}$ of the ellipsoïds are finite and non zero, see (33)). In the present case, the boundary condition on ∂B_{ρ^-} is a Neumann one, and the matrix $A(x)$ is

$$A(x) = \begin{cases} \beta I & \text{in } B_{\rho^+}\setminus B_{\rho^-}, \\ \sum_{i=1}^n \lambda_i e_i\otimes e_i & \text{in } I\!R^n \setminus B_{\rho^+}, \end{cases}$$

where (λ_i) is the minimizer in the right hand side of (62). Apart from this, the second step of the proof of Theorem 2.3 can be repeated *mutatis mutandis* to yield the existence of a minimizer of (49) over Σ_ξ.

Finally, consider a matrix ξ of rank less than or equal to $n-1$ and such that $\rho(\xi) = |\xi| < \sqrt{\lambda\beta}$. As noticed in Remark 3.2, $Qf(\xi) = Cf(\xi) = 2\sqrt{\frac{\lambda}{\beta}}|\xi|$ for such matrices ξ. In such a case, assume that $\sigma(x)$ is a minimizer for (49) over Σ_ξ. Then

$$Qf(\xi) = \frac{1}{|\Omega|}\int_\Omega f(\sigma)dx.$$

Define the set

$$\Omega_\beta = \{x\in\Omega \,|\, \sigma(x)\neq 0\}.$$

Then

$$Qf(\xi) = \frac{1}{|\Omega|}\int_{\Omega_\beta} \frac{1}{\beta}|\sigma(x)|^2 + \lambda\frac{|\Omega_\beta|}{|\Omega|}. \tag{69}$$

Note that

$$\int_\Omega \sigma_{ij}dx = \sum_{k=1}^n \int_\Omega D_k(\sigma_{ik}x_j)dx = \sum_{k=1}^n \int_{\partial\Omega} \xi_{ik}x_j n_k dH^{n-1}$$

$$= \sum_{k=1}^n \int_\Omega D_k(\xi_{ik}x_j)dx = |\Omega|\xi_{ij},$$

so that $\frac{1}{|\Omega|}\int_\Omega \sigma\, dx = \xi$.

The mapping ϕ defined on $I\!R^+\setminus\{0\}\times I\!R^{n^2}$ by

$$\phi(t,z) = \frac{|z|^2}{t}$$

is convex, and Jensen's inequality implies, in view of (69) that

$$\begin{aligned} Qf(\xi) &\ge \tfrac{|\Omega_\beta|}{|\Omega|}\phi\left(\beta, \tfrac{1}{|\Omega_\beta|}\textstyle\int_{\Omega_\beta}\sigma(x)dx\right) + \lambda\tfrac{|\Omega_\beta|}{|\Omega|} \\ &= \tfrac{|\Omega|}{\beta|\Omega_\beta|}|\xi|^2 + \lambda\tfrac{|\Omega_\beta|}{|\Omega|}. \end{aligned} \tag{70}$$

Hence, recalling that, since rank $\xi < n$,

$$Qf(\xi) = \min_{0\le\theta\le1} \left\{ \frac{1}{(1-\theta)\beta}|\xi|^2 + \lambda(1-\theta) \right\},$$

we deduce that

$$\frac{|\Omega_\beta|}{|\Omega|} = 1 - \theta_{\min},$$

and that equality holds in (70), *i.e.*,

$$\frac{1}{|\Omega_\beta|}\int_{\Omega_\beta} \phi\left(\beta, \sigma(x)\right) dx = \phi\left(\beta, \frac{1}{|\Omega_\beta|}\int_{\Omega_\beta} \sigma(x)dx\right). \tag{71}$$

But, for $t_0 > 0$, $z_0 \in I\!R^{n^2}$ and any $(t, z) \in (0, +\infty) \times I\!R^{n^2}$,

$$\phi(t, z) = \phi(t_0, z_0) + D\phi(t_0, z_0) \cdot (z - z_0, t - t_0) + \frac{1}{t}|z - \frac{t}{t_0}z_0|^2$$

so that, upon setting

$$\begin{cases} z = \sigma(x) & , \quad z_0 = \xi, \\ t = \beta & , \quad t_0 = \beta(1 - \theta_{\min}), \end{cases}$$

(71) implies that

$$\sigma(x) = \frac{1}{1 - \theta_{\min}}\xi \quad \text{for a.e. } x \in \Omega_\beta.$$

Then $\tilde\sigma(x) = \sigma(x) - \xi$ is a divergence-free field that satisfies

$$\begin{cases} \tilde\sigma(x) \cdot n = 0 \quad \text{on } \partial\Omega, \\ \int_\Omega \tilde\sigma(x)dx = 0, \\ \tilde\sigma(x) = \begin{cases} \frac{\theta_{\min}}{1-\theta_{\min}}\xi & \text{on } \Omega_\beta, \\ 0 & \text{on } \Omega \setminus \Omega_\beta. \end{cases} \end{cases}$$

But there are no such fields other than 0. Thus, for $\xi \neq 0$ and rank $\xi < n$, there are no minimizers for (49).

Proof of Theorem 3.5. The proof is very similar to that of Theorem 3.3. We start from the formula (57) for $Qf_\alpha(\eta)$,

$$Qf_\alpha(\eta) = \min_{0\le\theta\le1} \left\{ f_\alpha^\star(\theta, \eta) + \lambda(1-\theta) \right\}$$

where, denoting by $0 \le \eta_1 \le ... \le \eta_n$ the singular values of the matrix η,

$$f_\alpha^\star(\theta, \eta) = \inf_{\lambda_1\le...\le\lambda_n} \sum_{i=1}^{n} \frac{\eta_i^2}{\lambda_i}. \tag{72}$$

In (72) the infimum has to be taken on the following set :

$$\begin{cases} \underline{a}(\theta) = \left(\frac{\theta}{\alpha} + \frac{1-\theta}{\beta}\right)^{-1} \le \lambda_i \le \theta\alpha + (1-\theta)\beta = \overline{a}(\theta) \\ \sum_{i=1}^n (\lambda_i - \alpha)^{-1} \le (\underline{a}(\theta) - \alpha)^{-1} + (n-1)(\overline{a}(\theta) - \alpha)^{-1} \\ \sum_{i=1}^n (\beta - \lambda_i)^{-1} \le (\beta - \underline{a}(\theta))^{-1} + (n-1)(\beta - \overline{a}(\theta))^{-1} \end{cases}$$

If rank $\eta < n$, then $\eta_1 = 0$ and it is easily that the minimum in (72) is attained for

$$\lambda_1 = \left(\frac{\theta}{\alpha} + \frac{1-\theta}{\beta}\right)^{-1} \quad \text{and} \quad \lambda_i = \theta\alpha + (1-\theta)\beta, \quad 2 \le i \le n.$$

(In this case the constraint $\sum_{i=1}^n (\beta-\lambda_i)^{-1} \le (\beta - \underline{a}(\theta))^{-1} + (n-1)(\beta - \overline{a}(\theta))^{-1}$ is exactly satisfied.) Therefore, for rank $\eta < n$,

$$f_\alpha^\star(\theta, \eta) = \frac{1}{\theta\alpha + (1-\theta)\beta}|\eta|^2,$$

and a simple minimization in θ yields $Qf_\alpha(\eta) = Cf_\alpha(\eta)$, the convexification of f_α. From here on, the end of the proof follows that of Theorem 3.3.

4 A generalization of the Kohn-Strang functional; partial results on the possible existence of minimizers

In this short section the setting is that of Section 2, but a more general (non quadratic) functional is considered. Specifically, for η a matrix with lines $(\eta_1, ..., \eta_N)$ in $I\!R^n$,

$$f(\eta) = \inf \left\{ \lambda + \sum_{i=1}^N W_1(\eta_i), \ \sum_{i=1}^N W_2(\eta_i) \right\} \tag{73}$$

where W_1 and W_2 are convex C^1-function on $I\!R^n$, positively homogeneous of degree p ($1 < p < +\infty$) (*i.e.*, $W_i(\lambda a) = \lambda^p W_i(a)$, $\lambda \ge 0$, $i = 1, 2$); it is also assumed that

$$W_i(a) \ne 0, \quad a \ne 0, \quad i = 1, 2,$$

and that, for every b in $I\!R^n$, there exists a constant $\gamma(b) > 0$ such that

$$W_i(a) \ge W_i(b) + DW_i(b) \cdot (a - b) + \gamma(b)|a - b|^p. \tag{74}$$

We are unable, in this latter setting, to prove the exact analogue of Theorem 2.3 (or rather of the generalization of Theorem 2.3 to the non-degenerate case) because we lack an explicit construction of the type performed in the proof of Theorem 2.3 whenever rank $\xi = n$. Our result is the following

Theorem 4.1 *If rank $\xi = 1$, (5), (73) has a minimizer over D_ξ if and only if*

$$\begin{array}{l} \min_{\theta\in(0,1)} \left[\left(\sum_{i=1}^N \mu_i^p\right)(\theta W_1^\star + (1-\theta)W_2^\star)^\star(a) + \lambda\theta\right] \\ \geq \min\left[\left(\sum_{i=1}^N \mu_i^p\right) W_1(a) + \lambda, \left(\sum_{i=1}^N \mu_i^p\right) W_2(a)\right], \end{array} \quad (75)$$

where $\xi = \mu \otimes a$, $\mu \in \mathbb{R}^N$, $a \in \mathbb{R}^n$, and $\star$ denotes the Legendre transformation.

Remark 4.2 *The seemingly non explicit character of Theorem 4.1 can be cured whenever a more explicit form is available for W_1 and W_2. For example if $W_1(a) = (\alpha/p)|a|^p$, $W_2(a) = (\beta/p)|a|^p$, $\alpha < \beta$, then (75) is satisfied if and only if*

$$|\xi|^p \notin p^\lambda \left[\frac{1}{\beta^{\frac{1}{p-1}}}, \frac{1}{\alpha^{\frac{1}{p-1}}}\right].$$

Proof of Theorem 4.1. The functional (5) fails, once again, to be (sequentially) weakly lower semicontinuous over $H^1(\Omega, \mathbb{R}^N)$. Its lower semicontinuous envelope is given by

$$\int_\Omega Qf(\eta)dx,$$

where Qf, the quasiconvexification of f, is given by

$$Qf(\eta) = \inf_{\varphi\in W_\#^{1,p}(Y;\mathbb{R}^N)} \int_Y f(\eta + D\varphi)dy. \quad (76)$$

In (76) Y is the unit cube in $\mathbb{R}^n$ and $W_\#^{1,p}(Y;\mathbb{R}^N)$ denotes the subspace of $W^{1,p}(Y;\mathbb{R}^N)$ of periodic functions. An argument identical to that developed at the onset of Section 2 would lead to

$$Qf(\eta) = \inf_{0\leq\theta\leq 1}\{f(\theta,\eta) + \lambda\theta\}, \quad (77)$$

with

$$f(\theta,\eta) = \inf_{\substack{\chi\in L^\infty(Y\,;\,\{0,1\}) \\ \int_Y \chi(y)dy=\theta}} \left\{\sum_{i=1}^N W_\chi(\eta_i)\right\}. \quad (78)$$

In (78), W_χ is the homogenized energy associated to χ, *i.e.*,

$$W_\chi(a) = \inf_{\varphi\in W_\#^{1,p}(Y;\mathbb{R})} \int_Y (\chi(y)W_1 + (1-\chi(y))W_2)(a + D\varphi(y))dy \quad (79)$$

(see e.g. [12]).

If rank $\xi = 1$, then $\xi = \mu \otimes a$, $a \in I\!R^n$, $\mu \in I\!R^N$, and because of the homogeneous character of W_i, $i = 1, 2$, (78) becomes

$$f(\theta, \xi) = \left(\sum_{i=1}^{N} \mu_i^p \right) g(\theta, a), \tag{80}$$

with

$$g(\theta, a) = \inf_{\substack{\chi \in L^\infty(Y\,;\,\{0,1\}) \\ \int_Y \chi(y)dy = \theta}} \{W_\chi(a)\}. \tag{81}$$

A lower bound for $g(\theta, a)$ is easily obtained upon introduction of the dual problem for W_χ. Specifically, it is a classical result of the theory of homogenization –and a straightforward consequence of von Neumann's min-max theorem– that

$$W_\chi(a) = \sup_{b \in I\!R^n} \{b \cdot a - W_\chi^\star(b)\}, \tag{82}$$

where

$$W_\chi^\star(b) = \inf_{s \in \Sigma_\#} \left\{ \int_Y (\chi(y) W_1^\star + (1 + \chi(y)) W_2^\star)\,(b + s(y)) dy \right\}. \tag{83}$$

In (83), $W_i^\star$, $i = 1, 2$, are the Legendre transforms of W_1 and W_2, and $\Sigma_\#$ is defined by

$$\Sigma_\# = \left\{ s \in L^2(Y; I\!R^N) \mid \operatorname{div} s = 0 \text{ in } Y,\ s \cdot \vec{n} \text{ antiperiodic on } \partial Y,\ \int_Y s\,dy = 0 \right\},$$

Taking $s = 0$ as test function in (83) implies, in view of (82), that

$$g(\theta, a) \geq \sup_{b \in I\!R^n} \{b \cdot a - (\theta W_1^\star(b) + (1 - \theta) W_2^\star(b))\}. \tag{84}$$

Actually the inequality in (84) is an equality. This latter result is well known in the field of homogenization although we were enable to locate a complete proof in the available literature. A proof is given in Remark 4.3 below for the sake of completeness ; as such it can be safely skipped by a trusting reader.

Remark 4.3 *Inequality (84) is actually an equality. Indeed, consider the case of the homogenized energy associated to a characteristic function $\chi(y)$, defined on Y as*

$$\chi(y) = \tilde{\chi}(y_1),\ \text{with}\ \tilde{\chi}(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq \theta, \\ 0 & \text{if } 0 < t \leq 1. \end{cases}$$

Let $s \in I\!R$ and e_1 be the unit vector in the y_1-direction. Remark that if ϕ_1 and ϕ_2 are any two vectors of $I\!R^n$, with $\theta\phi_1 + (1 - \theta)\phi_2 = 0$, then the function

$$\varphi(y_1) = \begin{cases} \phi_1 y_1 & \text{if } 0 \leq y_1 \leq \theta, \\ \phi_2 y_1 + (\phi_1 - \phi_2)\theta & \text{if } \theta \leq y_1 \leq 1, \end{cases}$$

is an admissible test function in (79) specialized to the case at hand. Thus

$$W_\chi(se_1) \le I := \inf_{\substack{\phi_1,\phi_2 \in I\!\!R^n \\ \theta\phi_1+(1-\theta)\phi_2=0}} \{\theta W_1(se_1+\phi_1) + (1-\theta) W_2(se_1+\phi_2)\}. \quad (85)$$

The infimum I is computed as follows :

$$I = \inf_{\substack{\phi_1,\phi_2 \\ \theta\phi_1+(1-\theta)\phi_2=0}} \left\{\theta \sup_{q_1} [q_1\cdot(se_1+\phi_1) - W_1^\star(q_1)] + (1-\theta)\sup_{q_2}[q_2\cdot(se_1+\phi_2) - W_2^\star(q_2)]\right\}$$

and, upon application of a finite dimensional min-max theorem,

$$I = \sup_{q_1,q_2} \Big\{ se_1\cdot(\theta q_1 + (1-\theta)q_2) - (\theta W_1^\star(q_1) + (1-\theta)W_2^\star(q_2)) + \inf_{\substack{\phi_1,\phi_2 \\ \theta\phi_1+(1-\theta)\phi_2=0}} \{q_1\cdot\theta\phi_1 + q_2\cdot(1-\theta)\phi_2\}\Big\}. \quad (86)$$

But the infimum in (ϕ_1,ϕ_2) is $-\infty$ unless $q_1=q_2$. Thus

$$W_\chi(se_1) \le I = \sup_{q\in I\!\!R^n} \{se_1\cdot q - (\theta W_1^\star(q) + (1-\theta)W_2^\star(q))\}.$$

But, according to (81), (84),

$$W_\chi(se_1) \ge I.$$

Thus

$$W_\chi(se_1) = I.$$

Since we could always choose the y_1-direction to be in the direction of a given vector a, the choice of $a = se_1$ is not restrictive and

$$g(\theta,a) = \sup_{b\in I\!\!R^n} \{b\cdot a - (\theta W_1^\star(b) + (1-\theta)W_2^\star(b))\}. \quad (87)$$

Let us resume the proof of Theorem 4.1. In view of (80), (87), if rank $\xi = 1$, $\xi = \mu\otimes a$ and

$$f(\theta,\xi) = \left(\sum_{i=1}^N \mu_i^p\right)(\theta W_1^\star + (1-\theta)W_2^\star)^\star(a), \quad (88)$$

where $(\quad)^\star$ stands once again for the Legendre transform. Assume that (75) does not hold, or equivalently, that the infimum in (77) is attained for $\theta_{\min} \ne 0, 1$. (Note that $g(\theta,a)$ is a continuous function of θ; see [9], (3.21), (3.22) and Lemma 3.9). Then, if u is a minimizer for (5), (73) in D_ξ, let

$$\Omega_1 = \left\{ x\in\Omega \,|\, \sum_{i=1}^N W_1(Du_i(x)) + \lambda < \sum_{i=1}^N W_2(Du_i(x))\right\},$$

and

$$\Omega_2 = \Omega \setminus \Omega_1.$$

Then,

$$\int_\Omega f(Du(x))dx = \int_{\Omega_1} \sum_{i=1}^N W_1(Du_i(x))dx + \int_{\Omega_2} \sum_{i=1}^N W_2(Du_i(x))dx + \lambda|\Omega_1|. \quad (89)$$

Further, by virtue of Jensen's inequality,

$$Qf(\xi) = \frac{1}{|\Omega|}\int_\Omega f(Du(x))dx \geq \quad (90)$$

$$\frac{|\Omega_1|}{|\Omega|}\sum_{i=1}^N W_1\left(\frac{1}{|\Omega_1|}\int_{\Omega_1} Du_i(x)dx\right) + \frac{|\Omega_2|}{|\Omega|}\sum_{i=1}^N W_2\left(\frac{1}{|\Omega_2|}\int_{\Omega_2} Du_i(x)dx\right) + \lambda\frac{|\Omega_1|}{|\Omega|}.$$

Set

$$\theta = \frac{|\Omega_1|}{|\Omega|},\ \xi_1 = \frac{1}{|\Omega_1|}\int_{\Omega_1} Du(x)dx,\ \xi_2 = \frac{1}{|\Omega_2|}\int_{\Omega_2} Du(x)dx,$$

and remark that $\theta\xi_1 + (1-\theta)\xi_2 = \xi = \mu \otimes a$. Then (90) becomes

$$\begin{array}{rl} Qf(\xi) & \geq \theta \sum_{i=1}^N \sup_{b\in I\!R^n}\{b\cdot(\xi_1)_i - W_1^\star(b)\} \\ & \quad +(1-\theta)\sum_{i=1}^N \sup_{b\in I\!R^n}\{b\cdot(\xi_2)_i - W_2^\star(b)\} + \lambda\theta \\ & \geq \sum_{i=1}^N \sup_{b\in I\!R^n}\{b\cdot\xi_i - (\theta W_1^\star + (1-\theta)W_2^\star)(b)\} + \lambda\theta \\ & = \left(\sum_{i=1}^N \mu_i^p\right)\sup_{b\in I\!R^n}\{b\cdot a - (\theta W_1^\star + (1-\theta)W_2^\star)(b)\} + \lambda\theta. \end{array} \quad (91)$$

The homogeneous character of degree $\frac{p}{p-1}$ of $W_i^\star$ $(i = 1, 2)$ has been used in deriving the last equality of (91). In view of (77), (88), the equality holds in (90) or (91), *i.e.*, upon recalling (89),

$$\left\{ \begin{array}{l} \theta = \theta_{\min}, \\ \frac{1}{|\Omega_1|}\int_{\Omega_1} W_1(Du_i(x))dx = W_1((\xi_1)_i), \\ \frac{1}{|\Omega_2|}\int_{\Omega_2} W_2(Du_i(x))dx = W_2((\xi_2)_i). \end{array} \right.$$

Invoking (74) for the first (and last) time we conclude that

$$Du(x) = \left\{ \begin{array}{ll} \xi_1 \quad , & \text{a.e. on } \Omega_1, \\ \xi_2 \quad , & \text{a.e. on } \Omega_2, \end{array} \right.$$

which is impossible unless $\xi_1 = \xi_2$ by an argument identical to that used at the end of Step 1 of the proof of Theorem 2.3 in Section 2. But $\xi_1 = \xi_2 = \xi$ is not possible because $u = \xi \cdot x$ is not a minimizer since $\theta_{\min} \neq 0, 1$. The proof of Theorem 4.1 is complete.

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